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Multivariate Fuss-Catalan numbers and B-quasisymmetric functions

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Abstract. We study the ideal generated by constant-term free B-quasisymmetric polynomials, and prove that the quotient of the polynomial ring by this ideal has dimension given by $\frac{1}{2n+1}\binom{3n}{n}$, the number of ternary trees, or Fuss-Catalan number of order 3. This leads us to introduce and study multivariate Fuss-Catalan numbers, whose combinatorial interpretation is given by some statistics on ternary trees and plane paths.

Résumé. Nous étudions l'idéal engendré par les polynômes B-quasisymétriques (sans terme constant), et prouvons que le quotient de l'anneau des polynômes par cet idéal est de dimension $\frac{1}{2n+1}\binom{3n}{n}$, le nombre d'arbres ternaires, ou nombre de Fuss-Catalan d'ordre 3. Nous en profitons pour introduire et étudier combinatoirement certains nombres de Fuss-Catalan multivariés, ce qui fait apparaître une bi-statistique sur les arbres ternaires et certains chemins du plan.

1. Introduction

To start with, we recall a small part of the story of the study of ideals and quotients related to symmetric or quasisymmetric polynomials. The root of this work is a result of Artin [**1**]. Let us consider the set of variables $X_n = x_1, x_2, \ldots, x_n$. The space of polynomials in the variables X_n with rational coefficients is denoted by $\mathbb{Q}[X_n]$. The subspace of symmetric polynomials is denoted by Sym_n. If $\mathbf{V}$ is a subset of the polynomial ring, we denote by $\langle \mathbf{V}^+ \rangle$ the ideal generated by elements of a $\mathbf{V}$ with no constant term. Artin's result is given by:

$$\dim \mathbb{Q}[X_n]/\langle Sym_n^+ \rangle = n!\,. \tag{1.1}$$

Another, more recent, part of the story deals with quasisymmetric polynomials. The space $QSym_n \subset \mathbb{Q}[X_n]$ of quasisymmetric polynomials was introduced by Gessel [**13**] as generating functions for Stanley's P-partitions [**21**]. This is the starting point of many recent works in several areas of combinatorics [**9, 12, 16, 17, 22**].

In [**4, 5**], Aval *et. al.* study the problem analogous to Artin's work in the case of quasisymmetric polynomials. Their main result is that the dimension of the quotient is given by Catalan numbers:

$$\dim \mathbb{Q}[X_n]/\langle QSym_n^+ \rangle = C_n = \frac{1}{n+1}\binom{2n}{n}. \tag{1.2}$$

An interesting axis of research is the extension of these results to 2 sets of variables. Let $\mathcal{A}_n$ denote the alphabet

$$\mathcal{A}_n = x_1, y_1, x_2, y_2, \ldots, x_n, y_n.$$

Since symmetric (resp. quasisymmetric) polynomials may be seen as S_n-invariants under the action that permutes variables (resp. under Hivert's action [**16**]), one can define diagonal analogues by letting S_n

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act simulaneously on the x's and y's. We then obtain the space $DSym_n$ (resp. $DQSym_n$) of diagonally symmetric (resp. quasisymmetric) functions.

The diagonal coinvariant space $\mathbb{Q}[\mathcal{A}_n]/\langle DSym_n^+\rangle$ has been studied extensively in the last 15 years by several authors [**8, 11, 14**]. A great achievment in this area is Haiman's proof of the following equality (*cf.* [**14**]):

$$\dim \mathbb{Q}[\mathcal{A}_n]/\langle DSym_n^+\rangle = (n+1)^{n-1}\,.$$

The space $DQSym_n$ was introduced in [**19**], then recently studied in [**7**], [**18**], and [**6**], where the coinvariant space is investigated, and a conjectural basis is presented.

To end this presentation, we introduce the space $QSym_n(B)$ of B-quasisymmetric polynomials, which is the focus of this article. This space, whose definition appears implicitly in [**19**], is studied with more details in [**7**]. A precise definition will be given in the next section, and we only mention here that $QSym_n(B)$ is a subspace (and in fact a subalgebra, *cf.* [**7**]) of $DQSym_n$.

We now state the main result of this work, which appears as a generalization of equation (1.2).

THEOREM 1.1.

$$\dim \mathbb{Q}[\mathcal{A}_n]/\langle QSym_n(B)^+\rangle = \frac{1}{2n+1}\binom{3n}{n}. \tag{1.3}$$

Observe that in Equations (1.2) and (1.3), the dimensions $\frac{1}{n+1}\binom{2n}{n}$ and $\frac{1}{2n+1}\binom{3n}{n}$ are respectively the numbers of binary and ternary trees (*cf.* [**20**]). Since we deal with polynomials in two alphabets (and since the ideal $\langle QSym_n(B)^+\rangle$ is homogeneous), we can study the bigraded version of Equation (1.3). More precisely, we look at the subspace of $\mathbb{Q}[\mathcal{A}_n]/\langle QSym_n(B)^+\rangle$ of polynomials of degree k in $x_1,\dots,x_n$ and degree l in $y_1,\dots,y_n$, and consider its dimension, which we denote by $B(n,k,l)$. It appears that these numbers present their own interest, which led us to study them.

Let us now give the plan of this article. We have decided to deal first with the combinatorial part, *i.e.* the study of the numbers $B(n,k,l)$, which is the subject of the next section, and the algebraic part is developped in the last section of this paper.

Remark. This paper is the extended abstract of our work. More details and the complete proofs (here are ony given the sketches of some proofs) can be found in [**2, 3**].

2. Multivariate Fuss-Catalan numbers

2.1. Catalan triangle, binary trees, and Dyck paths.

The *Catalan numbers*

$$C(n) = \frac{1}{n+1}\binom{2n}{n}$$

are integers that appear in many combinatorial problems. These numbers first arose in the work of Catalan as the number of triangulations of a polygon by mean of non-intersecting diagonals. Stanley [**21, 23**] maintains a dynamic list of exercises related to Catalan numbers, including (at this date) 127 combinatorial interpretations.

Closely related to Catalan numbers are *ballot numbers*. To serve our purpose, we shall neither state the so-called *ballot problem*, nor give an explicit formula, but we introduce integers $B(n,k)$ for $(n,k)\in\mathbb{N}^*\times\mathbb{N}$ defined by the following recurrence:

- $B(1,0)=1$
- $\forall n>1$ and $0\le k<n$, $B(n,k)=\sum_{i=0}^{k} B(n-1,i)$
- $\forall k\ge n$, $B(n,k)=0$.

Observe that the recursive formula in the second condition is equivalent to:

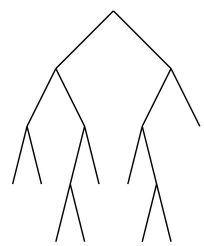
$$B(n,k) = B(n-1,k) + B(n,k-1). \tag{2.1}$$

We shall present the $B(n,k)$'s by the following triangular representation (zero entries are omitted) where moving down increases n and moving right increases k.

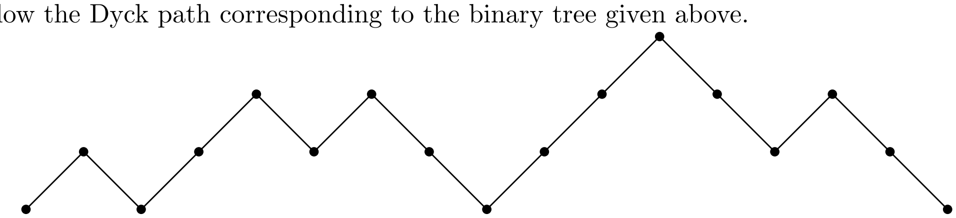
1					
1	1				
1	2	2			
1	3	5	5		
1	4	9	14	14	
1	5	14	28	42	42

The crucial observation is that computing the horizontal sums of these integers give : 1, 2, 5, 14, 42, 132. We recognize the first terms of the Catalan series, and this fact will be proven in Proposition 2.1, after introducing combinatorial objects.

A *binary tree* is a tree in which every internal node has exactly 2 sons. The number of binary trees with n internal nodes is given by the n-th Catalan number.



A *Dyck path* is a path consisting of steps $(1,1)$ and $(1,-1)$, starting from $(0,0)$, ending at $(2n,0)$, and remaining above the line $y=0$. The number of Dyck paths of length $2n$ is also given by the n-th Catalan number. More precisely, the depth-first search of the tree gives a bijection between binary trees and Dyck paths: we associate to each external node (except the left-most one) a $(1,1)$ step and to each internal node a $(1,-1)$ step by searching recursively the left son, then the right son, then the current node. As an example, we show below the Dyck path corresponding to the binary tree given above.



An inportant parameter in our study will be the length of the right-most sequence of $(1,-1)$ of the path. This parameter equals 2 in our example. Observe that under the correspondence between paths and trees, this parameter corresponds to the length of the right-most string of right sons in the tree. We shall use the expressions *last down sequence* and *last right string*, for these parts of the path and of the tree.

Now we come to the announced result. It is well-known and simple, but is the starting point of our work.

PROPOSITION 2.1. *We have the following equality:*

$$\sum_{k=0}^{n-1} B(n,k) = C(n) = \frac{1}{n+1}\binom{2n}{n}.$$

PROOF. Let us denote by $\mathcal{C}_{n,k}$ the set of Dyck paths of length $2n$ with a last down sequence of length equal to $n-k$.

We shall prove that $B(n,k)$ is the cardinality of $\mathcal{C}_{n,k}$.

The proof is done recursively on n. If $n=0$, this is trivial. If $n>0$, let us suppose that $B(n-1,k)$ is the cardinality of $\mathcal{C}_{n-1,k}$ for $0 \le k < n-1$. Let us consider an element of $\mathcal{C}_{n,k}$. If we erase the last step $(1,1)$ and the following step $(1,-1)$, we obtain a Dyck path of length $2(n-1)$ and with a last decreasing sequence of length $n-l \ge n-k$. If we keep track of the integer k, we obtain a bijection between $\mathcal{C}_{n,k}$ and $\cup_{l\le k}\mathcal{C}_{n-1,l}$. $\square$

2.2. Fuss-Catalan tetrahedron and ternary trees. This subsection, which is the heart of this part of the work, is the study of a 3-dimensional analogue of the Catalan triangle presented in the previous section. We consider exactly the same recurrence, and let the array grow, not in 2, but in 3 dimensions.

More precisely, we introduce the sequence $B_3(n,k,l)$ indexed by integers n, k and l, and defined recursively by:

- $B_3(1,0,0) = 1$
- $\forall n > 1,\ k+l < n,\ B_3(n,k,l) = \sum_{0\le i\le k, 0\le j\le l} B_3(n-1,i,j)$
- $\forall k+l \ge n,\ B_3(n,k,l) = 0.$

Observe that the recursive formula in the second condition is equivalent to:

$$B_3(n,k,l) = B_3(n-1,k,l) + B_3(n,k-1,l) + B_3(n,k,l-1) - B_3(n,k-1,l-1) \tag{2.2}$$

and this expression can be used to make some computations lighter, but the presentation above explains more about the generalization of the definition of the ballot numbers $B(n,k)$.

Because of the planar structure of the sheet of paper, we are led to present the tetrahedron of $B_3(n,k,l)$'s by its sections with a given n.

$$n=1 \longrightarrow \begin{bmatrix} 1 \end{bmatrix}$$
$$n=2 \longrightarrow \begin{bmatrix} 1 & 1 \\ 1 & \end{bmatrix}$$
$$n=3 \longrightarrow \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & \\ 2 & & \end{bmatrix}$$
$$n=4 \longrightarrow \begin{bmatrix} 1 & 3 & 5 & 5 \\ 3 & 8 & 10 & \\ 5 & 10 & & \\ 5 & & & \end{bmatrix}$$
$$n=5 \longrightarrow \begin{bmatrix} 1 & 4 & 9 & 14 & 14 \\ 4 & 15 & 30 & 35 & \\ 9 & 30 & 45 & & \\ 14 & 35 & & & \\ 14 & & & & \end{bmatrix}$$

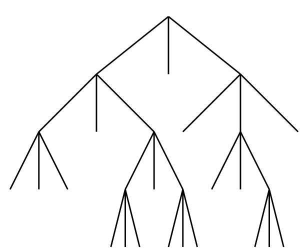
It is clear that $B_3(n,k,0) = B_3(n,0,k) = B(n,k)$. The reader may easily check that when we compute $\sum_{k,l} B_3(n,k,l)$, we obtain: 1, 3, 12, 55, 273. These integers are the first terms of the following sequence (*cf.* [**20**]):

$$C_3(n) = \frac{1}{2n+1}\binom{3n}{n}.$$

2.3. Combinatorial interpretation. Fuss[1]-Catalan numbers (*cf.* [**15**]) are given by the formula

$$C_p(n) = \frac{1}{(p-1)n+1}\binom{pn}{n}, \tag{2.3}$$

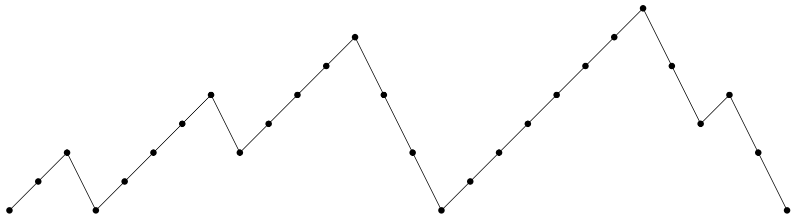
and $C_3(n)$ appear as order-3 Fuss-Catalan numbers. The integers $C_3(n)$ are known [**20**] to count *ternary trees*, *i.e.* trees in which every internal node has exactly 3 sons.



Ternary trees are in bijection with *2-Dyck paths*, which are defined as paths from $(0,0)$ to $(3n,0)$ with steps $(1,1)$ and $(1,-2)$, and remaining above the line $y=0$. The bijection between these objects is the same as in the case of binary trees, *i.e.* a depth-first search, with the difference that here an internal node

[1]Nikolai Fuss (Basel, 1755 – St Petersburg, 1826) helped Euler prepare over 250 articles for publication over a period on about seven years in which he acted as Euler's assistant, and was from 1800 to 1826 permanent secretary to the St Petersburg Academy.

is translated into a $(1,-2)$ step. To illustrate this bijection, we give the path corresponding to the previous example of ternary tree:



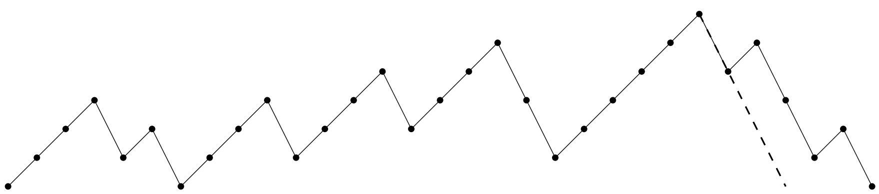
We shall consider these paths with respect to the position of their down steps. Let $\mathcal{D}_{n,k,l}$ denote the set of 2-Dyck paths of length $3n$, with k down steps at even height and l down steps at odd height. By convention, the last sequence of down steps is not considered (the number of these steps is by definition equal to $n-k-l$). In the previous example, $n=9$, $k=5$ and $l=2$.

Proposition 2.2. *We have*

$$\sum_{k,l} B_3(n,k,l) = C_3(n) = \frac{1}{2n+1}\binom{3n}{n}.$$

Moreover, $B_3(n,k,l)$ is the cardinality of $\mathcal{D}_{n,k,l}$.

Proof. Let k and l be fixed. Let us consider an element of $\mathcal{D}_{n,k,l}$. If we cut this path after its $(2n-2)$-th up step, and complete with down steps, we obtain a 2-Dyck path of length $3(n-1)$ (see figure below). It is clear that this path is an element of $\mathcal{D}_{n,i,j}$ for some $i \le k$ and $j \le l$. We can furthermore reconstruct the original path from the truncated one, if we know k and l. We only have to delete the last sequence of down steps (here the dashed line), to draw $k-i$ down steps, one up step, $l-j$ down steps, one up step, and to complete with down steps. This gives a bijection from $\mathcal{D}_{n,k,l}$ to $\cup_{0\le i\le k, 0\le j\le l}\mathcal{D}_{n-1,i,j}$, which implies Proposition 2.2.



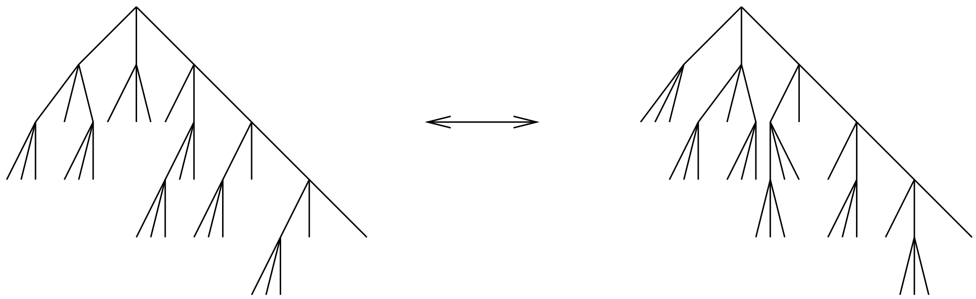
$\square$

Remark 2.1. It is interesting to translate the bi-statistics introduced on 2-Dyck paths to the case of ternary trees. As previously, we consider the depth-first search of the tree, and shall not consider the last right string. We define $\mathcal{T}_{n,k,l}$ as the set of ternary trees with n internal nodes, k of them being encountered in the search after an even number of leaves and l after and odd number of leaves. By the bijection between trees and paths, and Proposition 2.2, we have that the cardinality of $T_{n,k,l}$ is $B_3(n,k,l)$.

Remark 2.2. It is clear from the definition that:

$$B_3(n,k,l) = B_3(n,l,k).$$

But this fact is not obvious when considering trees or paths, since the statistics defined are not clearly symmetric. To explain this, we can introduce an involution on the set of ternary trees which sends an element of $\mathcal{T}_{n,k,l}$ to $\mathcal{T}_{n,l,k}$. To do this, we can exchange for each node of the last right string its left and its middle son, as in the following picture. Since the number of leaves of a ternary tree is odd, every "even" node becomes an "odd" one, and conversely.



2.4. Explicit formula. Now a natural question is to obtain explicit formulas for the $B_3(n,k,l)$. The answer is given by the following proposition.

Proposition 2.3. *The intergers $B_3(n,k,l)$ are given by*

$$B_3(n,k,l) = \binom{n+k-1}{k}\binom{n+l-1}{l}\frac{n-k-l}{n} \tag{2.4}$$

Proof. [sketch] The proof is a variation of the cycle lemma [**10**], used to enumerate $\mathcal{D}_{n,k,l}$. It is also possible, once we have the formula (2.4), to check the recurrence (2.2). ☐

3. Ideals of B-quasisymmetric functions

3.1. Definitions, notations and results. For these definitions, we follow [**7**], with some minor differences, for the sake of simplicity of the computations we will have to make.

Let $\mathbb{N}$ and $\bar{\mathbb{N}}$ denote two occurrrences of the set of nonnegative integers. We shall write $\bar{\mathbb{N}} = \{\bar{0}, \bar{1}, \bar{2}, \ldots\}$ and make no difference between the elements of $\mathbb{N}$ and $\bar{\mathbb{N}}$ in any arithmetical expression. We distinguish $\mathbb{N}$ and $\bar{\mathbb{N}}$ for the ease of reading.

A *bivector* is a vector $v = (v_1, v_2, \ldots, v_{2k-1}, v_{2k})$ such that the odd entries $\{c_{2i-1},\ i = 1..k\}$ are in $\mathbb{N}$, and the even entries $\{c_{2i},\ i = 1..k\}$ are in $\bar{\mathbb{N}}$.

A *bicomposition* is a bivector in which there is no consecutive zeros, *i.e.* no pattern $0\bar{0}$ or $\bar{0}0$.

The integer k is called the *size* of v. The *weight* of the vector v is by definition the couple $(|v|_{\mathbb{N}}, |v|_{\bar{\mathbb{N}}})$, where $|v|_{\mathbb{N}} = \sum_{i=1}^{k} v_{2i-1}$ and $|v|_{\bar{\mathbb{N}}} = \sum_{i=1}^{k} v_{2i}$. We also set $|v| = |v|_{\mathbb{N}} + |v|_{\bar{\mathbb{N}}}$.

For example $(1, \bar{0}, 2, \bar{1}, 0, \bar{2}, 3, \bar{0})$ is a bicomposition of size 4, and of weight $(6, 3)$.

To make notations lighter, we shall sometimes write bivectors or bicomposition as words, for example $1\bar{0}2\bar{1}0\bar{2}3\bar{0}$.

The *fundamental B-quasisymmetric functions*, indexed by bicompositions, are defined as follows

$$F_{c_1c_2\ldots c_{2k-1}c_{2k}}(\mathcal{A}_n) = \sum x_{i_1}\cdots x_{i_{|c|_{\mathbb{N}}}}\, y_{j_1}\cdots y_{j_{|c|_{\bar{\mathbb{N}}}}} \in \mathbb{Q}[\mathcal{A}_n]$$

where the sum is taken over indices i's and j's such that

$$i_1 \leq \cdots i_{c_1} \leq j_1 \leq \cdots j_{c_2} < i_{c_1+1} \leq \cdots i_{c_1+c_3} \leq j_{c_2+1} \leq \cdots \leq j_{c_2+c_4} < i_{c_1+c_3+1} \leq \cdots$$

We give some examples:
$F_{1\bar{2}} = \sum_{i\leq j\leq k} x_i y_j y_k$,
$F_{0\bar{2}1\bar{0}} = \sum_{i\leq j<k} y_i y_j x_k$.

It is clear from the definition that the bidegree (*i.e.* the couple (degree in x, degree in y)) of F_c in $\mathbb{Q}[\mathcal{A}_n]$ is the weight of c. If the size of c is greater than n, we shall set $F_c(\mathcal{A}_n) = 0$.

The space of B-quasisymmetric functions, denoted by $QSym_n(B)$ is the vector subspace of $\mathbb{Q}[\mathcal{A}_n]$ generated by the $F_c(\mathcal{A}_n)$, for all bicompositions c.

Let us denote by $\mathcal{I}_n^2$ the ideal $\langle QSym_n(B)^+\rangle$ generated by B-quasisymmetric functions with zero constant term.

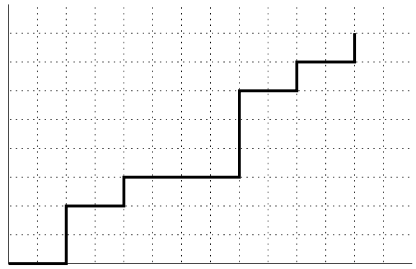
With these notations, our goal is to prove

$$\dim \mathbb{Q}[\mathcal{A}_n]/\langle QSym_n(B)^+\rangle = \frac{1}{2n+1}\binom{3n}{n}. \tag{3.1}$$

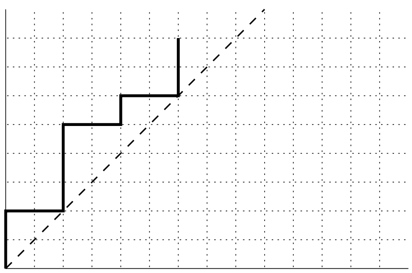
3.2. Paths and $\mathcal{G}$-set. The aim of this subsection is to construct a set $\mathcal{G}$ of polynomials, which will be proved in the next section to be a Gröbner basis of $\mathcal{I}_n^2$. This part of the work is greatly inspired from [**4, 5**].

Let $v = (v_1, v_2, \ldots, v_{2k-1}, v_{2k})$ be a bivector of size n. We associate to v a path $\pi(v)$ in the plane $\mathbb{N} \times \mathbb{N}$, with steps (0,1) or (2,0). We start from (0,0) and add for each entry v_i (read from left to right): v_i steps (2,0), followed by one step (0,1).

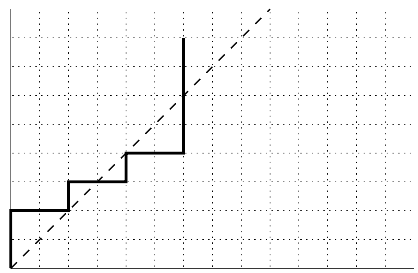
As an example, the path associated to $(1, \bar{0}, 1, \bar{2}, 0, \bar{0}, 1, \bar{1})$ is



We have two kinds of path, regarding their position to the diagonal $x = y$. If a path always remains above this line, we call it a *2-Dyck path*, and say that the corresponding vector is *2-Dyck*. Conversely, if the path enters the region $x < y$, we call both the path and the vector *transdiagonal*. For example, $v = (0, \bar{0}, 1, \bar{0}, 0, \bar{1}, 1, \bar{0})$ is 2-Dyck, whereas $w = (0, \bar{0}, 1, \bar{1}, 1, \bar{0}, 0, \bar{0})$ is transdiagonal.



π (v)



π (w)

A simple but important observation is that a vector $v = (v_1, v_2, \ldots, v_{2k-1}, v_{2k})$ is transdiagonal if and only if there exists $1 \leq l \leq k$ such that

$$v_1 + v_2 + \cdots + v_{2l-1} + v_{2l} \geq l. \tag{3.2}$$

Our next task is to construct a set $\mathcal{G}$ of polynomials, mentionned above. From now on, unless otherwise indicated, vectors are of size n. For w a vector of size $k < n$, $w0^*$ denotes the vector (of size n) obtained by adding the desired number of $0\bar{0}$ patterns. We shall define the *length* $\ell(v)$ of a vector v as the integer k such that $v = v_1\, v_2\, \ldots\, v_{2k-1}\, v_{2k}\, 0^*$ with $v_{2k-1}\, v_{2k} \neq 0\bar{0}$. In the case of bicompositions, the notions of size and length coincide.

For v a vector (of size n), we denote by $\mathcal{A}_n^v$ the monomial

$$\mathcal{A}_n^v = x_1^{v_1} y_1^{v_2} \cdots x_n^{v_{2n-1}} y_n^{v_{2n}}.$$

To deal with leading terms of polynomials, we will use the lexicographic order induced by the ordering of the variables:

$$x_1 > y_1 > x_2 > y_2 > \cdots > x_n > y_n.$$

The lexicographic order is defined on monomials as follows: $\mathcal{A}_n^v >_{\text{lex}} \mathcal{A}_n^w$ if and only if the first non-zero entry of $v - w$ (componentwise) is positive.

The set

$$\mathcal{G} = \{G_v\} \subset \mathcal{I}_n^2$$

is indexed by transdiagonal vectors. Let v be a transdiagonal vector.

For $v = c0^*$ with c a non-zero bicomposition of length $\geq n$ (which implies that v is transdiagonal), we define

$$G_v = F_c.$$

If v cannot be written as $c0^*$, the polynomial G_v is defined recursively. We look at the rightmost occurrence of two consecutive zeros (on the left of a non-zero entry: we do not consider the subword 0^*). Two cases are to be distinguished according to the parity of the position of this pattern:

- if $v = w0\bar{0}\alpha\beta c0^*$, with w a vector of size $k-1$, $\alpha \in \mathbb{N}$ (by definition non-zero), $\beta \in \bar{\mathbb{N}}$, c a bicomposition, we define

$$G_{w0\bar{0}\alpha\beta c0^*} = G_{w\alpha\beta c0^*} - x_k\, G_{w(\alpha-1)\beta c0^*}; \tag{3.3}$$

- if $v = w\alpha\bar{0}0\beta c0^*$, with w a vector of size $k-1$, $\alpha \in \mathbb{N}$, $\beta \in \bar{\mathbb{N}}$ (by definition non-zero), c a bicomposition, we define

$$G_{w\alpha\bar{0}0\beta c0^*} = G_{w\alpha\beta c0^*} - y_k\, G_{w\alpha(\beta-1)c0^*}. \tag{3.4}$$

We easily check that both terms on the right of (3.3) and (3.4) are indexed by vectors that are transdiagonal as soon as v is transdiagonal. We do it for (3.3) : let us denote $v' = w\alpha\beta c0^*$ and $v'' = w(\alpha-1)\beta c0^*$. Let l be the smallest integer such that (3.2) holds for v. If $l \geq k-1$ then w is transdiagonal thus so are v' and v'', and if not:

$$v'_1 + v'_2 + \cdots + v'_{2l-3} + v'_{2l-2} \geq l \quad \text{and} \quad v''_1 + v''_2 + \cdots + v''_{2l-3} + v''_{2l-2} \geq l-1.$$

Since v' and v'' are of length equal to $\ell(v)-1$, this defines any G_v for v transdiagonal by induction on $\ell(v)$.

It is interesting to develop an example, where we take $n=3$.

$$\begin{array}{rcl}
G_{0\bar{0}1\bar{0}0\bar{2}} & = & G_{0\bar{0}1\bar{2}0\bar{0}} - y_2\, G_{0\bar{0}1\bar{1}0\bar{0}} \\
& = & (G_{1\bar{2}0\bar{0}0\bar{0}} - x_1\, G_{0\bar{2}0\bar{0}0\bar{0}}) - y_2\,(G_{1\bar{1}0\bar{0}0\bar{0}} - x_1\, G_{0\bar{1}0\bar{0}0\bar{0}}) \\
& = & (F_{1\bar{2}} - x_1\, F_{0\bar{2}}) - y_2(F_{1\bar{1}} - x_1\, F_{0\bar{1}}) \\
& = & (x_1y_1^2 + x_1y_1y_2 + x_1y_1y_3 + x_1y_2^2 + x_1y_2y_3 + x_1y_3^2 + x_2y_2^2 + x_2y_2y_3 \\
& & \;+x_2y_3^2 + x_3y_3^2 - x_1(y_1^2 + y_1y_2 + y_1y_3 + y_2^2 + y_2y_3 + y_3^2)) \\
& & -y_2(x_1y_1 + x_1y_2 + x_1y_3 + x_2y_2 + x_2y_3 + x_3y_3 - x_1(y_1+y_2+y_3)) \\
& = & x_2y_3^2 - y_2x_3y_3 + x_3y_3^2
\end{array}$$

The monomials of the result are ordered with respect to the lexicographic order and we observe that the leading monomial (denoted LM) of $G_{0\bar{0}1\bar{0}0\bar{2}}$ is $\mathcal{A}_3^{0\bar{0}1\bar{0}0\bar{2}}$. The following proposition shows that this fact holds in general for the family $\mathcal{G}$.

Proposition 3.1. *Let v be a transdiagonal vector. The leading monomial of G_v is*

$$LM(G_v) = \mathcal{A}_n^v. \tag{3.5}$$

Proof. [sketch] It is done by induction on the length of v. □

3.3. Proof of the main theorem. The aim of this subsection is to prove Theorem 1.1, by showing that the set $\mathcal{G}$ constructed in the previous section is a Gröbner basis for $\mathcal{I}_n^2$. This will be achieved in several steps.

We introduce the notation $\mathcal{Q}_n = \mathbb{Q}[\mathcal{A}_n]/\mathcal{I}_n^2$ and define

$$\mathcal{B}_n = \{\mathcal{A}_n^v \;/\; \pi(v) \textit{ is a } 2\textit{-Dyck path}\}.$$

Lemma 3.1. *Any polynomial $P \in \mathbb{Q}[\mathcal{A}_n]$ is in the span of $\mathcal{B}_n$ modulo $\mathcal{I}_n^2$. That is*

$$P(\mathcal{A}_n) \equiv \sum_{\mathcal{A}_n^v \in \mathcal{B}_n} c_v \mathcal{A}_n^v. \tag{3.6}$$

Proof. It clearly suffices to show that (3.6) holds for any monomial $\mathcal{A}_n^v$, with v transdiagonal. We assume that there exists $\mathcal{A}_n^v$ not reducible of the form (3.6) and we choose $\mathcal{A}_n^w$ to be the smallest amongst them with respect to the lexicographic order. Let us write

$$\begin{array}{rcl}
\mathcal{A}_n^w & = & LM(G_w) \\
& = & (\mathcal{A}_n^w - G_w) + G_w \\
& \equiv & \mathcal{A}_n^w - G_w \quad (\mathrm{mod}\ \mathcal{I}_n^2).
\end{array}$$

All monomials in $(\mathcal{A}_n^w - G_w)$ are lexicographically smaller than $\mathcal{A}_n^w$, thus they are reducible. This contradicts our assuption and completes the proof. □

This lemma implies that $\mathcal{B}_n$ spans the quotient $\mathcal{Q}_n$. We will now prove its linear independence. The next lemma is a crucial step.

Lemma 3.2. *If we denote by $\mathcal{L}[S]$ the linear span of a set S, then*

$$\mathbb{Q}[\mathcal{A}_n] = \mathcal{L}[\mathcal{A}_n^v F_c \ / \ \mathcal{A}_n^v \in \mathcal{B}_n, \ |c| \geq 0]. \tag{3.7}$$

Proof. We have the following reduction for any monomial $\mathcal{A}_n^w$ in $\mathbb{Q}[\mathcal{A}_n]$:

$$\mathcal{A}_n^w = \sum_{\mathcal{A}_n^v \in \mathcal{B}_n} c_v \mathcal{A}_n^v + \sum_{|c|>0} Q_c F_c, \quad Q_c \in \mathbb{Q}[\mathcal{A}_n]. \tag{3.8}$$

We then apply the reduction (3.6) to each monomial of the Q_c's. Now we use the algebra structure of QSym(B) (*cf.* Proposition 37 of [**7**]) to reduce products of fundamental B-quasisymmetric functions as linear combinations of F_c's. We obtain (3.7) in a finite number of operations since degrees strictly decrease at each operation, because $|c| > 0$ implies $\deg Q_c < |w|$. □

Now we come to the final step in the proof. Before stating this lemma, we introduce some notation, and make an observation. For $v = (v_1, v_2, v \ldots, v_{2k-1}, v_{2k})$ a bivector, let $r(v)$ denote the reverse bivector: $r(v) = (v_{2k}, v_{2k-1}, \ldots, v_2, v_1)$. In the same way, let $R(\mathcal{A}_n)$ denote the reverse alphabet of $\mathcal{A}_n$: $R(\mathcal{A}_n) = y_n, x_n, \ldots, y_1, x_1$. Then one has for any bicomposition c:

$$F_c(R(\mathcal{A}_n)) = F_{r(c)}(\mathcal{A}_n). \tag{3.9}$$

Lemma 3.3. *The set $\mathcal{G}$ is a linear basis of $\mathcal{I}_n^2$, i.e.*

$$\mathcal{I}_n^2 = \mathcal{L}[G_w \ / \ w \ \textit{transdiagonal}]. \tag{3.10}$$

Proof. [sketch] We use Lemma 3.2, observation (3.9), and the algebra structure of $QSym_n(B)$ to write:

$$\begin{aligned} \mathcal{I}_n^2 &= \langle F_c, \ |c| > 0 \rangle_{\mathbb{Q}[\mathcal{A}_n]} = \mathcal{L}[\mathcal{A}_n^v \, F_c \, F_{c'} \ / \ R(\mathcal{A}_n)^v \in \mathcal{B}_n, \ |c| > 0, \ |c'| \geq 0] \\ &= \mathcal{L}[\mathcal{A}_n^v \, F_{c''} / \ R(\mathcal{A}_n)^v \in \mathcal{B}_n, \ |c''| > 0]. \end{aligned}$$

Then we prove that we can reduce any term $\mathcal{A}_n^v \, F_{c''}$ using the G polynomials, and we illustrate this on an example, where $n = 5$:

$$\begin{aligned} x_1 \, y_2 \, F_{1\bar{0}0\bar{1}} &= y_2(x_1 \, F_{1\bar{0}0\bar{1}}) \\ &= y_2(G_{2\bar{0}0\bar{1}0\bar{0}0\bar{0}0\bar{0}} - G_{0\bar{0}2\bar{0}0\bar{1}0\bar{0}0\bar{0}}) \\ &= y_2 \, G_{2\bar{0}0\bar{1}0\bar{0}0\bar{0}0\bar{0}} - y_2 \, G_{0\bar{0}2\bar{0}0\bar{1}0\bar{0}0\bar{0}} \\ &= G_{2\bar{0}0\bar{2}0\bar{0}0\bar{0}0\bar{0}} - G_{2\bar{0}0\bar{0}0\bar{2}} - G_{0\bar{0}2\bar{1}0\bar{1}0\bar{0}0\bar{0}} + G_{0\bar{0}2\bar{0}0\bar{1}0\bar{1}0\bar{0}}. \end{aligned}$$

□

Now we are able to complete the proof of Theorem 1.1. We can even state a more precise result.

Theorem 3.4. *A basis of the quotient $\mathcal{Q}_n$ is given by the set*

$$\mathcal{B}_n = \{\mathcal{A}_n^v \ / \ \pi(v) \ \textit{is a} \ 2\textit{-Dyck path}\},$$

which implies

$$\dim \mathcal{Q}_n = \frac{1}{2n+1}\binom{3n}{n}. \tag{3.11}$$

Since $\mathcal{I}_n^2$ is bihomogeneous, the quotient $\mathcal{Q}_n$ is bigraded and we can consider $\mathbf{H}_{k,l}(\mathcal{Q}_n)$ the subspace of $\mathcal{Q}_n$ consisting of polynomials of bidegree (k, l), then

$$\dim \mathbf{H}_{k,l}(\mathcal{Q}_n) = \binom{n+k-1}{k}\binom{n+l-1}{l}\frac{n-k-l}{n}. \tag{3.12}$$

PROOF. By Lemma 3.1, the set $\mathcal{B}_n$ spans $\mathcal{Q}_n$. Assume we have a linear dependence:

$$P = \sum_{\mathcal{A}_n^v \in \mathcal{B}_n} a_v \, \mathcal{A}_n^v \ \in \ \mathcal{I}_n^2.$$

By Lemma 3.3, the set $\mathcal{G}$ spans $\mathcal{I}_n^2$, thus

$$P = \sum_{u \text{ transdiagonal}} b_u \, G_u.$$

This implies $LM(P) = \mathcal{A}_n^u$, with u transdiagonal, which is absurd. Hence $\mathcal{B}_n$ is a basis of the quotient $\mathcal{Q}_n$.

The expressions (3.11) and (3.12) are consequences of Section 2's results.

□

Remark. This work admits direct generalization. We can define quasisymmetric polynomials in p sets of variables. In this case, the quotient of the polynomial ring by the ideal generated by p-quasisymmetric polynomials (without constant term) has dimension given by $\frac{1}{pn+1}\binom{(p+1)n}{n}$. These numbers are Fuss-Catalan numbers, which enumerate $(p+1)$-ary trees. The combinatorial part corresponds to let the "Catalan recurrence" grow in $(p+1)$ dimensions, and we obtain multivariate Fuss-Catalan numbers of order $(p+1)$. All details can be found in [**2, 3**].

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