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Schubert polynomials for the affine Grassmannian

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Abstract. Confirming a conjecture of Mark Shimozono, we identify polynomial representatives for the Schubert classes of the affine Grassmannian as the k-Schur functions in homology and affine Schur functions in cohomology. Our results rely on Kostant and Kumar's nilHecke ring, work of Peterson on the homology of based loops on a compact group, and earlier work of ours on non-commutative k-Schur functions.

Résumé. Nous prouvons une conjecture de Mark Shimozono en montrant que des representations polynomiales pour les classes de Schubert des Grassmanniennes affines sont des fonctions k-Schur en homologie, et des fonctions de Schur affines en cohomologie. Nous utilisons l'anneau nilHecke de Kostant et Kumar, le travail de Peterson sur l'homologie des circuits basés sur un groupe compact, et notre travail antérieur sur les fonctions de k-Schur non-commutatives.

1. Introduction

This article is an extended abstract of the paper [**11**] with the same title. Some results and many details have been omitted.

In [**3**], Bott calculated the homology and cohomology rings of the based loop spaces ΩK, where K is a compact Lie group. In type A, both $H^*(\Omega SU_n)$ and $H_*(\Omega SU_n)$ can be identified with a ring of symmetric functions: in cohomology as a quotient of the ring of symmetric functions and in homology as a subring of the ring of symmetric functions. Separately, Kostant and Kumar [**8**] have calculated the cohomology rings $H^*(\mathcal{G}/\mathcal{P})$ of homogeneous spaces of Kac-Moody groups in terms of the Schubert classes $\sigma^w \in H^*(\mathcal{G}/\mathcal{P})$. It is well known that when $\mathcal{G}$ is of affine type and $\mathcal{P}$ a maximal parabolic, then $\mathcal{G}/\mathcal{P}$ is homotopy-equivalent to the based loops on the finite-dimensional compact group associated to $\mathcal{G}$. Thus in type $\hat{A}_{n-1}$, we have $H^*(\mathcal{G}/\mathcal{P}) = H^*(\Omega SU(n))$. While some of our results generalize to all Dynkin types, we will restrict ourselves to type A for the remainder of this article.

Our main result is the identification of the Schubert classes $\sigma^w \in H^*(\mathcal{G}/\mathcal{P})$ and $\sigma_w \in H_*(\mathcal{G}/\mathcal{P})$ as explicit symmetric functions. In the homology case, these polynomials are known as the *k-Schur functions*, originally introduced by Lapointe, Lascoux and Morse [**16**] and studied thoroughly by Lapointe and Morse [**13, 14**]. In the cohomology case, these polynomials were introduced by Lapointe and Morse in [**15**] where they were called *dual k-Schur functions* and also studied by myself in [**10**] where they were called *affine Schur functions*. These results were conjectures of Mark Shimozono (in the cohomology case, the conjecture was made precise by Jennifer Morse).

Thus the k-Schur functions $s_\lambda^{(k)}(x)$ and the affine Schur functions $\tilde{F}_\lambda(x)$ can be considered affine homology and cohomology Schubert polynomials respectively. Schubert polynomials for the flag variety were introduced by Lascoux and Schützenberger [**17**] and has led to numerous developments in algebra, geometry and combinatorics. It should be expected that affine Schubert polynomials lead to many exciting developments as well. Note that since $\Omega SU(n)$ is a loop space, its homology $H_*(\Omega SU(n)) = H_*(\mathcal{G}/\mathcal{P})$ is a Hopf-algebra. Our

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I am indebted to my coauthors Luc Lapointe, Jennifer Morse and Mark Shimozono, with whom I have studied k-Schur functions and the affine Grassmannian for nearly a year. I began working on k-Schur and dual k-Schur functions more than a year ago when Jennifer first introduced them to me, and Mark explained his geometric conjectures to me.

identification of Schubert classes is actually an isomorphism of Hopf-algebras, and gives an interpretation of the Hall inner product as the natural pairing between homology and cohomology. This feature of the affine theory is lacking in the classical finite case. We will only briefly discuss the Hopf-structures in this article.

Our results rely heavily on the *nilHecke ring* $\mathbb{A}$ introduced by Kostant and Kumar [**8**], results of Peterson [**19**] on the homology of based loop spaces, and the *non-commutative k-Schur functions* by the author in [**10**]. The non-commutative k-Schur functions are elements $s_w^{(k)}$ of a commutative subalgebra $\mathbb{B} \subset \mathbb{A}$, which we call the *affine Fomin-Stanley algebra* (since it is closely related to the work in [**6**]), of the nilHecke ring. We showed in [**10**] that $\mathbb{B}$ was isomorphic to a subring of the ring of symmetric functions which can be identified via Bott's result with $H_*(\mathcal{G}/\mathcal{P})$. Peterson has constructed an isomorphism $j : H_*^T(\mathcal{G}/\mathcal{P}) \to Z_{\mathbb{A}}(S)$ of the equivariant homology $H_*^T(\mathcal{G}/\mathcal{P})$ with a certain centraliser subalgebra $Z_{\mathbb{A}}(S) \subset \mathbb{A}$ of the nilHecke ring. We show here that "evaluation at 0" takes $Z_{\mathbb{A}}(S)$ onto $\mathbb{B}$ and that the composition with Peterson's j-homomorphism takes the Schubert classes $\sigma_{(w)}$ to the non-commutative k-Schur functions $s_w^{(k)}$. Kostant and Kumar have calculated the structure constants of $H^*(\mathcal{G}/\mathcal{P})$ in terms of a coproduct Δ on $\mathbb{A}$ and we compute directly that this coproduct, when restricted to the subalgebra $\mathbb{B}$, agrees with the usual coproduct of the symmetric functions. This shows that $\mathbb{B}$, when viewed as a ring of symmetric functions, is Hopf-isomorphic to $H_*(\mathcal{G}/\mathcal{P})$.

There are many open problems related to this work, and we mention a couple: it is natural to ask for representatives in K-theory, in equivariant (co)homology and in quantum cohomology. It is also natural to ask to generalize our work from the affine Grassmannian $\mathcal{G}/\mathcal{P}$ to the affine flag variety $\mathcal{G}/\mathcal{B}$ and to generalize from type A to all Weyl types. Together with Luc Lapointe, Jennifer Morse and Mark Shimozono, we have been developing an affine version of Schensted insertion and an affine Pieri rule [**12**].

2. Equivariant homology and cohomology of $\mathcal{G}/\mathcal{P}$

Let $\mathcal{G}$ be the affine Kac-Moody Group of type $\hat{A}_{n-1}$ over $\mathbb{C}$ and let T be a Cartan subgroup of $\mathcal{G}$. Let $\mathcal{B}$ be a Borel subgroup of $\mathcal{G}$. Let $\mathcal{P}$ be a parabolic subgroup of $\mathcal{G}$. The homogeneous space $\mathcal{G}/\mathcal{P}$ is not a finite dimensional variety but an ind-variety (see [**9**]). The group $\mathcal{G}$ possesses a *Bruhat decomposition* $\mathcal{G} = \bigcup_{w \in W} \mathcal{B}w\mathcal{B}$ where W denotes the *affine symmetric group*. The Bruhat decomposition induces a decomposition of $\mathcal{G}/\mathcal{P}$ into *Schubert cells*:

$$\mathcal{G}/\mathcal{P} = \bigcup_{w \in W^P} X_w$$

where P is the parabolic subgroup of W associated to $\mathcal{P}$ and W^P denotes the elements of shortest length in W/P (see [**7**]). The Schubert classes $\sigma_w = [X_w]$ representing X_w in $H_*(\mathcal{G}/\mathcal{P})$ form a basis of the homology. We will denote the Schubert classes in homology, cohomology, equivariant homology and equivariant cohomology as follows

$$\sigma_w \in H_*(\mathcal{G}/\mathcal{P}) \ , \ \sigma^w \in H^*(\mathcal{G}/\mathcal{P}) \ , \ \sigma_{(w)} \in H_*^T(\mathcal{G}/\mathcal{P}) \ , \ \sigma^{(w)} \in H^*_T(\mathcal{G}/\mathcal{P}).$$

Throughout this paper, all homology and cohomology rings will be with $\mathbb{Z}$-coefficients.

From now on we shall assume that $\mathcal{P}$ is a maximal parabolic subgroup. The corresponding parabolic subgroup $W_0 \subset W$ is the usual symmetric group S_n and we denote the minimal-length representatives of W/W_0 by W^0. We call the elements of W^0 *Grassmannian elements*. The homogeneous space $\mathcal{G}/\mathcal{B}$ is known as the *affine flag variety* and $\mathcal{G}/\mathcal{P}$ is known as the *affine Grassmannian*. The isomorphism type of $\mathcal{G}/\mathcal{P}$ does not depend on the choice of maximal parabolic $\mathcal{P}$. It is in fact homeomorphic to $GL_n(\mathcal{K})/GL_n(\mathcal{O})$ where $\mathcal{K} = \mathbb{C}((t))$ denotes the field of Laurent series and $\mathcal{O} = \mathbb{C}[[t]]$ denotes the subring of power series.

A special feature of $\mathcal{G}/\mathcal{P}$ is that it is a group as follows. Let $K = U_n \subset GL_n$ be the compact group of type A_{n-1}. Then it is well known that $\mathcal{G}/\mathcal{P}$ is homotopy equivalent to (the identity component of) ΩK, the space of based loops into K. The group structure of ΩK induces a multiplication on (equivariant) homology, so that $H_*(\mathcal{G}/\mathcal{P})$ and $H^*(\mathcal{G}/\mathcal{P})$ are dual Hopf-algebras. Thus one can sensibly ask for *homology Schubert polynomials* representing the Schubert classes $\sigma_w \in H_*(\mathcal{G}/\mathcal{P})$. This is a feature not present in classical Schubert calculus.

The homology and cohomology rings (and their Hopf-algebra structures) of ΩK were earlier computed by Bott.

Theorem 2.1 ([**3**]). *We have the isomorphisms*

$$H_*(\mathcal{G}/\mathcal{P}) = \mathbb{Z}[\sigma_1, \sigma_2, \ldots, \sigma_{n-1}]$$

and

$$H^*(\mathcal{G}/\mathcal{P}) = \mathcal{S}H^*(\mathbb{CP}^{n-1})$$

where $\mathcal{S}$ *denotes an infinite symmetric power.*

These rings can be identified respectively with a subring and a quotient ring of the ring of symmetric functions. The aim of this paper is thus to identify the Schubert classes $\sigma_w \in H_*(\mathcal{G}/\mathcal{P})$ and $\sigma^w \in H^*(\mathcal{G}/\mathcal{P})$ as explicit symmetric functions.

3. NilHecke Ring

Let $\{r_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ denote the simple generators of W and let $\{\alpha_i \mid i \in \mathbb{Z}/n\mathbb{Z}\}$ denote the simple roots of the root system of type $\hat{A}_{n-1}$ and for a real root α we let $\alpha^\vee$ denote the corresponding coroot. For each root α, we denote the corresponding reflection by r_α. Let $h^*_\mathbb{Z}$ denote the $\mathbb{Z}$-span of the fundamental weights, and let $S = \mathrm{Sym}(h^*_\mathbb{Z})$ denote the ring of polynomials in the weights so that $S = H^*_T(\text{point})$.

Let $\mathbb{A}$ denote the *affine nilHecke ring* of type $\hat{A}_{n-1}$ (see [**8**]). (Note that Kostant and Kumar define $\mathbb{A}$ over the rationals, but we have found it more convenient, following Peterson [**19**], to work over $\mathbb{Z}$.) It is the ring with a 1 given by generators $\{A_i \mid i \in \mathbb{Z}/n\mathbb{Z}\} \cup \{\lambda \mid \lambda \in h^*_\mathbb{Z}\}$ and the relations

$$\begin{aligned}
A_i\, \lambda &= (r_i \cdot \lambda)\, A_i + \langle \lambda, \alpha_i^\vee \rangle \cdot 1 && \text{for } \lambda \in h^*_\mathbb{Z} \\
A_i\, A_i &= 0 && \\
A_i\, A_j &= A_j\, A_i && \text{if } |i-j| \geq 2 \\
A_i\, A_{i+1}\, A_i &= A_i\, A_{i+1}\, A_i. &&
\end{aligned}$$

The ring $\mathbb{A}$ acts as generalized BGG-Demazure operators on $H^*_T(X)$ for any LK-space X (here LK is the space of all loops into the unitary group U_n). The element A_i corresponds to the map $H^*_T(\mathcal{G}/\mathcal{B}) \to H^{*-2}_T(\mathcal{G}/\mathcal{B})$ obtained by integration along the fibers of the $\mathbb{P}^1$-fibration $\mathcal{G}/\mathcal{B} \to \mathcal{G}/\mathcal{P}_i$ where $\mathcal{P}_i$ are the minimal parabolic subgroups. In fact Peterson [**19**] has shown that $\mathbb{A}$ is exactly the ring of *"compact characteristic operators"*; see also [**9**]. Combinatorially, in the classical case the elements A_i act as *divided difference operators* on the Schubert polynomials.

Let $w \in W$ and let $w = s_{i_1} \cdots s_{i_l}$ be a reduced decomposition of w. Then $A_w := A_{i_1} \cdots A_{i_l}$ is a well defined element of $\mathbb{A}$. We let $A_0 := 1$. By [**8**] or [**19**, Proposition 2-7], $\{A_w \mid w \in W\}$ is an S-basis of $\mathbb{A}$. We will also identify r_i with the element $1 - \alpha_i A_i \in \mathbb{A}$ and abusing notation, we write $w \in \mathbb{A}$ for the element in the nilHecke ring corresponding to $w \in W$.

Let $\mathbb{A}_0 \subset \mathbb{A}$ denote the subring over $\mathbb{Z}$ of $\mathbb{A}$ generated by the A_i only. I called this the *affine nilCoxeter algebra* in [**10**]. There is a specialization map $\phi_0 : \mathbb{A} \to \mathbb{A}_0$ given by

$$\phi_0 : \sum_w a_w A_w \longmapsto \sum_w \phi_0(a_w) A_w$$

where ϕ_0 evaluates a polynomial $s \in S$ by setting all α_i to 0.

For later use, we note the following straightforward result, whose proof we omit; see [**8**, Proposition 4.30].

Lemma 3.1. *Let* $w \in W$ *and* $\lambda \in S$ *be of degree 1. Then*

$$A_w \lambda = (w \cdot \lambda) A_w + \sum_{r_\alpha w \lessdot w} \langle \lambda, \alpha^\vee \rangle\, A_{r_\alpha w}.$$

Here $\lessdot$ *denotes a cover in strong Bruhat order.*

The coefficients $\langle \lambda, \alpha^\vee \rangle$ are known as *Chevalley coefficients*.

4. The coproduct on $\mathbb{A}$

Define the coproduct map $\Delta : \mathbb{A} \to \mathbb{A} \otimes_S \mathbb{A}$ by

$$\begin{aligned} \Delta(s) &= 1 \otimes s = s \otimes 1 && \text{for } s \in S \\ \Delta(A_i) &= A_i \otimes 1 + r_i \otimes A_i = 1 \otimes A_i + A_i \otimes r_i \\ &= A_i \otimes 1 + 1 \otimes A_i - A_i \otimes \alpha_i A_i. \end{aligned}$$

This is a well defined map, which in addition is cocommutative. One can deduce from these relations that $\Delta(w) = w \otimes w$. (In the original work of [**8**], this last relation was used to define Δ, but we shall follow the set up of [**19**]).

One should be careful since the tensor product $\mathbb{A} \otimes_S \mathbb{A}$ is *not* a ring. For example,

$$(A_i \otimes 1).(1 \otimes \alpha_i) \neq (A_i \otimes 1).(\alpha_i \otimes 1)$$

However, it is shown in [**19**] that the action of $\mathbb{A}$ on $\mathbb{A} \otimes_S \mathbb{A}$ given by the above formulae still give a well defined action of $\mathbb{A}$ on $\mathbb{A} \otimes_S \mathbb{A}$. That is, $\Delta(a) = a \cdot (1 \otimes 1)$ for any $a \in \mathbb{A}$.

Note that ϕ_0 also sends $\mathbb{A} \otimes_S \mathbb{A}$ to $\mathbb{A}_0 \otimes_{\mathbb{Z}} \mathbb{A}_0$ by evaluating the coefficients at 0 when writing in the basis $\{A_w \otimes A_v\}_{w,v \in W}$.

Theorem 4.1 ([**8**]). *Let*

$$\Delta(A_w) = \sum_{u,v \in W} a_w^{u,v} A_u \otimes A_v.$$

Then $a_w^{u,v}$ are the (Schubert) structure constants of $H_T^*(\mathcal{G}/\mathcal{B})$, so that

$$\sigma^{(u)} \cdot \sigma^{(v)} = \sum_{w \in W} a_w^{u,v} \sigma^{(w)}.$$

Theorem 4.1 is in fact valid for all symmetrizable Kac-Moody groups. Since the product of two Grassmannian classes $\sigma^{(u)}$ and $\sigma^{(v)}$ (where $u, v \in W^0$) in $H^T(\mathcal{G}/\mathcal{P})$ is Grassmannian, we have the following simple result.

Lemma 4.2. *If $w \notin W^0$ and $u, v \in W^0$ then $a_w^{u,v} = 0$.*

5. Symmetric functions

We refer to [**18**] for details concerning the material of this section. Let $\Lambda = \Lambda_{\mathbb{Z}}$ denote the ring of symmetric functions over $\mathbb{Z}$ in infinitely many variables $x_1, x_2, \ldots$. We write $h_i(x)$ for the *homogeneous symmetric functions* and for a partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$, we write $h_\lambda(x) = h_{\lambda_1(x)} h_{\lambda_2(x)} \cdots$. The elements $h_1(x), h_2(x), \ldots \in \Lambda$ form a set of algebraically independent set of generators of Λ. We let $m_\lambda(x) \in \Lambda$ denote the *monomial symmetric functions*. They form a basis of the ring of symmetric functions over the integers.

Let $\Lambda_n \subset \Lambda$ denote the subring of the symmetric functions generated by $h_i(x)$ for $i \in [0, n-1]$. Let Λ^n denote the quotient of Λ given by $\Lambda^n = \Lambda / \langle m_\lambda(x) \mid \lambda_1 \geq n \rangle$. Clearly the set $\{m_\lambda(x) \mid \lambda_1 < n\}$ forms a basis of Λ^n. When giving an element $\bar{f} \in \Lambda^n$ we will usually just give a representative $f \in \Lambda$ without further comment.

The *Hall inner product*, denoted $\langle ., . \rangle : \Lambda \times \Lambda \to \mathbb{Z}$, is a symmetric non-degenerate pairing defined by $\langle h_\lambda(x), m_\mu(x) \rangle = \delta_{\lambda\mu}$. It induces a non-degenerate pairing $\langle ., . \rangle : \Lambda_n \times \Lambda^n \to \mathbb{Z}$.

It is not too difficult to see from Theorem 2.1 that $\Lambda_n \cong H_*(\mathcal{G}/\mathcal{P})$ and $\Lambda^n \cong H^*(\mathcal{G}/\mathcal{P})$.

In fact the ring of symmetric functions Λ is a Hopf algebra with coproduct given by $\Delta(h_i(x)) = \sum_{j \leq i} h_j(x) \otimes h_i(x)$. This Hopf-algebra structure gives Λ_n and Λ^n the structures of dual Hopf algebras.

6. Affine Schur functions and k-Schur functions

An integral orthonormal basis of Λ is given by the set of Schur functions $s_\lambda(x)$. We will be concerned with a set of dual bases $\{s_\lambda^{(k)}(x)\}$ of Λ_n and $\{F_\lambda(x)\}$ of Λ^n called respectively the *k-Schur functions*, and *affine Schur functions* or *dual k-Schur functions*. The k-Schur functions $\{s_\lambda^{(k)}(x)\}$ were introduced in [**16**], and were further studied in [**13, 14**]. We will give a quick "dual" definition of these functions.

Definition 6.1. Let $a = a_1a_2\cdots a_k$ be a word with letters from $\mathbb{Z}/n\mathbb{Z}$ so that $a_i \neq a_j$ for $i \neq j$. Let $A = \{a_1, a_2, \ldots, a_k\} \subset [0, n-1]$. The word a is *cyclically decreasing* if for every i such that $i, i+1 \in A$, the letter $i+1$ precedes i in a. A permutation w is *cyclically decreasing* if $w = s_{a_1}\cdots s_{a_k}$ for some cyclically decreasing sequence $a_1a_2\cdots a_k$.

Now define, following [**10**], the elements $h_i \in \mathbb{A}_0 \subset \mathbb{A} : i \in [0, n-1]$ by the formula

$$h_i = \sum_w A_w$$

where the sum is over *cyclically decreasing* permutations w with length $l(w) = i$. If $I \subset [0, n-1]$ and w be the corresponding cyclically decreasing permutation. Then we will write A_I for A_w.

Let $\mathbb{B}$ denote the subalgebra of $\mathbb{A}_0 \subset \mathbb{A}$ generated by the h_i for $i \in [0, n-1]$, which we call the *affine Fomin-Stanley subalgebra*.

Theorem 6.2 ([**10**]). *The algebra $\mathbb{B}$ is commutative. It is isomorphic to the subalgebra Λ_n of the symmetric functions generated by the homogeneous symmetric functions $h_i(x)$ for $i \in [0, n-1]$, under the map $\psi : h_i(x) \mapsto h_i$.*

Let $\langle ., . \rangle : \mathbb{A}_0 \times \mathbb{A}_0 \to \mathbb{Z}$ denote the bilinear pairing defined by $\langle A_w, A_v\rangle = \delta_{wv}$.

Definition 6.3 ([**10**]). Let $w \in W$. Define the *affine Stanley symmetric functions* $\tilde{F}_w(x) \in \Lambda$ by

$$\tilde{F}_w(x) = \sum_{a=(a_1,a_2,\ldots,a_t)} \left\langle h_{a_t}h_{a_{t-1}}\cdots h_{a_1}\cdot 1, A_w\right\rangle x_1^{a_1}x_2^{a_2}\cdots x_t^{a_t},$$

where the sum is over compositions of $l(w)$ satisfying $a_i \in [0, n-1]$.

The (image in Λ^n of the) set $\{\tilde{F}_w(x) \mid w \in W^0\}$ forms a basis of Λ^n (see [**10**]). We called these functions *affine Schur functions* in [**10**]. They were earlier introduced in a different manner in [**15**], where they were called *dual k-Schur functions*. The *k-Schur functions* $\{s_w^{(k)}(x) \mid w \in W^0\}$ are the dual basis of Λ_n to the affine Schur functions under the Hall inner product. There is a bijection $w \leftrightarrow \lambda(w)$ from Grassmannian permutations $\{w \in W^0\}$ to partitions $\{\lambda \mid \lambda_1 < n\}$ obtained by taking the *code* of the permutation; see [**2**]. We make the identifications $\tilde{F}_w(x) = \tilde{F}_{\lambda(w)}(x)$ and $s_w^{(k)}(x) = s_{\lambda(w)}^{(k)}(x)$ under this bijection. Note that in the terminology of [**16**], $k = n-1$.

7. Non-commutative k-Schurs

Recall that we have an isomorphism $\psi : \Lambda_n \to \mathbb{B}$. Define $\Delta_{\mathbb{B}} : \mathbb{B} \to \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{B}$ by

$$\Delta_{\mathbb{B}}(h_i) = \sum_{j\leq i} h_j \otimes h_{i-j}$$

and extending $\Delta_{\mathbb{B}}$ to a ring homomorphism. This is just the natural coproduct of the symmetric functions as explained in Section 5. The following definition is inspired by work of Fomin and Greene [**5**].

Definition 7.1. Let $w \in W^0$. The *non-commutative k-Schur functions* are given by

$$s_w^{(k)} := \psi(s_w^{(k)}(x)) \in \mathbb{B}.$$

The main result we need concerning the non-commutative k-Schur functions is the following.

Theorem 7.2 ([**10**]). *The non-commutative k-Schurs can be written in the A_w basis as*

$$s_w^{(k)} = A_w + \sum_{v\notin W^0} b_{w,v}A_v$$

where w is a Grassmannian permutation and the second term is a summation over non-Grassmannian permutations.

8. The Main Theorem

Our main theorem is the following.

Theorem 8.1. *The map $\theta : H_*(\mathcal{G}/\mathcal{P}) \to \Lambda_n$ given by*

$$\theta : \sigma_w \longmapsto s_w^{(k)}(x)$$

is an isomorphism of Hopf-algebras. The map $\theta' : H^*(\mathcal{G}/\mathcal{P}) \to \Lambda^n$ given by

$$\theta' : \sigma^w \longmapsto \tilde{F}_w(x)$$

is an isomorphism of Hopf-algebras.

In the homology case, this theorem was a conjecture of Mark Shimozono. The conjecture in the cohomology case was made precise by Jennifer Morse.

We shall prove the following technical result in Section 13.

Theorem 8.2. *The two coproducts $\Delta, \Delta_{\mathbb{B}}$ agree on $\mathbb{B}$ up to specialisation at 0:*

$$\phi_0 \circ \Delta = \Delta_{\mathbb{B}}.$$

The following theorem proves half of Theorem 8.1. Recall that $a_w^{u,v}$ are the multiplicative structure constants of $H^*(\mathcal{G}/\mathcal{P})$.

Theorem 8.3. *We have*

$$\phi_0(\Delta(s_w^{(k)})) = \sum_{u,v \in W^0 \,:\, l(u)+l(v)=l(w)} a_w^{u,v}\ s_u^{(k)} \otimes s_v^{(k)}.$$

Note that since the k-Schur functions $s_w^{(k)}(x)$ are Hall-dual to the affine Schur functions $\tilde{F}_w(x)$, Theorem 8.3 immediately implies that multiplication of $\tilde{F}_w(x)$ in Λ^n agrees with the multiplication of σ^w in $H^*(\mathcal{G}/\mathcal{P})$. See also the discussion in [**10**].

Proof. By Theorems 4.1 and 7.2, we have

$$\begin{aligned}\Delta(s_w^{(k)}) &= \Delta(A_w + \sum_v b_{w,v} A_v) \\ &= \sum_{u,v} a_w^{u,x} A_u \otimes A_x + \sum_v b_{w,v} \sum_{y,z} a_v^{y,z} A_y \otimes A_z\end{aligned}$$

The polynomials $a_w^{u,x}$ are known to have (homogeneous) degree $l(u) + l(x) - l(w)$, so we get

$$\phi_0(\Delta(s_w^{(k)})) = \sum_{\substack{u,x \\ l(u)+l(x)=l(w)}} a_w^{u,x} A_u \otimes A_x + \sum_v b_{w,v} \sum_{\substack{y,z \\ l(y)+l(z)=l(v)}} a_v^{y,z} A_y \otimes A_z.$$

By Lemma 4.2, we may actually write

$$\phi_0(\Delta(s_w^{(k)})) = \sum_{u,v \in W^0 : l(u)+l(v)=l(w)} a_w^{u,x} A_u \otimes A_x + \text{other terms}. \tag{8.1}$$

The other terms involve $A_y \otimes A_z$ where one of y or z is not Grassmannian.

Now by Theorem 8.2, we have $\phi_0(\Delta(s_w^{(k)})) \in \mathbb{B} \otimes_{\mathbb{Z}} \mathbb{B}$ so we may write it as

$$\phi_0(\Delta(s_w^{(k)})) = \sum_{u,x \in W^0} c_w^{u,x} s_u^{(k)} \otimes s_x^{(k)}$$

where $c_w^{u,x}$ are some integers. Using Theorem 7.2 again, we have

$$\phi_0(\Delta(s_w^{(k)})) = \sum_{u,x \in W^0} c_w^{u,x} A_u \otimes A_x + \text{other terms},$$

where as before the other terms involve the basis elements $A_y \otimes A_z$ where one of y or z is not Grassmannian. Comparing with (8.1) we have $c_w^{u,x} = a_w^{u,x}$, as required. □

9. $\mathbb{B}$ nearly annihilates S

To prove Theorem 8.2, and also to obtain the multiplicative constants of the homology $H_*(\mathcal{G}/\mathcal{P})$ we first prove a technical property of the Fomin-Stanley subalgebra $\mathbb{B}$.

THEOREM 9.1. *Let $b \in \mathbb{B}$ and $s \in S$. Then*

$$\phi_0(bs) = \phi_0(s)b.$$

PROOF. We show that $\phi_0(h_i \cdot \alpha_j) = 0$ for each i and the theorem follows since h_i generate $\mathbb{B}$. Without loss of generality we will assume that $j = 1$. Let $I \subset \mathbb{Z}/n\mathbb{Z}$ be of size i. We calculate $\phi_0(A_I\alpha_1)$ explicitly. In the following $[2, r]$ is the largest interval of its form (possibly empty) contained in I which contains 2. It is possible that $[2, r]$ contains 0 but it cannot contain 1 (since then it will have size n). Also the subset I' never contains any of $0, 1, 2$. The sums over a are always over $a \in [2, r]$. The (A),(B),(C) are for marking the terms only, for later use.

I	$\phi_0(A_I\alpha_1)$
$I' \cup [2, r]$	$-\sum_a A_{I-\{a\}}$(A)
$I' \cup [2, r] \cup \{1\}$	$2A_{I-\{1\}}$(A) $+ \sum_a A_{I-\{a\}}$(C)
$I' \cup [2, r] \cup \{0\}$	$-A_{I-\{0\}}$(A) $- \sum_a A_{I-\{a\}}$(B)
$I' \cup [2, r] \cup \{0, 1\}$	$-A_{I-\{0\}}$(C) $+ A_{I-\{1\}}$(B)

For example

$$\begin{aligned} &A_{[2,r]}A_1A_0\alpha_1 \\ &\qquad = A_{[2,r]}A_1((\alpha_1 + \alpha_0)A_0 - 1) \\ &\qquad = -A_{[2,r]}A_1 + A_{[2,r]}(-\alpha_1A_1A_0 + 2A_0 + (\alpha_1 + \alpha_0)A_1A_0 - A_0) \\ &\qquad = -A_{[2,r]-\{0\}} + A_{[2,r]-\{1\}} + \alpha_0A_{[2,r]}A_1A_0. \end{aligned}$$

The A_t factors for $t \in I'$ always commute in these calculations.

One observes that the terms marked (A) or (B) or (C) when grouped together cancel out. We have: (A) corresponds to subsets J of size $i-1$ such that J contains neither 1 nor 0; and (B) corresponds to subsets J of size $i-1$ such that J contains 0 but not 1; and (C) corresponds to subsets J of size $i-1$ such that J contains 1 but not 0. Every such subset in say case (A) will appear in all 3 case (A) terms. No other subsets (those containing both 0 and 1) appear in the sum $\sum_I A_I\alpha_1$.

For example, the subset $J = [2, 4] \cup [5, 7]$ will appear in $\phi_0(A_I\alpha_1)$ for $I = [2, 7]$ or $[1, 4] \cup [5, 7]$ or $\{0\} \cup [2, 4] \cup [5, 7]$. The multiplicities will be -1, 2, and -1 respectively, which cancel out. □

10. An identity for finite Weyl groups

Let W^{fin} be a finite Weyl group and $H^*(K/T)$ be the cohomology of the corresponding flag variety. Also let w° denote the longest element of W^{fin}.

PROPOSITION 10.1. *Suppose that for some coefficients $\{b_u \in \mathbb{Z}\}_{u \in W^{\mathrm{fin}}}$ the following identity holds in $\mathbb{Z}W^{\mathrm{fin}}$ for all integral weights $\lambda \in h^*_{\mathbb{Z}}$*

$$\sum_{u \in W^{\mathrm{fin}};\ l(u)>0} b_u \sum_{ur_\alpha \lessdot u} \langle \lambda, \alpha^\vee \rangle\, ur_\alpha = 0.$$

Then $b_u = 0$ for all u.

PROOF. First apply the transformation $u \mapsto w^\circ u$ to the identity of the Proposition. Then reindexing the b_u, we obtain

$$\sum_{u \in W^{\mathrm{fin}};\ u \neq w^\circ} b_u \sum_{ur_\alpha \gtrdot u} \langle \lambda, \alpha^\vee \rangle\, ur_\alpha = 0$$

for all λ.

Let $\sigma^{(0)}_u \in H^*(K/T)$ denote the Schubert classes in the finite flag variety. By the Chevalley-Monk formula [**1**] we have

$$[\lambda] \cdot \sigma^{(0)}_u = \sum_{ur_\alpha \gtrdot u} \langle \lambda, \alpha^\vee \rangle\, \sigma^{(0)}_{ur_\alpha}$$

where $[\lambda] \in H^*(K/T)$ denotes the image of λ under the characteristic homomorphism $S(h_{\mathbb{Z}}^*) \to H^*(K/T)$. For example, if $\lambda = \omega_i$ is a fundamental weight then $[\omega_i] = \sigma_{s_i}^{(0)}$. It is well known that $\sigma_{s_1}^{(0)}, \sigma_{s_2}^{(0)}, \ldots, \sigma_{s_{n-1}}^{(0)}$ generate $H^*(K/T)$ or alternatively that the characteristic homomorphism is surjective.

Suppose that $[\lambda] \cdot \sigma = 0$ for some $\sigma \in H^*(K/T)$ and all $\lambda \in h_{\mathbb{Z}}^*$. If $l(v) + l(u) = l(w_\circ)$ we have $\sigma_v^{(0)} \cdot \sigma_u^{(0)} = \delta_{v,w_\circ u}\sigma_{w_\circ}^{(0)}$. Since $\sigma_u^{(0)} \cdot \sigma = 0$ for all $u \neq \mathrm{id}$, we find that σ must be a multiple of the class $\sigma_{w_\circ}^{(0)}$. Letting $\sigma = \sum_u b_u \sigma_u^{(0)}$ and applying the Chevalley-Monk formula we obtain the proposition. □

11. The subalgebra $\mathbb{B}'$

Define a subalgebra $\mathbb{B}' \subset \mathbb{A}_0$ as follows:

$$\mathbb{B}' = \{a \in \mathbb{A}_0 \mid \phi_0(as) = \phi_0(s)a \text{ for all } s \in S\}.$$

Thus Theorem 9.1 says that $\mathbb{B} \subset \mathbb{B}'$. It turns out that $\mathbb{B}'$ is always a commutative subalgebra for all affine types, though we will not need such generality here.

Proposition 11.1. *Let $b \neq 0 \in \mathbb{B}'$ and write $b = \sum_w b_w A_w$ with $b_w \in \mathbb{Z}$. Then $b_w \neq 0$ for some $w \in W^0$.*

Proof. Let $D = \{w \in W \mid b_w \neq 0\}$. For each $w \in W$ we may uniquely write $w = x_w y_w$ where $x_w \in W^0$ and $y_w \in W_0$. Let $d = \{\min(l(y_w)) \mid w \in D\}$. We write $l_0(w) := l(y_w)$.

Suppose $d \neq 0$ and let $w \in D$ minimize $l_0(w)$. Let $\lambda \in S$ be of degree 1. Then by Lemma 3.1, $\phi_0(A_w\lambda) = \sum_{wr_\alpha \lessdot w} \langle \lambda, \alpha^\vee \rangle A_{wr_\alpha}$. We know that $w \gtrdot v$ if and only if a reduced decomposition of v is obtained from a reduced decomposition of w by removing a simple generator. Since $w = x_w y_w$, each such v satisfies $l_0(v) \geq l_0(w) - 1$. Let $D_w = \{v \lessdot w \mid l_0(v) = l_0(w) - 1\}$. Then $v \in D_w$ if and only if $v = x_v y_v$ where $x_v = x_w$ and $y_v \lessdot y_w$.

Now write $\phi_0(b\lambda) = \sum_v b'_v A_v$ and focus only on the coefficients of b'_v satisfying $l_0(v) = d-1$ and $v = xy_v$ for some fixed $x \in W^0$. If $b \in \mathbb{B}'$ then $b'_v = 0$. Thus in particular, for every $\lambda \in S$ of degree 1, we have

$$\sum_{u \in W_0} b_{xu} \sum_{ur_\alpha \lessdot u} \langle \lambda, \alpha^\vee \rangle A_x A_{ur_\alpha} = 0.$$

Factorizing A_x to the front, we see that this is impossible by Proposition 10.1. Since this is true for all $x \in W^0$ we conclude that we must have $d = 0$. □

12. Peterson's j-homomorphism

To further understand the non-commutative k-Schur functions, we require a result of Peterson. Let $Z_{\mathbb{A}}(S)$ denote the centralizer of S in $\mathbb{A}$.

Theorem 12.1 ([**19**]). *There is an isomorphism $j : H_*^T(\Omega K) \to Z_{\mathbb{A}}(S)$ such that*

$$j(\sigma_{(x)}) = A_x \mod I$$

where x is a Grassmannian permutation and

$$I = \sum_{w \in W_0\,;\, w \neq \mathrm{id}} \mathbb{A} \cdot A_w.$$

Recall that $W_0 = S_n$ is the usual symmetric group.

Theorem 12.2. *We have $\phi_0(Z_{\mathbb{A}}(S)) = \mathbb{B}'$. More precisely, $\{\phi_0(j(\sigma_{(u)})) \mid u \in W^0\}$ forms a basis of $\mathbb{B}'$ over $\mathbb{Z}$.*

Proof. The fact that $\phi_0(Z_A(S)) \subset \mathbb{B}'$ is a trivial calculation. Now let $b \in \mathbb{B}'$. By Proposition 11.1 it contains a Grassmannian term A_u with non-zero coefficient b_u. By Theorem 12.1, $b - b_u\phi_0(j(\sigma_{(u)}))$ has strictly fewer Grassmannian terms and also lies in $\mathbb{B}'$. Repeating, we see that one can write b uniquely as a $\mathbb{Z}$-linear combination of the elements $\phi_0(j(\sigma_{(u)}))$. □

Corollary 12.3. *The two algebras $\mathbb{B}$ and $\mathbb{B}'$ are identical (as subalgebras of $\mathbb{A}_0$) and we have*

$$\phi_0(j(\sigma_{(u)})) = s_u^{(k)}.$$

Proof. This follows immediately from Theorems 12.2 and 7.2 together with Proposition 11.1: both $\phi_0(j(\sigma_{(u)}))$ and $s_u^{(k)}$ lie in $\mathbb{B}'$ and have a unique Grassmannian term A_u. □

Finally, we can complete the proof of our main theorem.

Proof of Theorem 8.1. Let $x, y \in W^0$. If $\sigma_{(x)}\sigma_{(y)} = \sum_{z \in W^0} c_{x,y}^z \sigma_{(z)}$ in $H_T(\mathcal{G}/\mathcal{P})$ then $\sigma_x \sigma_y = \sum_{z \in W^0} \phi_0(c_{x,y}^z)\sigma_z$ in $H_*(\mathcal{G}/\mathcal{P})$. Thus $\mathbb{B} = \mathbb{B}'$ is isomorphic to $H_*(\mathcal{G}/\mathcal{P})$ and we have

$$s_x^{(k)}\, s_y^{(k)} = \sum_{z \in W^0} \phi_0(c_{x,y}^z) s_z^{(k)}.$$

This, together with Theorem 8.3 shows that θ and θ' are both algebra and co-algebra homomorphisms. The agreement of the remainder of the Hopf algebra structures is straightforward to verify. □

13. Proof of Theorem 8.2

We now return to the proof of Theorem 8.2. It will follow quickly from the following computation.

Proposition 13.1. *We have*

$$\phi_0(\Delta(h_i)) = \sum_j h_j \otimes h_{i-j}$$

Proof. Let $\beta_i = -\alpha_i$ be the negative simple roots. We use $\Delta(A_i) = A_i \otimes 1 + 1 \otimes A_i + A_i \otimes \beta_i A_i$.

Let $i_1, i_2, \ldots, i_l$ be a cyclically decreasing sequence. Thus

$$\Delta(A_{i_1} A_{i_2} \cdots A_{i_l}) = \prod_j \Delta(A_{i_j})$$
$$= (A_{i_1} \otimes 1 + 1 \otimes A_{i_1} + A_{i_1} \otimes \beta_{i_1} A_{i_1}) \cdots (A_{i_l} \otimes 1 + 1 \otimes A_{i_l} + A_{i_l} \otimes \beta_{i_l} A_{i_l})$$

Let us expand the product, by picking one of the three terms in each parentheses. (Strictly speaking we cannot multiply within $\mathbb{A} \otimes_S \mathbb{A}$, instead we are calculating the action of $\mathbb{A}$ on $\mathbb{A} \otimes_S \mathbb{A}$ via the coproduct: $\Delta(A_i) \cdot (\Delta(A_j) \cdot (1 \otimes 1)) = \Delta(A_i A_j)$).

Because of the cyclically decreasing assumption, the only times we encounter a factor looking like $A_{i_a}\beta_{i_b}$ (where $a < b$) we have either

$$A_{i_a}\beta_{i_b} = \beta_{i_b} A_{i_a} \tag{13.1}$$

or we will have $a = b - 1$ and $i_{a+1} = i_a - 1$ and

$$A_{i_a}\beta_{i_a - 1} = (\beta_{i_a - 1} + \beta_{i_a}) A_{i_a} + 1. \tag{13.2}$$

If (13.1) ever occurs, then β_{i_b} commutes with all A_{i_c} where $c < b$ and we may ignore the term since eventually we will apply ϕ_0. Similarly, if (13.2) occurs, the contribution of the term involving $\beta_{i_a - 1}$ is 0 after applying ϕ_0.

Also we perform the calculation

$$A_{i+1}(\beta_i)^m = \beta_{i+1}^m A_{i+1} + \beta_{i+1}^{m-1} + \text{other terms}, \tag{13.3}$$

where the other terms involve β_i on the left somewhere (and would be killed by ϕ_0 later).

Let B and C be two subsets of $[0, n-1]$ with total size equal to $k \leq n - 1$. We will first describe how to obtain the term $A_B \otimes A_C$ (which occurs in $h_{|B|} \otimes h_{|C|}$) from $\Delta(h_k)$. Define a sequence of integers ("current degree") $(\mathrm{cd}(i) : i \in \mathbb{Z}/n\mathbb{Z})$ by $\mathrm{cd}(i) = \max_t\{|I \cap [i - t, i]| + |J \cap [i - t, i]| - t - 1\}$. Since $|B| + |C| < n$ we can find i so that $\mathrm{cd}(i) = 0$ and $i \notin B \cup C$.

We may assume that $i = 0$. Let $B = (b_1 > \cdots > b_g)$ and $C = (c_1 > \cdots > c_h)$. Define a sequence $(t_1, t_2, \ldots, t_{n-1}) \in \{L, R, B, E\}^{n-1}$ as follows (E = empty, L = left, R = right and B = both):

$$t_i = \begin{cases} E & \text{if } \mathrm{cd}(i) = 0 \text{ and } E \notin B \cup C \\ L & \text{if } \mathrm{cd}(i) = 0 \text{ and } E \in B \text{ but } E \notin C \\ R & \text{if } E \notin B \text{ and } (\mathrm{cd}(i) > 0 \text{ or } E \in C) \\ B & \text{otherwise.} \end{cases}$$

Now let $I = \{i \in [1, n-1] \mid t \neq E\} \subset [1, n-1]$. Then $A_B \otimes A_C$ is obtained from $\Delta(A_I)$ by picking the term $A_{i_s} \otimes 1$ if $t_{i_s} = L$, the term $1 \otimes A_{i_s}$ if $t_{i_s} = R$ and $A_{i_s} \otimes \beta_{i_s} A_{i_s}$ if $t_{i_s} = B$.

The sequence of integers $(\mathrm{cd}(i))$ tells us the current degree (in the second factor of the tensor product) in S of the term that we want to pick whenever we encounter the situation of (13.3).

For example if $\mathrm{cd}(t) = 3$ and $\mathrm{cd}(t+1) = 3$ then $t+1 \in B$ or $t+1 \in C$. In the first case we will have $(A_{t+1} \otimes 1) \cdot (a \otimes \beta_i^3 b)$, for some a and b not involving S, and there is no further choice. In the second case we get

$$(1 \otimes A_{t+1}) \cdot (a \otimes \beta_i^3 b) = a \otimes (\beta_{i+1}^3 A_{t+1} + \beta_{i+1}^2) b,$$

modulo terms involving β_i on the right. One must make a further choice between $\beta_{i+1}^3 A_{t+1}$ and β_{i+1}^2. We pick the first term since we want $t+1 \in C$ and this agrees with the degree being $\mathrm{cd}(t+1) = 3$.

Thus every term of the form $A_B \otimes A_C$ appears in the expansion of $\phi_0(\Delta(h_i))$. Conversely, one can reverse the description given above to see that every term in the expansion is indeed of that form. $\square$

Proof of Theorem 8.2. From Proposition 13.1, we have $\Delta_{\mathbb{B}}(h_i) = \phi_0(\Delta(h_i))$. Now let $a \in \mathbb{B}$ and $b \in \mathbb{B}$ and suppose we have shown that $\Delta_{\mathbb{B}}(a) = \phi_0(\Delta(a))$ and $\Delta_{\mathbb{B}}(b) = \phi_0(\Delta(b))$. Let $\Delta(a) = \sum_{w,v} A_w \otimes a_{w,v} A_v$ and $\Delta(b) = \sum_{x,y} A_x \otimes b_{x,y} A_y$, where $a_{w,v}, b_{x,y} \in S$. Then

$$\begin{aligned}
\phi_0(\Delta(ab)) &= \phi_0(\Delta(a)\Delta(b)) \\
&= \phi_0\Big(\sum_{w,v,x,y} A_w A_x \otimes a_{w,v} A_v b_{x,y} A_y\Big) \\
&= \sum_{w,v,x,y} A_w A_x \otimes \phi_0(a_{w,v}) A_v \phi_0(b_{x,y}) A_y \qquad \text{by Theorem 9.1.} \\
&= \phi_0(\Delta(a))\phi_0(\Delta(b)) \\
&= \Delta_{\mathbb{B}}(a)\Delta_{\mathbb{B}}(b) \\
&= \Delta_{\mathbb{B}}(ab).
\end{aligned}$$

Since the h_i generate $\mathbb{B}$ this completes the proof. $\square$

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