Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Bijections of trees arising from Voiculescu's free probability theory

Artur Jeż and Piotr Śniady

Abstract. We present a bijective proof of the multidimensional generalizations of the Cauchy identity. Our bijection uses oriented planar trees equipped with some linear orders. The considered identities play an important role in the theory of operator algebras and our bijective prove can be used to prove multidimensional analogues of the arc-sine law in classical probability theory.

Résumé. Nous présentons une preuve bijective des généralisations multidimensionnelles de l'identité de Cauchy. Notre bijection emploie les arbres planaires orientés équipés de quelques ordres linéaires. Les identités considérés jouent un rôle important dans la théorie d'algèbres d'opérateur et notre bijection peut être employé pour prouver des analogues multidimensionnels de la loi d'arcsinus dans la théorie des probabilités classique.

1. Introduction

1.1. How to generalize the Cauchy identity? Cauchy identity states that for each nonnegative integer l

$$2^{2l} = \sum_{p+q=l} \binom{2p}{p}\binom{2q}{q}, \tag{1}$$

where the sum runs over nonnegative integers p, q. Cauchy identity and its bijective proof have important implications to the classical probability theory since they can be used to extract some information about random walks and arc-sine law [**Śni04**], it is therefore very tempting to look for some more identities which would share some resemblance to the Cauchy identity. Such identities could shed some light on the properties of the random walks in higher dimensions.

Guessing how the left-hand side of (1) could be generalized is not difficult and something like m^{ml} is a reasonable candidate. Unfortunately, it is by no means clear which sum should replace the right-hand side of (1). The strategy of writing down lots of wild and complicated sums with the hope of finding the right one by accident is predestined to fail. It is much more reasonable to find some combinatorial objects which are counted by the right-hand side of (1) and then to find a reasonable generalization of these objects.

For fixed integers $p, q \geq 0$ we consider the tree from Figure 1. Every edge of this tree is oriented and it is a good idea to regard these edges as one-way-only roads: if vertices x and y are connected by an edge and the arrow points from y to x then the travel from y to x is permitted but the travel from x to y is not allowed. This orientation defines a partial order $\prec$ on the set of the vertices: we say that $x \prec y$ if it is possible to travel from the vertex y to the vertex x by going through a number of edges (in order to remember this convention we suggest the Reader to think that $\prec$ is a simplified arrow $\leftarrow$). Let $<$ be a total order on the set of the vertices. We say that $<$ is compatible with the orientations of the edges if for all pairs of vertices x, y such that $x \prec y$ we also have $x < y$. It is very easy to see that for the tree from Figure 1 there are $\binom{2p}{p}\binom{2q}{q}$ total orders $<$ which are compatible with the orientations of the edges; this cardinality coincides with the

2000 *Mathematics Subject Classification.* Primary 05C05; Secondary 05A19, 05C30.

Key words and phrases. trees, bijections.

The second author thanks Kenneth Dykema for introducing him into the subject. Research supported by EU Research Training Network "QP-Applications", contract HPRN-CT-2002-00279 and European Commission Marie Curie Host Fellowship for the Transfer of Knowledge "Harmonic Analysis, Nonlinear Analysis and Probability" MTKD-CT-2004-013389. During the research the second author was a holder of a scholarship of European Post-Doctoral Institute for Mathematical Sciences.

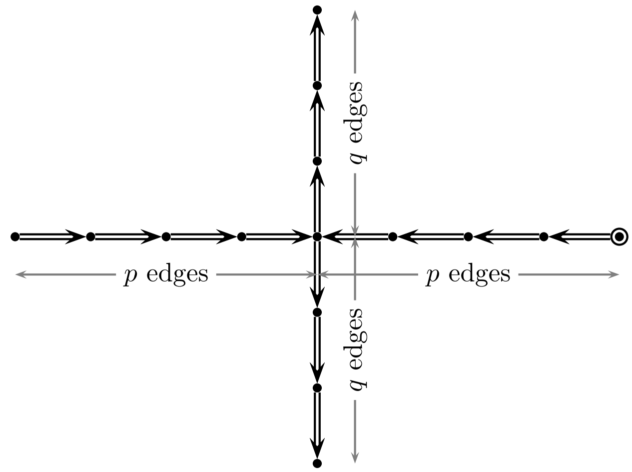

Figure 1. There are $\binom{2p}{p}\binom{2q}{q}$ total orders $<$ on the vertices of this oriented tree which are compatible with the orientation of the edges.

summand on the right-hand side of (1). It remains now to find some natural way of generating the trees of the form depicted on Figure 1 with the property $p+q=l$. We shall do it in the following.

1.2. Quotient graphs and quotient trees. We recall now the construction of Dykema and Haagerup [**DH04a**]. For integer $k \geq 1$ let G be an oriented k–gon graph with consecutive vertices $v_1, \dots, v_k$ and edges $e_1, \dots, e_k$ (edge e_i connects vertices v_i and v_{i+1}). The vertex v_1 is distinguished, see Figure 2. We encode the information about the orientations of the edges in a sequence $\epsilon(1), \dots, \epsilon(k)$ where $\epsilon(i) = +1$ if the arrow points from v_{i+1} to v_i and $\epsilon(i) = -1$ if the arrow points from v_i to v_{i+1}. The graph G is uniquely determined by the sequence ϵ and sometimes we will explicitly state this dependence by using the notation G_ϵ.

Let $\sigma = \big\{ \{i_1, j_1\}, \dots, \{i_{k/2}, j_{k/2}\} \big\}$ be a pairing of the set $\{1, \dots, k\}$, i.e. pairs $\{i_m, j_m\}$ are disjoint and their union is equal to $\{1, \dots, k\}$. We say that σ is compatible with ϵ if

$$\epsilon(i) + \epsilon(j) = 0 \qquad \text{for every } \{i,j\} \in \sigma. \tag{2}$$

It is a good idea to think that σ is a pairing between the edges of G, see Figure 2. For each $\{i,j\} \in \sigma$ we identify (or, in other words, we glue together) the edges e_i and e_j in such a way that the vertex v_i is identified with v_{j+1} and vertex v_{i+1} is identified with v_j and we denote by T_σ the resulting quotient graph. Since each edge of T_σ origins from a pair of edges of G, we draw all edges of T_σ as double lines. The condition (2) implies that each edge of T_σ carries a natural orientation, inherited from each of the two edges of G it comes from, see Figure 3.

From the following on, we consider only the case when the quotient graph T_σ is a tree. One can show [**DH04a**] that the latter holds if and only if the pairing σ is non–crossing [**Kre72**]; in other words it is not possible that for some $p < q < r < s$ we have $\{p, r\}, \{q, s\} \in \sigma$. The name of the non–crossing pairings comes from their property that on their graphical depictions (such as Figure 2) the lines do not cross. Let the root R of the tree T_σ be the vertex corresponding to the distinguished vertex v_1 of the graph G.

1.3. How to generalize the Cauchy identity? (continued). Let us come back to the discussion from Section 1.1. We consider the polygon G_ϵ corresponding to

$$\epsilon = (\underbrace{-1}_{l \text{ times}}, \underbrace{+1}_{l \text{ times}}, \underbrace{-1}_{l \text{ times}}, \underbrace{+1}_{l \text{ times}}).$$

All possible non-crossing pairings σ which are compatible with ϵ are depicted on Figure 4 and it easy to see that the corresponding quotient tree T_σ has exactly the form depicted on Figure 1.

In this way we managed to find relatively natural combinatorial objects, the number of which is given by the right-hand side of the Cauchy identity (1). After some guesswork we end up with the following conjecture (please note that the usual Cauchy identity (1) corresponds to $m = 2$).

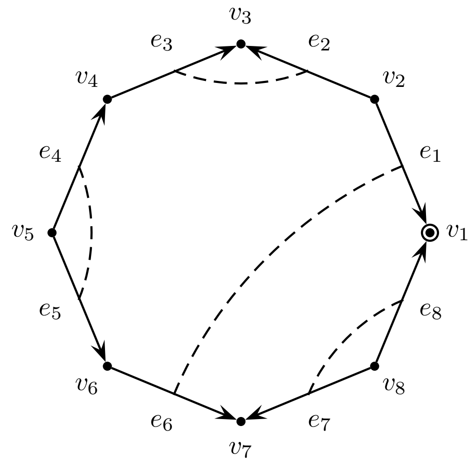

FIGURE 2. A graph G_ϵ corresponding to the sequence $\epsilon = (+1, -1, +1, +1, -1, -1, +1, -1)$. The dashed lines represent the pairing $\sigma = \big\{\{1, 6\}, \{2, 3\}, \{4, 5\}, \{7, 8\}\big\}$.

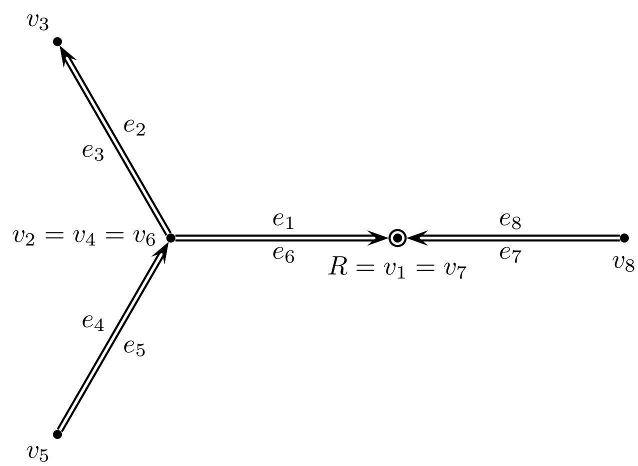

FIGURE 3. The quotient graph T_σ corresponding to the graph from Figure 2. The root R of the tree T_σ is encircled.

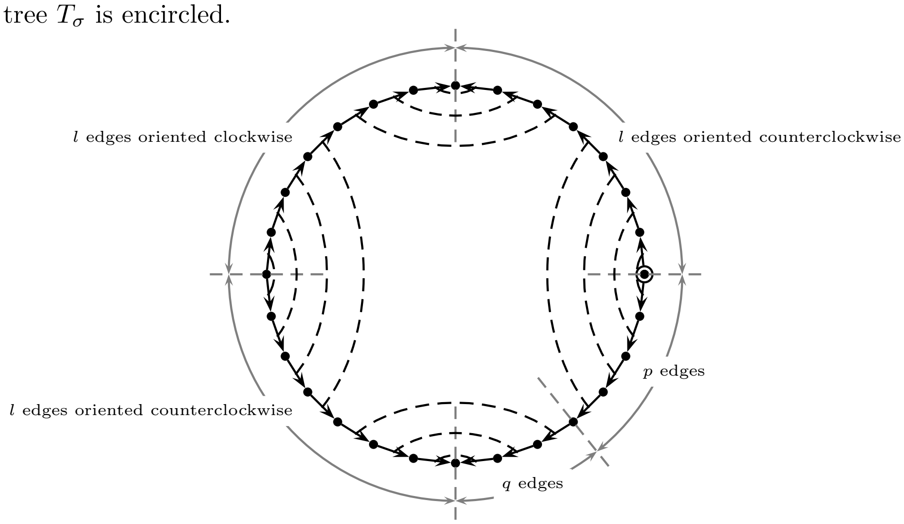

FIGURE 4. A graph T corresponding to sequence $\epsilon = (\underbrace{-1}_{l \text{ times}}, \underbrace{+1}_{l \text{ times}}, \underbrace{-1}_{l \text{ times}}, \underbrace{+1}_{l \text{ times}})$. The dashed lines denote a pairing σ for which the quotient graph T_σ is depicted on Figure 1.

THEOREM 1 (Generalized Cauchy identity). *For integers $l, m \geq 1$ there are exactly m^{ml} pairs $(\sigma, <)$, where σ is a non-crossing pairing compatible with*

$$\epsilon = (\underbrace{\underbrace{-1}_{l \text{ times}}, \underbrace{+1}_{l \text{ times}}, \underbrace{-1}_{l \text{ times}}, \underbrace{+1}_{l \text{ times}}, \dots}_{2m \text{ blocks, i.e. total of } 2ml \text{ elements}}) \tag{3}$$

and $<$ is a total order on the vertices of T_σ which is compatible with the orientations of the edges.

Above we provided only vague heuristical arguments why the above conjecture could be true. Surprisingly, as we shall see in the following, Theorem 1 is indeed true.

The formulation of Theorem 1 is combinatorial and therefore appears to be far from its motivation, the usual Cauchy identity (1), which is formulated algebraically, nevertheless for each fixed value of m one can enumerate all 'classes' of pairings compatible with (3) and for each class count the number of compatible orders $<$. To give to the Reader a flavor of the algebraic implications of Theorem 1, we present the case of $m = 3$ [**DY03**]:

$$3^{3l} = \sum_{p+q=l} \binom{3p}{p,p,p} \binom{3q}{q,q,q} + +3 \sum_{\substack{p+q+r=l-1 \\ r'+q'=r+q+1 \\ p''+r''=p+r+1}} \binom{2p+p''}{p,p,p''} \binom{2q+q'}{q,q,q'} \binom{r+r'+r''}{r,r',r''}. \tag{4}$$

and the case of $m = 4$ [**Śni03**]:

$$\begin{aligned} 4^{4k} = {} & \sum_{p+q=k} \binom{4p}{p,p,p,p} \binom{4q}{q,q,q,q} + 8 \sum_{\substack{p+q+r=k-1 \\ p'+q'=p+q+1 \\ p''+q''=p+q+1 \\ q'''+r'''=q+r+1}} \binom{2p+p'+p''}{p,p,p',p''} \binom{q+q'+q''+q'''}{q,q',q'',q'''} \binom{3r+r'''}{r,r,r,r'''} + \\ & + 4 \sum_{\substack{p+q'+r'=k-1 \\ p+q''+r''=k-1 \\ p'''+q'''=p+q'+1 \\ p''''+q''''=p+q''+1}} \binom{2p+p'''+p''''}{p,p,p''',p''''} \binom{q'+q'''}{q',q'''} \binom{q''+q''''}{q'',q''''} \binom{2r''}{r'',r''} \binom{2r'}{r',r'} \binom{q'+q''+q'''+q''''+2r'+2r''+2}{q'+q'''+2r'+1, q''+q''''+2r''+1} + \\ & + 8 \sum_{\substack{p+q+r+s=k-2 \\ q'+r'=q+r+s+2 \\ p''+r''=p+q+r+2}} \binom{2p}{p,p} \binom{q+q'}{q,q'} \binom{r+r''}{r,r''} \binom{2s}{s,s} \binom{3p+p''+2q+q'+2}{2p+q+q'+1, p+q+1, p''} \binom{2r+r'+r''+3s+2}{r+r''+2s+1, r+s+1, r'}. \end{aligned} \tag{5}$$

1.4. Bijective proof of generalized Cauchy identities. Theorem 1 was conjectured by Dykema and Haagerup [**DH04a**] and its first proof (analytic one) was given by the second-named author [**Śni03**]. Another analytic proof was given by Aagaard and Haagerup [**AH04**]. The main result of this article (which is a shortened and edited version of [**Śni04**]) is the first bijective proof of Theorem 1, formulated explicitly as the following theorem.

THEOREM 2 (The main result). *Let integers $l, m \geq 1$ be given. We set $L = lm + 1$ and*

$$\epsilon_i = (\underbrace{\underbrace{(-1)^{i-1}}_{l \text{ times}}, \underbrace{(-1)^i}_{l \text{ times}}}_{i \text{ times, i.e. a total of } 2li \text{ elements}}) \qquad \text{for } 1 \leq i \leq m.$$

Note that ϵ_m coincides (up to a possible sign change) with (3). The function described in this article provides a bijection between

(α) *the set of pairs $(\sigma, <)$, where σ is a pairing compatible with ϵ_m and $<$ is a total order on the vertices of T_σ which is compatible with the orientations of the edges;*

(β) *the set of tuples $(B_1, \dots, B_m)$, where $B_1, \dots, B_m$ are disjoint sets such that $B_1 \cup \dots \cup B_m = \{1, 2, \dots, L\}$ and*

$$|B_1| + \dots + |B_n| \leq ln$$

holds true for each $1 \leq n \leq m - 1$;

Alternatively, set (β) can be described as

(γ) *the set of sequences* $(a_1,\dots,a_L)$ *such that* $a_1,\dots,a_L \in \{1,\dots,m\}$ *and for each* $1 \le n \le m-1$ *at most* ln *elements of the sequence* (a_i) *belong to the set* $\{1,\dots,n\}$;

where the bijection between sets (β) and (γ) is given by $B_j = \{k : a_k = j\}$.

From the Raney lemma [**Ran60**] it follows that the set (β) has m^{ml} elements [**Śni03**] hence Theorem 1 indeed follows from Theorem 2.

2. Quotient trees

In the following we shall discuss some aspects of the quotient trees which were not included in Section 1.2. Sometimes, with a very small abuse of notation, we will denote by the same symbol T_σ the set of the vertices of the tree T_σ.

2.1. Structure of a planar tree. Order $\lhd$. For a non–crossing pairing σ we can describe the process of creating the quotient graph as follows: we think that the edges of the graph G are sticks of equal lengths with flexible connections at the vertices. Graph G is lying on a flat surface in such a way that the edges do not cross. For each pair $\{i,j\} \in \sigma$ we glue together edges e_i and e_j by bending the joints in such a way that the sticks should not cross. In this way T_σ has a structure of a planar tree, i.e. for each vertex we can order the adjacent edges up to a cyclic shift (just like points on a circle). We shall provide an alternative description of this planar structure in the following.

Let us visit the vertices of G in the usual cyclic order $v_1, v_2, \dots, v_k, v_1$ by going along the edges $e_1,\dots,e_k$; by passing to the quotient graph T_σ we obtain a journey on the graph T_σ which starts and ends in the root R. The structure of the planar tree defined above can be described as follows: if we travel on the graphical representation of T_σ by touching the edges by our left hand, we obtain the same journey. For each vertex of T_σ we mark the time we visit it for the first time; comparison of these times gives us a total order $\lhd$, called preorder [**Sta99**], on the vertices of T_σ. For example, in the case of the tree from Figure 3 we have $v_1 \lhd v_2 \lhd v_3 \lhd v_5 \lhd v_8$.

2.2. Catalan sequences. We say that $\epsilon = \big(\epsilon(1),\dots,\epsilon(k)\big)$ is a Catalan sequence if $\epsilon(1),\dots,\epsilon(k) \in \{-1,+1\}$, $\epsilon(1)+\cdots+\epsilon(k)=0$ and all partial sums are non-negative: $\epsilon(1)+\cdots+\epsilon(l) \ge 0$ for all $1 \le l \le k$. We say that ϵ is anti-Catalan if $-\epsilon$ is Catalan.

Lemma 3. *For a Catalan sequence ϵ there exists a unique compatible pairing σ with the property that $R \preceq v$ for every vertex $v \in T_\sigma$. For an anti-Catalan sequence ϵ there exists a unique compatible pairing σ with the property that $R \succeq v$ for every vertex $v \in T_\sigma$.*

3. Proof of almost the main result

3.1. Statement of the result. The following result will be crucial for the bijective proof of generalized Cauchy identities in Section 4.

Theorem 4. *Let $\epsilon = \big(\epsilon(1),\dots,\epsilon(k)\big)$ be a Catalan sequence. The function described in this section provides a bijection between*

(A) *the set of pairs $(\sigma,<)$, where σ is a pairing compatible with ϵ and $<$ is a total order on the vertices of T_σ compatible with the orientation of the edges;*

(B) *the set of pairs $(\sigma,<)$, where σ is a pairing compatible with ϵ and $<$ is a total order on the vertices of T_σ with the following two properties:*
- *on the set $\{x \in T_\sigma : x \succeq R\}$ the orders $<$ and $\lhd$ coincide;*
- *for all pairs of vertices $v,w \in T_\sigma$ such that $R \not\preceq v$ and $R \not\preceq w$ we have*

$$v \prec w \implies v < w.$$

Proof. In this article we will present only the bijection without presenting its inverse and without any proofs which can be found in [**Śni04**].

Our bijection will be given by repeating the following procedure: if the pair $(\sigma,<)$ is as in (B) then we our algorithm finishes. Otherwise, let D be the maximal element (with respect to the order $<$) such that $D \succ R$ and such that on the subtree $U = \{x : x \succeq R \text{ and } x < D\}$ the orders $<$ and $\lhd$ coincide. The vertex D is a leaf of the tree U which is not maximal in U (with respect to the order $\lhd$); otherwise this would contradict the maximality of D. We start in D a walk on the graph U with the first step going

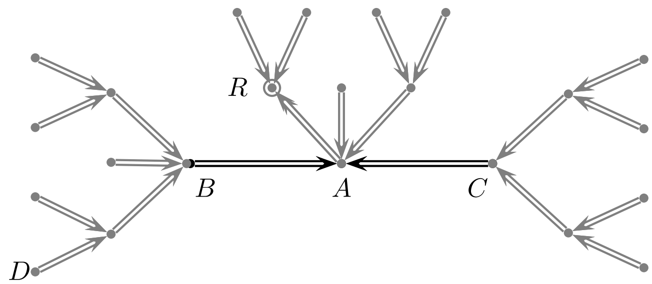

FIGURE 5. The case $D \neq B$. The order of the vertices is given by $R \leq A < B < C < D$. Note that only the edges belonging to the subtree U are displayed.

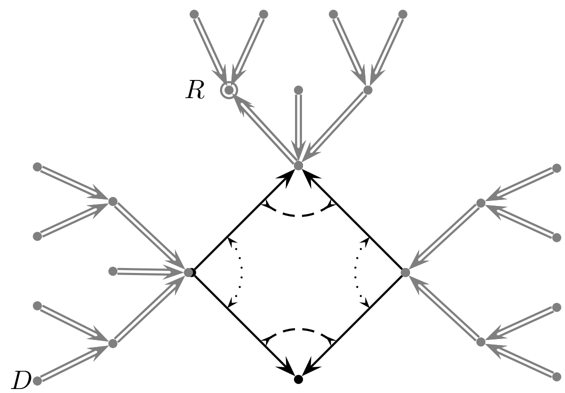

FIGURE 6. The tree from Figure 5 after ungluing the edges BA and CA.

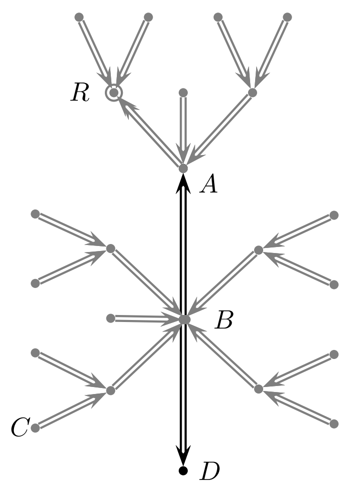

FIGURE 7. The tree from Figure 5 after regluing the edges BA and CA in a different way. Please notice the change of the labels of the vertices A, B, C, D.

towards the root R, always touching the edges by our left hand (as we did in Section 2.1) and we denote by $w_0 = D, w_1, w_2, \dots$ the consecutive vertices we visit on our journey. Let n be the smallest number for which the arrow on the edge connecting w_n and w_{n+1} points from w_{n+1} towards w_n; we denote $B = w_{n-1}$, $A = w_n$, $C = w_{n+1}$.

Let us consider the case when $B \neq D$, cf. Figure 5. Each of the edges BA and CA of the quotient graph T_σ was created by gluing a pair of edges of the graph G; let us unglue these four edges of G, cf. Figure 6 and let us glue these four edges in pairs in a different way, cf Figure 7. In this way we obtain a quotient

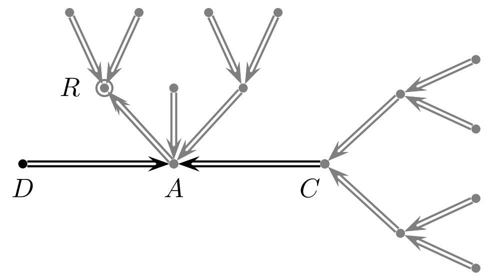

FIGURE 8. The case $D = B$. The order of vertices is given by $R \leq A < C < D$.

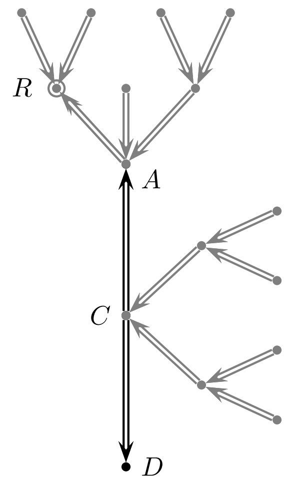

FIGURE 9. The tree from Figure 8 after regluing the edges DA and CA in a different way. Please notice the change of the labels of the vertices A, C, D.

graph $T_{\sigma'}$, where σ' is a pairing of edges obtained from σ by changing connections between certain four edges. Figure 5 and Figure 7 show an identification between the vertices of T_σ and $T_{\sigma'}$; please note that this identification is nontrivial only on the vertices A, B, C, D. We define the order $<$ on $T_{\sigma'}$ to be the inherited order $<$ from T_σ under the above identification of the vertices.

We consider now the case when $B = D$, cf. Figure 8. Similarly as above, we unglue and reglue in a different way edges DA and CA and thus we obtain a tree $T_{\sigma'}$ depicted on Figure 9. Figure 8 and Figure 9 show the identification between the vertices of T_σ and the vertices of $T_{\sigma'}$ and we define the order $<$ on $T_{\sigma'}$ to be the inherited order $<$ from T_σ.

After a finite number of steps the above procedure will eventually stop. □

REMARK 5. For each pair $(\sigma, <)$ from the set (A) and the corresponding pair $(\sigma', <)$ from the set (B) there is a canonical unique bijection j mapping the vertices of T_σ onto the vertices of $T_{\sigma'}$ with the property that for all $v, w \in T_\sigma$ the condition $v < w$ holds if and only if $j(v) < j(w)$. In fact this identification is very easy to see since the bijection from Theorem 4 is a composition of a number of elementary operations. Each such operation is either a replacement of Figure 5 by Figure 7 or replacement of Figure 8 by Figure 9 and for each such a replacement the corresponding identification preserves the labels of the vertices.

4. Proof of the main result

PROOF. We shall construct now the main result of the article: the bijection announced in Theorem 2. In this article we will present only the bijection without presenting its inverse and without any proofs which can be found in [**Śni04**].

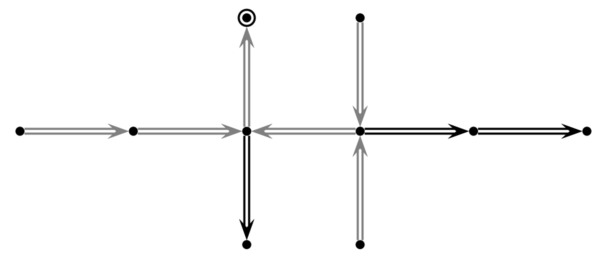

Figure 10. Example of a tree $T_{\tilde{\sigma}}$. A subtree $\{v : R \preceq v\}$ was marked in gray.

Firstly, observe that the order $<$ on the vertices of the tree T_σ can be alternatively described by labeling the vertices by the numbers from the set $\{1, 2, \dots, L\}$ in such a way that each number appears exactly once and the order of the labels coincides with the order $<$ on the vertices.

Our algorithm consists of $m-1$ steps; in the first step the variable i takes the value $i := m$ and after each step its value decreases by one. At the beginning of each step we start with a tree T_σ, where σ is a pairing compatible with ϵ_i such that some of the vertices are labeled by the numbers from the set $\{1, 2, \dots, L\}$ and some vertices might be unlabeled (in the first step $i = m$ there are no unlabeled vertices) and in this step we will construct the set B_i.

Let us consider the case when i is odd. We define a total order $<$ on the vertices of T_σ as follows: for a pair of vertices v, w which carry some labels we set $v < w$ if and only if the label of v is smaller than the label of w; if v has no label and w has a label then $v < w$; if both v and w have no labels then $v < w$ if and only if $v \lhd w$. In this way $(\sigma, <)$ is as prescribed in point (A) of Theorem 4.

Let $(\tilde{\sigma}, <)$ denote the corresponding element of the point (B). We consider the canonical identification of the vertices of the tree T_σ with the vertices of the tree $T_{\tilde{\sigma}}$, as described in Remark 5; in this way some of the vertices of the tree $T_{\tilde{\sigma}}$ are labeled by the numbers from the set $\{1, \dots, L\}$. We consider a subtree $U = \{x \in T_{\tilde{\sigma}} : R \preceq x\}$. We define B_i to be the set of the labels on the vertices of U and we remove all labels from the vertices of U.

Each edge of $T_{\tilde{\sigma}}$ consists of two edges of the graph G; let us unglue all the edges belonging to the tree U. We denote by T' the resulting graph, cf Figure 10 and Figure 11. The sequence ϵ_{i-1} can be obtained from the sequence ϵ_i by removal of the first l and the last l elements therefore the polygonal graph $G_{\epsilon_{i-1}}$ can be obtained from the graph G_{ϵ_i} by removing two groups (of l edges each) surrounding the distinguished vertex R from both sides; clearly these $2l$ edges must be among the unglued ones in the graph T'. We denote by T'' the graph obtained from T' by the removal of these $2l$ edges, cf Figure 12.

Please note that T'' can be obtained from the polygonal graph $G_{\epsilon_{i-1}}$ by gluing some pairs of edges hence it can be viewed as a certain polygonal graph $G_{\epsilon'}$ with a number of trees attached to it. The sequence ϵ' can be obtained from ϵ_{i-1} by a removal of a number of blocks of consecutive elements, provided the sum of elements of each block is equal to zero. Since ϵ_{i-1} is anti-Catalan, ϵ' is anti-Catalan as well. We denote by $T_{\sigma'}$ the tree resulting from T'' by gluing the edges constituting $G_{\epsilon'}$ by the pairing given by Lemma 3 applied to ϵ'; please note that in this way we defined implicitly the pairing σ' compatible with ϵ_{i-1}, cf Figure 13. Thus, the description of the step of the algorithm in the case when i is odd is finished.

To cover the case when i is even we can simply reverse the orientations on all edges (which corresponds to a change of signs in the sequence ϵ_i) and consider the opposite order on the set $\{1, \dots, L\}$; since sequence $-\epsilon_i$ is Catalan and $-\epsilon_{i-1}$ is anti-Catalan we reduced the situation to the case considered previously.

Our algorithm takes a particularly simple form for $i = 1$; we simply set B_1 to be the set of the labels of the tree T_σ and the algorithm stops. □

5. Combinatorial calculus: how to convert an analytic proof into a bijection?

The bijection presented in this article might look artificial and it is by no means clear how the authors invented it. It turns out that there is a very systematic way of constructing this bijection given by careful analysis of the analytic proof of generalized Cauchy identities given by the second-named author [**Śni03**]. In this analytic proof we associated to oriented trees certain polynomials and we proved that these polynomials fulfill recursion relation analogous to the one fulfilled by Abel polynomials. It turns out that if we replace the

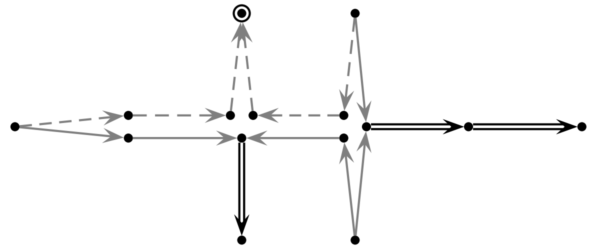

FIGURE 11. The graph T' obtained for $T_{\sigma'}$ depicted on Figure 10.

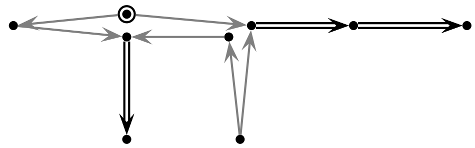

FIGURE 12. The graph T'' is obtained from T' depicted on Figure 11 by removal of the dashed edges. The graph T'' can be regarded as a certain polygonal graph $G_{\epsilon'}$ with a number of trees attached to it.

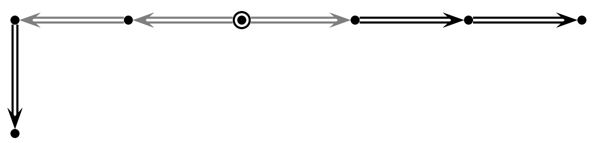

FIGURE 13. Tree $T_{\sigma'}$ is obtained from the graph T'' depicted on Figure 12 by gluing edges as prescribed in Lemma 3.

usual differential calculus by a *combinatorial calculus* in which the role of polynomials is played by certain graphs and oriented sets then the analytic proof from [**Śni03**] is valid also in this more general setup and it determines uniquely the bijection presented in this article [**JŚ06a**].

6. Postscript: operator algebras, free probability and triangular operator T

The story presented in Sections 1.1 and 1.3 is too beautiful to be true. In fact, it is not how the generalized Cauchy identities were discovered. In this section we will present the true story which also gives very strong motivations for studying these identities.

6.1. Invariant subspace conjecture. The *Voiculescu's free probability* [**VDN92, HP00**] is a non-commutative probability theory with the classical notion of independence replaced by the notion of *freeness*. Natural examples which fit nicely into the framework of the free probability include large random matrices, free products of von Neumann algebras and asymptotics of large Young diagrams. Families of operators which arise in the free probability are, informally speaking, very non-commutative and for this reason they are perfect candidates for counterexamples to the conjectures in the theory of operator algebras [**Voi96**].

Dykema and Haagerup [**DH04a**] suggested that free probability could be used to construct a counterexample for the famous *invariant subspace conjecture* (this conjecture asks if for every bounded operator x acting on an infinite-dimensional Hilbert space $\mathcal{H}$ there exists a closed subspace $\mathcal{K} \subset \mathcal{H}$ such that $\mathcal{K}$ is nontrivial in the sense that $\mathcal{K} \neq \{0\}$, $\mathcal{K} \neq \mathcal{H}$ and which is an invariant subspace of x). They also described explicitly a very good candidate for such a counterexample, namely *the triangular operator* T [**DH04a**].

For more details on the history of the search of such a counterexample within free probability theory we refer to [**Śni04**].

6.2. Combinatorics of the triangular operator T. Even though the primary description of the triangular operator T was purely analytic as a limit of certain random matrices, already in the original article [**DH04a**] Dykema and Haagerup gave a purely combinatorial description of this operator and we will present it in the following.

The triangular operator T is an element of a certain algebra (*finite von Neumann algebra*) equipped with a functional ϕ. The elements of such algebras can be uniquely determined by the values of ϕ on all polynomials in T and $T^\star$ therefore we need to specify the numbers $\phi(T^{\epsilon(1)} \cdots T^{\epsilon(n)})$ for any sequence $\epsilon(1), \dots, \epsilon(n) \in \{1, \star\}$. Dykema and Haagerup proved that that $(n/2+1)!\ \phi(T^{\epsilon(1)} \cdots T^{\epsilon(n)})$ is equal to the number of pairs $(\sigma, <)$ such that σ is a pairing compatible with ϵ and $<$ is a total order on the vertices of T_σ which is compatible with the orientation of the edges (please notice that the sequence ϵ considered above takes the values 1 and $\star$ while in the rest of this article we used the convention that ϵ takes the values $+1$ and -1, this difference is irrelavant). The Reader may easily see that the latter definition of T is very closely related to the results presented in this paper; in particular Theorem 1 can be now equivalently stated as follows (in fact it is the form in which Dykema and Haagerup stated originally their conjecture [**DH04a**]):

Theorem 6. *If $l, m \geq 1$ are integers then*

$$\phi\left[\left(T^l (T^\star)^l\right)^m\right] = \frac{m^{ml}}{(ml+1)!}.$$

Theorem 1 and Theorem 6 were conjectured by Dykema and Haagerup [**DH04a**] in the hope that they might be useful in the study of spectral properties of T. Literally speaking, this hope turned out to be wrong since the later construction of the hyperinvariant subspaces of T by Dykema and Haagerup [**DH04b, Haa02**] did not make use of Theorem 1 and Theorem 6, however it made use of one of the auxiliary results used in our proof [**Śni03**] of these theorems. In this way, indirectly, Theorem 1 and Theorem 6 turned out to be indeed helpful for their original purpose. Later on Aagaard and Haagerup [**AH04**] gave a different analytic proof of the generalized Cauchy identities based on very clever matrix manipulations.

As we already mentioned, Dykema and Haagerup [**DH04b, Haa02**] constructed a family of hyperinvariant subspaces of T and in this way the original motivation for studying the operator T (as a possible counterexample for the invariant subspace conjecture) ended up as a failure. Nevertheless, operator T is still regarded as a canonical example of a quasinilpotent operator and its deep understanding may give us an insight into the structure of all quasinilpotent operators.

6.3. Applications in classical probability theory. The generalized Cauchy identities and their bijective proof can be used [**JŚ06b**] to extract some information about multidimensional random walks and Brownian motions.

References

[AH04] Lars Aagaard and Uffe Haagerup, *Moment formulas for the quasi-nilpotent DT-operator*, Internat. J. Math. **15** (2004), no. 6, 581–628. MR MR2078883

[DH04a] Ken Dykema and Uffe Haagerup, *DT-operators and decomposability of Voiculescu's circular operator*, Amer. J. Math. **126** (2004), no. 1, 121–189. MR 2 033 566

[DH04b] ———, *Invariant subspaces of the quasinilpotent DT-operator*, J. Funct. Anal. **209** (2004), no. 2, 332–366. MR 2 044 226

[DY03] Kenneth Dykema and Catherine Yan, *Generating functions for moments of the quasi–nilpotent DT operator*, Adv. in Appl. Math. **30** (2003), no. 3, 545–561.

[Haa02] U. Haagerup, *Random matrices, free probability and the invariant subspace problem relative to a von Neumann algebra*, Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002) (Beijing), Higher Ed. Press, 2002, pp. 273–290. MR 2004e:46002

[HP00] Fumio Hiai and Dénes Petz, *The semicircle law, free random variables and entropy*, American Mathematical Society, Providence, RI, 2000. MR 2001j:46099

[JŚ06a] Artur Jeż and Piotr Śniady, *Generalized Cauchy identities, trees and multidimensional Brownian motions. Part II: Combinatorial differential calculus*, In preparation, 2006.

[JŚ06b] ———, *Generalized cauchy identities, trees and multidimensional Brownian motions. Part III: Multidimensional arc-sine laws*, In preparation, 2006.

[Kre72] G. Kreweras, *Sur les partitions non croisées d'un cycle*, Discrete Math. **1** (1972), no. 4, 333–350. MR 46 #8852

[Ran60] George N. Raney, *Functional composition patterns and power series reversion*, Trans. Amer. Math. Soc. **94** (1960), 441–451.

[Śni03] Piotr Śniady, *Multinomial identities arising from the free probability theory*, J. Combin. Theory Ser. A **101** (2003), no. 1, 1–19.

[Śni04] ———, *Generalized Cauchy identities, trees and multidimensional Brownian motions. Part I: bijective proof of generalized Cauchy identities*, Preprint `arXiv:math.CO/0412043`, 2004.

[Sta99] Richard P. Stanley, *Enumerative combinatorics. Vol. 2*, Cambridge Studies in Advanced Mathematics, vol. 62, Cambridge University Press, Cambridge, 1999. MR 2000k:05026

[VDN92] D. V. Voiculescu, K. J. Dykema, and A. Nica, *Free random variables*, American Mathematical Society, Providence, RI, 1992. MR 94c:46133

[Voi96] D. Voiculescu, *The analogues of entropy and of Fisher's information measure in free probability theory. III. The absence of Cartan subalgebras*, Geom. Funct. Anal. **6** (1996), no. 1, 172–199. MR 96m:46119

Institute of Mathematics, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland
E-mail address: `s141494@math.uni.wroc.pl`

Institute of Mathematics, University of Wroclaw, pl. Grunwaldzki 2/4, 50-384 Wroclaw, Poland
E-mail address: `Piotr.Sniady@math.uni.wroc.pl`
URL: `http://www.math.uni.wroc.pl/~psnia`