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Nonnegativity properties of the dual canonical basis

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ABSTRACT. Using Du's characterization of the dual canonical basis of the coordinate ring $\mathcal{O}(GL_n(\mathbb{C}))$, we show that all basis elements may be expressed in terms of immanants. We then give a new factorization of permutations avoiding the patterns 3412 and 4231, which in turn yields a factorization theorem for the corresponding Kazhdan-Lusztig basis of the Hecke algebra $H_n(q)$. Using this factorization, we show that for every totally nonnegative immanant $\mathrm{Imm}_f(x)$ and its expansion $\sum d_w \mathrm{Imm}_w(x)$ with respect to the basis of Kazhdan-Lusztig immanants, the coefficient d_w must be nonnegative when w avoids the patterns 3412 and 4231.

RÉSUMÉ. En utilissant les résultats de Du, nous démontrons que chacque élement du base dual canonique de $\mathcal{O}(GL_n(\mathbb{C}))$, se peut réalise en terme d'immanants. Nous factorisent les permutations qui evitent le 3412 et le 4231, et aussi les élements du base de Kazhdan-Lusztig pour l'algébre de Hecke $H_n(q)$. En utilissant cette factorisation, nous montrons que pour chacque immanant totalement nonnegatif $\mathrm{Imm}_f(x)$ et l'expression $\sum d_w \mathrm{Imm}_w(x)$ en terme de base dual canonique, le coefficient d_w est nonnegatif quand w evite le 3412 et le 4231.

1. Introduction

Searching for solutions of the quantum Yang-Baxter equation, Drinfeld [**Dri85**] and Jimbo [**Jim85**] introduced a quantization $U_q(\mathfrak{sl}_n\mathbb{C})$ of the universal enveloping algebra $U(\mathfrak{sl}_n\mathbb{C})$. An explosion of mathematical research soon led to a quantization $\mathcal{O}_q(SL_n\mathbb{C})$ of the coordinate ring $\mathcal{O}(SL_n\mathbb{C})$, related by Hopf algebra duality to $U_q(\mathfrak{sl}_n\mathbb{C})$, and to a development of the representation theory of these algebras now known as quantum groups. In particular, Kashiwara [**Kas91**] and Lusztig [**Lus90**] discovered a *canonical* (or *crystal*) *basis* of $U_q(\mathfrak{sl}_n\mathbb{C})$ which has many interesting representation theoretic properties. The corresponding dual basis of $\mathcal{O}_q(SL_n\mathbb{C})$ is known as the *dual canonical basis* and is perhaps best understood as the projection of another *dual canonical basis* of the quantum polynomial ring $\mathbb{C}_q[x_{1,1},\dots,x_{n,n}]$. (See [**Du92**].) An elementary description of the canonical and dual canonical bases has been somewhat elusive, especially in the nonquantum ($q=1$) setting.

In [**Lus94**] Lusztig proved that when we specialize $q=1$, the elements of the dual canonical basis of $\mathbb{C}[x_{1,1},\dots,x_{n,n}]$ are *totally nonnegative* (TNN) polynomials in the following sense. We define a matrix to be *totally nonnegative* (TNN) if each of its minors is nonnegative. (See, e.g. [**FZ00**].) We define a polynomial $p(x)\in\mathbb{C}[x_{1,1},\dots,x_{n,n}]$ to be *totally nonnegative* (TNN) if for each $n\times n$ TNN matrix $A=(a_{i,j})$, we have

$$p(A) \underset{\mathrm{def}}{=} p(a_{1,1},\dots,a_{n,n}) \geq 0.$$

While it is not true that a polynomial is TNN only if it belongs to the dual canonical cone, we will show that certain coordinates of the polynomial with respect to the dual canonical cone must be nonnegative. Our criterion involves avoidance of the patterns 3412 and 4231 in permutations and thus links total nonnegativity to smoothness in Schubert varieties.

In Section 2 we will review Du's formulation of the dual canonical basis and show that these elements can be expressed in terms of functions called immanants. In Section 3 we will state a factorization theorem

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for 3412-avoiding, 4231-avoiding permutations and for the corresponding Kazhdan-Lusztig basis elements. In Section 4 we will use the factorization and immanant results to prove that for each TNN homogeneous element $p(x)$ of the coordinate ring $\mathcal{O}(SL_n(\mathbb{C}))$, certain coordinates with respect to the dual canonical basis must be nonnegative.

2. Kazhdan-Lusztig immanants and the dual canonical basis

The canonical bases of $\mathcal{O}(SL_n(\mathbb{C}))$ and $\mathcal{O}(GL_n(\mathbb{C}))$ may be obtained easily from a basis of the polynomial ring $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$. We will call this basis too *the dual canonical basis.*

Before explicitly describing the dual canonical basis, let us look at a multigrading of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ in terms of multisets. The polynomial ring has a traditional grading by degree,

$$\mathbb{C}[x] = \bigoplus_{r \geq 0} \mathcal{A}_r,$$

where $\mathcal{A}_r$ is the complex span of degree-r monomials. We may refine this grading by defining a multigrading of $\mathcal{A}_r$ indexed by pairs of r-element multisets. Let $\mathcal{M}(n, r)$ be the set of r-element multisets of n. Then we have

$$\mathcal{A}_r = \bigoplus_{M, M' \in \mathcal{M}(n,r)} \mathcal{A}_r(M, M'),$$

where we define a polynomial to be homogeneous of multidegree (M, M') if in each of its monomials, the multiset of row indices is M and the multiset of column indices is M'. For example, the polynomial $x_{1,1}x_{2,1}^2x_{3,3} - x_{1,1}x_{2,1}x_{2,3}x_{3,1}$ belongs to the component $\mathcal{A}_3(1223, 1113)$ of $\mathbb{C}[x_{1,1}, \ldots, x_{3,3}]$.

Closely related to this multigrading are *generalized submatrices* of x. Given two r-element multisets $M = m_1 \cdots m_r$, $M' = m'_1 \cdots m'_r$ of $[n]$ (written as weakly increasing words), define the (M, M') *generalized submatrix* of x to be the matrix

$$x_{M,M'} = \begin{bmatrix} x_{m_1,m'_1} & x_{m_1,m'_2} & \cdots & x_{m_1,m'_r} \\ x_{m_2,m'_1} & x_{m_2,m'_2} & \cdots & x_{m_2,m'_r} \\ \vdots & \vdots & & \vdots \\ x_{m_r,m'_1} & x_{m_r,m'_2} & \cdots & x_{m_r,m'_r} \end{bmatrix}.$$

Letting $y = x_{M,M'}$, we see that for every permutation w in S_r, the monomial

$$y_{1,w(1)} \cdots y_{r,w(r)} = x_{m_1,m'_{w(1)}} \cdots x_{m_n,m'_{w(r)}} \tag{2.1}$$

belongs to $\mathcal{A}_r(M, M')$. We obtain the polynomial in the preceding paragraph from the matrix $y = x_{1223,1113}$ as $y_{1,1}y_{2,2}y_{3,3}y_{4,4} - y_{1,1}y_{2,2}y_{3,4}y_{4,3}$.

The multigrading is also closely related to parabolic subgroups of S_r as follows. Associate to M a subset $\iota(M)$ of the generators $\{s_1, \ldots, s_{r-1}\}$ of S_r by

$$\iota(M) = \{s_j \,|\, m_j = m_{j+1}\}.$$

Let $I = \iota(M)$ and $J = \iota(M')$ be the subsets of generators of S_r corresponding to multisets M, M'. Letting the parabolic subgroups W_I and W_J act by left and right multiplication on all $r \times r$ matrices (restricting the defining representation of S_r to the parabolic subgroups), we see that $x_{M,M'}$ is fixed by this action.

The dual canonical basis of $\mathbb{C}[x_{1,1}, \ldots, x_{n,n}]$ consists of homogeneous elements with respect to the multigrading above. Du gives a formula for the elements of this basis in terms of the following polynomials $\widetilde{Q}_{u,w}(q)$ which are alternating sums of (inverse) Kazhdan-Lusztig polynomials,

$$\widetilde{Q}_{u,w}(q) = \sum_{\substack{v \in W_I w W_J \\ u \leq v \leq w}} (-1)^{\ell(w) - \ell(v)} P_{w_0 v, w_0 u}(q),$$

where u and w are maximal representatives of cosets in $W_I \backslash W / W_J$, and $\leq$ is the Bruhat order on S_r. These are generalizations of Deodhar's q-parabolic Kazhdan-Lusztig polynomials [**Deo91**], for when $I = \emptyset$ we have

$$\widetilde{Q}_{u,w}(q) = \widetilde{P}^J_{w_0 w w'_0, w_0 u w'_0}(q),$$

where w_0 and w'_0 are the longest elements of W and W_J, respectively.

We will express the dual canonical basis in terms of Kazhdan-Lusztig immanants $\{\mathrm{Imm}_u(x) \,|\, u \in S_n\}$ introduced in [**RS05a**],

$$\mathrm{Imm}_u(x) = \sum_{w \geq u} (-1)^{\ell(w)-\ell(u)} P_{w_0 w, w_0 u}(1) x_{1,w(1)} \cdots x_{n,w(n)},$$

and in terms of generalized submatrices as defined above.

THEOREM 2.1. *Let M, M' be two r-element multisets of $[n]$. The nonzero polynomials in the set $\{\mathrm{Imm}_v(x_{M,M'}) \,|\, v \in S_r\}$ are the dual canonical basis of $\mathcal{A}_r(M, M')$. In particular, the permutations v corresponding to nonzero polynomials are maximal length representatives of double cosets in $W_{\iota(M)} \backslash W / W_{\iota(M')}$.*

PROOF. Let $I = \iota(M)$, $J = \iota(M')$. By [**Du92**, Lem. 2.2], the canonical basis elements of $\mathcal{A}_r(M, M')$ are in bijective correspondence with cosets in $W_{\iota(M)} \backslash W / W_{\iota(M')}$, and each has the form

$$Z_u = \sum_{z \geq u} (-1)^{\ell(z')-\ell(u')} \widetilde{Q}_{u',z'}(1) x_{1,1}^{\alpha(z,1,1)} \cdots x_{i,j}^{\alpha(z,i,j)} \cdots x_{n,n}^{\alpha(z,n,n)},$$

where u, z are minimal representatives of double cosets in $W_I \backslash W / W_J$, u', z' are the respective maximal coset representatives, and

$$\alpha(z, i, j) = |\{z(k) \,|\, m_k = i\} \cap \{k \,|\, m'_k = j\}|.$$

It is straightforward to show that $u \leq z$ if and only if $u' \leq z'$ for any pairs (u, z) and (u', z') of minimal coset representatives and corresponding maximal coset representatives. (See [**HS05**] and references listed there.) We may therefore rewrite Du's description by summing over only maximal coset representatives,

$$Z_u = \sum_{z' \geq u'} (-1)^{\ell(z')-\ell(u')} \widetilde{Q}_{u',z'}(1) x_{1,1}^{\alpha(z,1,1)} x_{1,2}^{\alpha(z,i,j)} \cdots x_{n,n}^{\alpha(z,n,n)}.$$

Let $y = x_{M,M'}$. Then for any function $f : S_r \to \mathbb{C}$ we have

$$\mathrm{Imm}_f(y) = \sum_{w \in S_r} f(w) y_{1,w(1)} \cdots y_{n,w(n)}. \tag{2.2}$$

Since each permutation u in the double coset $W_I w W_J$ satisfies

$$y_{1,u(1)} \cdots y_{n,u(n)} = y_{1,w(1)} \cdots y_{n,w(n)},$$

we may sum over these double cosets,

$$\mathrm{Imm}_f(y) = \sum_{D \in W_I \backslash W / W_J} \left(\sum_{v \in D} f(v) \right) y_{1,w(1)} \cdots y_{n,w(n)},$$

where w is any representative of the double coset D. Note that $y_{i,w(i)} = x_{j,\ell}$ if $m_i = j$ and $m'_{w(i)} = \ell$. Thus the exponent of $x_{j,\ell}$ in $y_{1,w(1)} \cdots y_{n,w(n)}$ is equal to the number of indices i which satisfy

$$m_i = j, \quad m'_{w(i)} = \ell.$$

Since this is just $\alpha(w, i, j)$, we have

$$y_{1,w(1)} \cdots y_{n,w(n)} = x_{1,1}^{\alpha(w,1,1)} \cdots x_{n,n}^{\alpha(w,n,n)}.$$

Now consider the function $f_u : v \mapsto (-1)^{\ell(v)-\ell(u)} P_{w_0 v, w_0 u}(1)$ and the corresponding immanant of y, $\mathrm{Imm}_u(y) = \mathrm{Imm}_{f_u}(y)$. If u is not a maximal representative of a double coset in $W_I \backslash W / W_J$, then by [**Cur85**, Thm. 1.2] we have $su > u$ for some transposition s in I, or we have $us > u$ for some transposition s in J. By [**RS05a**, Cor. 6.4] either of these conditions implies that $\mathrm{Imm}_u(y) = 0$. Suppose therefore that u' is a maximal coset representative. Then by (2.2) we have

$$\mathrm{Imm}_{u'}(y) = \sum_{D \in W_I \backslash W / W_J} \left(\sum_{\substack{v \in D \\ v \geq u'}} (-1)^{\ell(v)-\ell(u')} P_{w_0 v, w_0 u'}(1) \right) x_{1,1}^{\alpha(w',1,1)} \cdots x_{n,n}^{\alpha(w',n,n)},$$

where w' is the maximal representative of D, and we include the inequality $v \geq u'$ because the number $P_{w_0v,w_0u'}(1)$ is zero otherwise. For each coset D and its maximal representative w', the inner sum is equal to

$$(-1)^{\ell(w')-\ell(u')} \sum_{\substack{v \in W_I w' W_J \\ u' \leq v \leq w'}} (-1)^{\ell(w')-\ell(v)} P_{w_0v,w_0u'}(1) = (-1)^{\ell(w')-\ell(u')} \widetilde{Q}_{u',w'}(1),$$

and we have

$$\mathrm{Imm}_{u'}(y) = \sum_{D \in W_I \backslash W / W_J} (-1)^{\ell(w')-\ell(u')} \widetilde{Q}_{u',w'}(1) x_{1,1}^{\alpha(w,1,1)} \cdots x_{n,n}^{\alpha(w,n,n)}.$$

Note that for any double coset whose maximal representative w' satisfies $u' \leq w'$, we have $\widetilde{Q}_{u',w'}(1) = 0$ and the contribution to the sum is zero. The sum therefore may be taken over double cosets D whose maximal element w' satisfies $w' \geq u'$, and we have

$$Z_u = \mathrm{Imm}_{u'}(x_{M,M'}),$$

as desired. $\square$

Quantizing the Kazhdan-Lusztig immanants by

$$\mathrm{Imm}_v(x;q) = \sum_{w \geq v} (-q^{-1/2})^{\ell(w)-\ell(v)} \widetilde{Q}_{v,w}(q) x_{1,w(1)} \cdots x_{n,w(n)},$$

one constructs the quantum dual canonical basis of $\mathcal{A}_r(M,M')$ by taking all of the polynomials $(q^{1/2})^{\ell(w_0^J)-\ell(w_0^I)}\mathrm{Imm}_v(x_{M,M'};q$ where $I = \iota(M)$, $J = \iota(M')$, v is a maximal length coset representative in $W_I \backslash W / W_J$, and w_0^I, w_0^J are the maximal length elements of W_I, W_J. Details will appear in [**Ska05**]. (See [**Bru05**], [**Du92**] for other descriptions of this basis.)

Letting $\mathbf{B}$ be the dual canonical basis of $\mathbb{C}[x_{1,1}, \dots, x_{n,n}]$, we have the following formulas for the dual canonical bases of the coordinate rings of $GL_n(\mathbb{C})$ and $SL_n(\mathbb{C})$. The dual canonical basis of

$$\mathcal{O}(GL(n,\mathbb{C})) \cong \mathbb{C}[x_{1,1}, \dots, x_{n,n}, t]/(\det(x)t - 1)$$

is obtained by dividing elements of $\mathbf{B}$ by powers of the determinant,

$$\cup_{r \geq 0} \cup_{(M,M') \in \mathcal{M}(n,r)} \{\mathrm{Imm}_w(x) \det(x)^{-k} \,|\, k \geq 0; w \text{ maximal in } W_{\iota M} \backslash S_r / W_{\iota(M')}\}.$$

The dual canonical basis of

$$\mathcal{O}(SL(n,\mathbb{C})) \cong \mathbb{C}[x_{1,1}, \dots, x_{n,n}]/(\det(x) - 1)$$

is obtained by projecting $\mathbb{C}[x_{1,1}, \dots, x_{n,n}]$ or $\mathcal{O}(GL(n,\mathbb{C}))$ onto $\mathcal{O}(SL(n,\mathbb{C}))$.

3. A Factorization Theorem

While each nonnegative linear combination of dual canonical basis elements is a totally nonnegative polynomial, the converse of this statement is false. Intimately related to this fact is the vector space duality between the component $\mathcal{A}_n([n],[n])$ of the polynomial ring $\mathbb{C}[x_{1,1}, \dots, x_{n,n}]$ and and the group algebra $\mathbb{C}[S_n]$, defined by

$$\langle x_{1,u(1)} \cdots x_{n,u(n)}, T_v \rangle = \delta_{u,v}.$$

In particular, Kazhdan and Lusztig [**KL79**] defined a basis $\{C'_w(q) \,|\, w \in S_n\}$ of the Hecke algebra $H_n(q)$ by

$$C'_v(q) = q^{-1/2} \sum_{u \leq v} P_{u,v}(q) T_u,$$

where $\{P_{u,v}(q) \,|\, u, v \in S_n\}$ are certain polynomials for which no elementary formula is known. Dual to this basis is the basis of Kazhdan-Lusztig immanants,

$$\langle \mathrm{Imm}_u(x), C'_v(1) \rangle = \delta_{u,v}.$$

Since no elementary formula is known for the Kazhdan-Lusztig polynomials, it is not surprising that we also have no elementary formula for the Kazhdan-Lusztig basis of the Hecke algebra or for the Kazhdan-Lusztig immanants. Nevertheless, we can deduce certain properties of the Kazhdan-Lusztig immanants by studying Kazhdan-Lusztig basis elements which have a rather simple form and others which factor as products of these. The basis elements we shall consider correspond to permutations whose one-line notations avoid certain patterns. The factorization of these basis elements closely resembles the factorization of the corresponding permutations.

Given a word $u = u_1 \cdots u_k$ on a totally ordered alphabet and a permutation v in S_k with one-line notation $v_1 \cdots v_k$, we will say that u *matches the pattern* v if the letters of u appear in the same relative order as those of v. We will also say that u_1 *matches the* v_1, u_2 matches the v_2, etc. For example, a word $u_1u_2u_3$ with $u_2 < u_3 < u_1$ matches the pattern 312, with u_1 matching the 3, u_2 matching the 1, and u_3 matching the 2.

We will say that a permutation w in S_n *avoids the pattern* v if no subword $w_{i_1} \cdots w_{i_k}$ with $i_1 < \cdots < i_k$ matches the pattern v. We will also call such a permutation v*-avoiding.* In particular, we will be interested in permutations which avoid the patterns 3412 and 4231. Note that a permutation w avoids these patterns if and only if w^{-1} does, since the patterns are involutions. In particular, corresponding to each adjacent transposition s_i is the basis element $C'_{s_i}(q) = q^{-1/2}(T_e + T_{s_i})$, and we have the following factorization result of Billey and Warrington [**BW01**].

THEOREM 3.1. *Let $s_{i_1} \cdots s_{i_\ell}$ be a reduced expression for w. Then we have*

$$C'_w(q) = C'_{s_{i_1}}(q) \cdots C'_{s_{i_\ell}}(q)$$

if and only if the one-line notation for w avoids the patterns 321*,* 56781234*,* 46781235*,* 56718234*,* 46718235*.*

Other permutations w for which $C'_w(q)$ has a particularly nice form are known as *reversals*. Write $s_{[i,j]}$ for the permutation which fixes indices $1, \dots, i-1, j+1, \dots, n$ and reverses the remaining indices. Corresponding to reversals are the Kazhdan-Lusztig basis elements

$$C'_{s_{[i,j]}}(q) = (q^{-1/2})^{\binom{j-i+1}{2}} \sum_{v \leq s_{[i,j]}} T_v$$

We will show in Theorem 3.3 that permutations which avoid the patterns 3412 and 4231 factor as products of these basis elements.

To begin, we define the map $\oplus : S_n \times S_m \to S_{n+m}$, as is somewhat customary, by

$$s_{i_1} \cdots s_{i_\ell} \oplus s_{j_1} \cdots s_{j_k} = s_{i_1} \cdots s_{i_\ell} s_{j_1+n} \cdots s_{j_k+n}.$$

OBSERVATION 3.2. *If u and v are* 3412*-avoiding,* 4231*-avoiding permutations in S_m and S_n, then $u \oplus v$ is a* 3412*-avoiding,* 4231*-avoiding permutation in S_{m+n}.*

We will say that a permutation w has an *irreducible zig-zag factorization* if there exist a positive integer r, a sequence of nonnegative integers

$$j_1, \dots, j_r, k_1, \dots, k_r,$$

all odd except possibly for j_1 and k_r which may also be zero, and a sequence of intervals

$$(3.1) \quad \begin{aligned} &a_0, b_{1,1}, \dots, b_{1,j_1}, a_1, c_{1,1}, \dots, c_{1,k_1}, d_1, \\ &\qquad \dots, b_{i,1}, \dots, b_{i,j_i}, a_i, c_{i,1}, \dots, c_{i,k_i}, d_i, \dots, \\ &\qquad\qquad b_{r,1}, \dots, b_{r,j_r}, a_r, c_{r,1}, \dots, c_{r,k_r}, d_r, \end{aligned}$$

all nonempty except possibly for a_0, a_r, such that w is equal to the product of the reversals on these intervals in the order listed,

$$w = s_{a_0} \cdots s_{d_r},$$

and the endpoints of the intervals, which we denote by

$$a_i = [\lambda(a_i), \rho(a_i)], \quad b_{i,j} = [\lambda(b_{i,j}), \rho(b_{i,j})], \quad c_{i,k} = [\lambda(c_{i,k}), \rho(c_{i,k})], \quad d_i = [\lambda(d_i), \rho(d_i)],$$

satisfy the following conditions.

(1) $j_1 = 0$ if and only if $a_0 = s_\emptyset$.
(2) $k_r = 0$ if and only if $a_r = s_\emptyset$.
(3) For each i satisfying $a_{i-1} \neq s_\emptyset$ we have

$$\begin{aligned} \lambda(a_{i-1}) < \lambda(b_{i,1}) &= \lambda(b_{i,2}) < \lambda(b_{i,3}) = \cdots < \lambda(b_{i,j_i}) = \lambda(d_i), \\ \rho(a_{i-1}) = \rho(b_{i,1}) &< \rho(b_{i,2}) = \rho(b_{i,3}) < \cdots = \rho(b_{i,j_i}) < \rho(d_i). \end{aligned}$$

(4) For each i satisfying $a_i \neq s_\emptyset$ we have

(5) and

$$\lambda(a_i) = \lambda(c_{i,1}) > \lambda(c_{i,2}) = \lambda(c_{i,3}) > \cdots = \lambda(c_{i,k_i}) > \lambda(d_i),$$
$$a'_i > \rho(c_{i,1}) = \rho(c_{i,2}) > \rho(c_{i,3}) = \cdots > \rho(c_{i,k_i}) = \rho(d_i),$$

(6) For $i = 1, \ldots, r$ we have

$$\rho(b_{i,j_i}) < \lambda(c_{i,k_i}).$$

(7) For $i = 1, \ldots, r-1$ we have

$$\rho(c_{i,1}) < \rho(b_{i+1,1}).$$

Note that the intervals

$$\{a_i \,|\, 0 \leq i \leq r\} \cup \{b_{i,j} \,|\, 1 \leq i \leq r, j \text{ even }\} \cup \{c_{i,j} \,|\, 1 \leq i \leq r, j \text{ even }\} \cup \{d_i \,|\, 1 \leq i \leq r\}$$

have length at least two when they are nonempty.

Note that the lexicographic order on the set (3.1) of intervals (padded with zeros at the end) is

$$a_0, b_{1,1}, \ldots, b_{1,j_1}, d_1, c_{1,k_1}, \ldots, c_{1,1}, a_1, \ldots, b_{i,1}, \ldots, b_{i,j_i}, d_i, c_{i,k_i}, \ldots, c_{i,k_1}, a_i, \ldots, b_{r,1}, \ldots, b_{r,j_r}, d_r, c_{r,1}, \ldots, c_{r,k_r}, a_r, \tag{3.2}$$

If in this factorization we have $r = 1$ and $k_1 = 0$, then the only intervals to appear are

$$a_0, b_{1,1}, \ldots, b_{1,j_1}, d_1,$$

(with $a_0, b_{1,1}, \ldots, b_{1,j_1}$ not appearing if $j_1 = 0$) and we will say that the irreducible zig-zag factorization is *lexicographically increasing*. Given a reversal factorization $W_1 \cdots W_p$ in which each W_i is an irreducible zigzag factorization and the intervals don't overlap, we will call this a *zig-zag factorization*. Note that each permutation u possessing an irreducible zig-zag factorization decomposes as

$$u = e \oplus \cdots \oplus e \oplus v \oplus e \oplus \cdots \oplus e,$$

where e is the identity element of S_1, and v possesses the same irreducible zig-zag factorization as u.

PROPOSITION 3.1. *A permutation avoids the patterns* 3412*,* 4231 *if and only if it has a zig-zag factorization.*

PROOF. Omitted. □

Two examples of 3412-avoiding, 4231-avoiding permutations and zig-zag factorizations are

$$654213 = s_{[1,5]}s_{[3,5]}s_{[3,6]}, \qquad 621354 = s_{[1,3]}s_{[3,4]}s_{[4,6]}.$$

PROPOSITION 3.2. *Let $s_{I_1} \cdots s_{I_p}$ be a zig-zag factorization of $w \in S_n$, let $(t_1, \ldots, t_p)$ be a subexpression of this factorization, and define the permutation $u = t_1 \cdots t_p$. Then $u \leq w$ in the Bruhat order.*

PROOF. Omitted. □

The above factorization results for permutations translate into the following factorization result for Kazhdan-Lusztig basis elements. Given a sequence of intervals $I = (I_1, \ldots, I_r)$, define the $H_n(q)$ algebra element

$$\Phi(I_1, \ldots, I_r; q) = C'_{s_{I_1}}(q) \cdots C'_{s_{I_r}}(q).$$

THEOREM 3.3. *Let w avoid the patterns* 3412 *and* 4231 *and have zig-zag factoriztion (3.1), define the sequence of intervals*

$$I = (a_0, b_{1,2}, b_{1,4}, \ldots, b_{1,j_1-1}, a_1, c_{1,2}, c_{1,4}, \ldots, c_{1,k_1-1}, d_1, \ldots, b_{i,2}, \ldots, b_{i,j_i-1}, a_i, c_{i,2}, \ldots, c_{i,k_i-1}, d_i, \ldots, b_{r,2}, \ldots, b_{r,j_r-1}, a_r, c_{r,2}, \ldots, c_{r,k_r-1}, d_r) \tag{3.3}$$

and define the number

$$\gamma = \prod_{i=1}^{r} \prod_{\substack{j=1 \\ j \text{ odd}}} |b_{i,j}|! \prod_{\substack{k=1 \\ k \text{ odd}}} |c_{i,k}|!.$$

Then the Kazhdan-Lusztig basis element $C'_w(1)$ factors as

$$C'_w(1) = \tfrac{1}{\gamma}\Phi(I;1),$$

and the Kazhdan-Lusztig basis element $C'_w(q)$ factors as

$$C'_w(q) = q^{-\ell(w)/2}C'_w(1),$$

or equivalently

$$C'_w(q) = \tfrac{1}{\gamma}q^{\delta/2}\Phi(I;q),$$

where

$$\delta = \sum_{i=0}^{r}\binom{|a_i|}{2} + \sum_{i=0}^{r}\binom{|d_i|}{2} + \sum_{\substack{j=1 \\ j\ even}}\binom{|b_{i,j}|}{2} + \sum_{\substack{k=1 \\ k\ even}}\binom{|c_{i,k}|}{2} - \ell(w).$$

Proof. Omitted. □

Corresponding to previoius examples of $3412-avoiding$,$4231-avoiding$ permutations and zig-zag factorizations are the factorizations of Kazhdan-Lusztig basis elements,

$$654213 = s_{[1,5]}s_{[3,5]}s_{[3,6]}, \qquad 621354 = s_{[1,3]}s_{[3,4]}s_{[4,6]},$$

$$C'_{654213}(q) = C'_{s_{[1,5]}}(q)\frac{q^{3/2}}{3_q!}C'_{s[3,6]}(q), \qquad C'_{621354}(q) = C'_{s_{[1,3]}}(q)C'_{s[3,4]}(q)C'_{s[4,6]}(q).$$

4. The dual cone of total nonnegativity

In [**RS05a**, Sec. 7], we have cones of TNN and SNN elements of $\mathrm{span}_{\mathbb{C}}\{x_{1,w(1)}\cdots x_{n,w(n)} \,|\, w \in S_n\}$ were defined. Virtually all of the known TNN and SNN polynomials belong to these cones. (See [**RS05a**, Sec. 1].) Generalizing these definitions a bit, we will define the following cones of functions on $n \times n$ matrices. Define the *dual canonical cone*, the *dual cone of total nonnegativity*, and the *dual cone of Schur nonnegativity*, which we will denote by $\check{C}_B$, $\check{C}_{\mathrm{TNN}}$, and $\check{C}_{\mathrm{SNN}}$, respectively, to be the cones whose extreme rays are homogeneous elements of $\mathbb{C}[x_{1,1},\ldots,x_{n,n}]$ belonging to B, having the TNN property, and having the SNN property, respectively. Our use of the term *dual* refers to the relationship of this point of view to that of Stembridge [**Ste92**], who define the *cone of total nonnegativity* to be the smallest cone in $\mathbb{C}[S_n]$ containing all of elements of the form $\sum_{w\in S_n} a_{1,w(1)}\cdots a_{n,w(n)}$, where $A = (a_{i,j})$ is a totally nonnegative matrix.

Using this terminology, we have the following.

Corollary 4.1. *The dual canonical cone is contained in the intersection of the dual cones of total nonnegativity and Schur nonnegativity.*

Proof. The main results of [**RS05a**] and [**RS05b**] show that Kazhdan-Lusztig immanants of generalized submatrices of $x = (x_{i,j})_{i,j=1}^n$ are TNN and SNN. Since the cone generated by these functions is $\check{C}_B$, we have the desired result. □

The author and A. Zelevinsky have verified that the containment of $\check{C}_B$ in $\check{C}_{\mathrm{TNN}}$ is strict. In particular, the homogeneous element

$$\mathrm{Imm}_{3214}(x) + \mathrm{Imm}_{1432}(x) - \mathrm{Imm}_{3412}(x) \tag{4.1}$$

belongs to $\check{C}_{\mathrm{TNN}} \smallsetminus \check{C}_B$. Moreover we have used cluster algebras and Maple to show that this element is equal to a subtraction-free rational expression in matrix minors. Thus the cone of functions which have this subtraction-free rational function (SFR) property must also properly contain $\check{C}_B$. On the other hand, the element (4.1) does not belong to $\check{C}_{\mathrm{SNN}}$, for its evaluation on the Jacobi-Trudi matrix H_{2222} expands in the Schur basis as

$$2s_{62} + 2s_{53} + 2s_{521} - s_{44} + 2s_{431} + 2s_{422}.$$

Thus $\check{C}_B$ and $\check{C}_{\mathrm{SNN}}$ are not known to be different. Let us examine the difference $\check{C}_{\mathrm{TNN}} \smallsetminus \check{C}_B$ more closely.

THEOREM 4.2. *Let H be the planar network corresponding to a zig-zag factorization of a* 3412*-avoiding,* 4231*-avoiding permutation w in S_n, and let A_w be the path matrix of H. Then there exists a nonnegative integer c such that we have*

$$\mathrm{Imm}_v(A_w) = \begin{cases} c & \text{if } v = w, \\ 0 & \text{otherwise.} \end{cases}$$

PROOF. Omitted. □

The existence of the matrices specified by the previous theorem allows us to compare the dual canonical cone with the dual cone of total nonnegativity as follows.

THEOREM 4.3. *Let $\mathrm{Imm}_f(x)$ be totally nonnegative and let its expansion in terms of Kazhdan-Lusztig immanants be given by*

$$\mathrm{Imm}_f(x) = \sum_{w \in S_n} d_w \mathrm{Imm}_w(x).$$

Then c_u is nonnegative for each 3412*-avoiding,* 4231*-avoiding permutation u.*

PROOF. Let u be a 3412-avoiding, 4231-avoiding permutation in S_n, and suppose that d_u is negative. Let G_u be the planar network corresponding to the reversal factorization of u, and let A_u be the path matrix of G_u. Then we have

$$\mathrm{Imm}_f(A_u) = cd_u < 0,$$

contradicting the total nonnegativity of $\mathrm{Imm}_f(x)$. □

Theorem 4.3 suggests several problems. Recalling Lakshmibai and Sandhya's result [**LS90**] that a permutation w's avoidance of the patterns 3412 and 4231 is equivalent to smoothness of the Schubert variety Γ_w, we have the following.

PROBLEM 4.4. Find an intuitive reason for the connection between total nonnegativity, the dual canonical basis, and smoothness of Schubert varieties.

It would also be interesting to understand precisely how the cones mentioned earlier are related.

PROBLEM 4.5. Find the extremal rays of $\check{C}_{\mathrm{TNN}}$, $\check{C}_{\mathrm{SNN}}$, and the cone of SFR functions in $\mathbb{C}[x_{1,1}, \dots, x_{n,n}]$, or describe the containments satisfied by these cones.

Since the factorizations given in Theorems 3.1 and 3.3 agree on permutations which avoid all seven of the forbidden patterns, the author believes that there is a simple generalization of the two results. It would be interesting to understand in even greater generality which elements of the Kazhdan-Lusztig basis factor as products of others.

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