On the characteristic map of finite unitary groups

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Abstract. In his classic book on symmetric functions, Macdonald describes a remarkable result by Green relating the character theory of the finite general linear group to transition matrices between bases of symmetric functions. This connection allows us to analyze the representation theory of the general linear group via symmetric group combinatorics. Using the work of Ennola, Kawanaka, Lusztig and Srinivasan, this paper describes the analogous setting for the finite unitary group. In particular, we explain the connection between Deligne-Lusztig theory and Ennola’s efforts to generalize Green’s work, and from this we deduce various representation theoretic results. Applications include finding certain sums of character degrees, and a model of Deligne-Lusztig type for the finite unitary group, which parallels results of Klyachko and Inglis and Saxl for the finite general linear group.

1. Introduction

In his seminal work [7], Green described a remarkable connection between the class functions of the finite general linear group \( \text{GL}(n, \mathbb{F}_q) \) and a generalization of the ring of symmetric functions of the symmetric group \( S_n \). In particular, Green defines a map, called the characteristic map, that takes irreducible characters to Schur-like symmetric functions, and recovers the character table of \( \text{GL}(n, \mathbb{F}_q) \) as the transition matrix between these Schur functions and Hall-Littlewood polynomials [14, Chapter IV]. Thus, we can use the combinatorics of the symmetric group \( S_n \) to understand the representation theory of \( \text{GL}(n, \mathbb{F}_q) \). Some of the implications of this approach include an indexing of irreducible characters and conjugacy classes of \( \text{GL}(n, \mathbb{F}_q) \) by multi-partitions and a formula for the degrees of the irreducible characters in terms of these partitions.

This paper describes the parallel story for the finite unitary group \( \text{U}(n, \mathbb{F}_{q^2}) \) by collecting known results for this group and examining some applications of the unitary characteristic map. Inspired by Green, Ennola [4, 5] used results of Wall [16] to construct the appropriate ring of symmetric functions and characteristic map. Ennola was able to prove that the analogous Schur-like functions correspond to an orthonormal basis for the class functions, and conjectured that they corresponded to the irreducible characters. He theorized that the representation theory of \( \text{U}(n, \mathbb{F}_{q^2}) \) should be deduced from the representation theory of \( \text{GL}(n, \mathbb{F}_q) \) by...
substituting $-q$ for every occurrence of $q$. The general phenomenon of obtaining a polynomial invariant in $q$ for $U(n, \mathbb{F}_q^2)$ by this substitution has come to be known as “Ennola duality”.


This paper begins by describing some of the combinatorics and group theory associated with the finite unitary groups. Section 2 defines the finite unitary groups, outlines the combinatorics of multi-partitions, and gives a description of some of the key subgroups. Section 2.4 analyzes the conjugacy classes of $U(n, \mathbb{F}_q^2)$ and gives a description of the conjugacy classes of $U(2, \mathbb{F}_q)$ in terms of Deligne-Lusztig characters. Kawanaka [7] and Inglis and Saxl [12] found by Klyachko [10] and Henderson [8] to adapt a model for the general linear group, found by Klyachko [12] and Inglis and Saxl [10], to the finite unitary group. The main results are

I. (Theorem 3.2) The Deligne-Lusztig characters correspond to power-sum symmetric functions via the characteristic map of Ennola.

II. (Corollary 3.2) The multiplicative structure that Ennola defined on $C$ is Deligne-Lusztig induction.

Section 4 computes the degrees of the irreducible characters, and uses this result to evaluate various sums of character degrees (see [14, IV.6, Example 5] for the GL($n, \mathbb{F}_q$) analogue of this method). The main results are

III. (Theorem 4.1) An irreducible $\chi^\lambda$ character of $U(m, \mathbb{F}_q^2)$ corresponds to

$$(-1)^{[m/2]+n(\lambda)}s_\lambda \text{ and } \chi^\lambda(1) = q^{n(\lambda')} \prod_{\square \in \lambda} \frac{(q^h(\square) - (-1)^{h(\square)})}{(q^1 - (-1)^1)},$$

where $s_\lambda$ is a Schur-like function, and both $n(\lambda)$ and $h(\square)$ are combinatorial statistics on the multi-partition $\lambda$.

IV. (Corollary 4.2) If $\mathcal{P}_n^{\Theta}$ indexes the irreducible characters $\chi^\lambda$ of $U(n, \mathbb{F}_q)$, then

$$\sum_{\lambda \in \mathcal{P}_n^{\Theta}} \chi^\lambda(1) = \{g \in U(n, \mathbb{F}_q^2) \mid g \text{ symmetric}\}.$$

V. (Theorem 4.3) We give a subset $\mathcal{X} \subseteq \mathcal{P}_n^{\Theta}$ such that

$$\sum_{\lambda \in \mathcal{X}} \chi^\lambda(1) = (q+1)q^2(q^3+1) \cdots q^{2n-2}(q^{2n-1}+1) = \frac{|U(2n, \mathbb{F}_q)|}{|Sp(2n, \mathbb{F}_q)|}.$$

Section 5 uses results by Ohmori [15] and Henderson [8] to adapt a model for the general linear group, found by Klyachko [12] and Inglis and Saxl [10], to the finite unitary group. The main result is

VI. (Theorem 5.2) Let $U_m = U(m, \mathbb{F}_q^2)$, where $q$ is odd, and let $\Gamma_m$ be the Gelfand-Graev character of $U_m$. $1$ be the trivial character of the finite symplectic group $Sp_{2r} = Sp(2r, \mathbb{F}_q)$, and $R^u_{ \Gamma_m}$ be the Deligne-Lusztig induction functor. Then

$$\sum_{0 \leq 2r \leq m} R^u_{\Gamma_m \otimes \text{Ind}^{Sp_{2r}}_{Sp_{2r}}(1)} = \sum_{\lambda \in \mathcal{P}_m^{\Theta}} \chi^\lambda.$$

That is, in the theorem of Klyachko, one may replace parabolic induction by Deligne-Lusztig induction to obtain a theorem for the unitary group.
These results give considerable combinatorial control over the representation theory of the finite unitary group, and there are certainly more applications to these results than what we present in this paper. Furthermore, this characteristic map gives some insight as to how a characteristic map might look in general type, using the invariant rings of other Weyl groups.

2. Preliminaries

2.1. The unitary group and its underlying field. Let \( K = \overline{\mathbb{F}_q} \) be the algebraic closure of the finite field with \( q \) elements and let \( K_m = \mathbb{F}_{q^m} \) denote the finite subfield with \( q^m \) elements. Let \( \text{GL}(n, K) \) denote the general linear group over \( K \), and define Frobenius maps

\[
F : \text{GL}(n, K) \rightarrow \text{GL}(n, K) \quad \text{and} \quad F' : \text{GL}(n, K) \rightarrow \text{GL}(n, K)
\]

\[
(a_{ij}) \mapsto (a_{ij}^q)^{-1}, \quad \text{and} \quad (a_{ij}) \mapsto (a_{n-j,n-i}^q)^{-1}.
\]

Then the unitary group \( U_n = \text{U}(n, K_2) \) is given by

\[
U_n = \text{GL}(n, K)_F = \{ a \in \text{GL}(n, K) \mid F(a) = a \}
\]

\[
\cong \text{GL}(n, K)_{F'} = \{ a \in \text{GL}(n, K) \mid F'(a) = a \}.
\]

We define the multiplicative groups \( \mathbb{M}_m \) as

\[
\mathbb{M}_m = \text{GL}(1, K)^{F_m} = \{ x \in K \mid x^{q^m - (-1)^m} = 1 \}.
\]

Note that \( \mathbb{M}_m \cong K^* \) only if \( m \) is even. We identify \( K^* \) with the inverse limit \( \lim_{\rightarrow} \mathbb{M}_m \) with respect to the norm maps

\[
N_{mr} : \mathbb{M}_m \rightarrow \mathbb{M}_r, \quad x \mapsto x^{-q} \cdots x^{-q^{m/r-1}}, \quad \text{where} \ m, r \in \mathbb{Z}_{\geq 1} \text{ with } r \mid m.
\]

If \( \mathbb{M}^*_m \) is the group of characters of \( \mathbb{M}_m \), then the direct limit \( K^* = \lim_{\rightarrow} \mathbb{M}^*_m \) gives the group of characters of \( K^* \).

Let

\[
\Theta = \{ F\text{-orbits of } K^* \}.
\]

A polynomial \( f(t) \in K_2[t] \) is \( F \)-irreducible if there exists an \( F \)-orbit \( \{ x, x^{-q}, \ldots, x^{(-q)^d} \} \) of \( K^* \) such that \( f(t) = (t-x)(t-x^{-q}) \cdots (t-x^{(-q)^d}) \).

Let

\[
(2.4) \quad \Phi = \{ f \in K_2[t] \mid f \text{ is } F\text{-irreducible} \} \overset{1-1}{\rightarrow} \{ F\text{-orbits of } K^* \}.
\]

2.2. Combinatorics of \( \Phi \)-partitions and \( \Theta \)-partitions. Fix an ordering of \( \Phi \) and \( \Theta \), and let

\[
\mathcal{P} = \{ \text{partitions} \} \quad \text{and} \quad \mathcal{P}_n = \{ \nu \in \mathcal{P} \mid |\nu| = n \}.
\]

Let \( \mathcal{X} \) be either \( \Phi \) or \( \Theta \). An \( \mathcal{X} \)-partition \( \nu = (\nu(x_1), \nu(x_2), \ldots) \) is a sequence of partitions indexed by \( \mathcal{X} \).

The size of an \( \mathcal{X} \)-partition \( \nu \) is

\[
(2.5) \quad ||\nu|| = \sum_{x \in \mathcal{X}} |x||\nu(x)|, \quad \text{where} \quad |x| = \begin{cases} |x| & \text{if } \mathcal{X} = \Theta, \\ d(x) & \text{if } \mathcal{X} = \Phi, \end{cases}
\]

\(|x|\) is the size of the orbit \( x \in \Theta \), and \( d(x) \) is the degree of the polynomial \( x \in \Phi \). Let

\[
(2.6) \quad \mathcal{P}^\mathcal{X}_n = \{ \mathcal{X}\text{-partitions } \nu \mid ||\nu|| = n \}, \quad \text{and} \quad \mathcal{P}^\mathcal{X} = \bigcup_{n=1}^{\infty} \mathcal{P}^\mathcal{X}_n.
\]

For \( \nu \in \mathcal{P}^\mathcal{X} \), let

\[
(2.7) \quad n(\nu) = \sum_{x \in \mathcal{X}} |x|n(\nu(x)), \quad \text{where} \quad n(\nu) = \sum_{i=1}^{\ell(\nu)} (i-1)\nu_i.
\]

The conjugate \( \nu' \) of \( \nu \) is the \( \mathcal{X} \)-partition \( \nu' = (\nu(x_1)', \nu(x_2)', \ldots) \), where \( \nu' \) is the usual conjugate partition for \( \nu \in \mathcal{P} \).

The semisimple part \( \nu_s \) of \( \nu = (\nu(x_1), \nu(x_2), \ldots) \in \mathcal{P}^\mathcal{X}_n \) is

\[
(2.8) \quad \nu_s = ((1^{\nu(x_1)}), (1^{\nu(x_2)}), \ldots) \in \mathcal{P}^\mathcal{X}_n,
\]
and the unipotent part \( \nu_u \) of \( \nu \in P_n^X \) is given by
\[
\nu_u(1) \quad \text{has parts } \{ |x|\nu(x) | \ x \in X, i = 1, \ldots, \ell(\nu(x)) \}
\]
where
\[
1 = \begin{cases} \{1\} & \text{if } X = \Theta, \\ t - 1 & \text{if } X = \Phi, \end{cases}
\]
1 is the trivial character in \( \mathbb{K}^* \), and \( \nu_u(x) = \emptyset \) for \( x \neq 1 \).

### 2.3. Levi subgroups and maximal tori

Let \( X \) be either \( \Phi \) or \( \Theta \) as in Section 2.2. For \( \nu \in P_n^X \), let
\[
L_\nu = \bigoplus_{x \in X_\nu} L_\nu(x), \quad \text{where } X_\nu = \{ x \in X | \nu(x) \neq \emptyset \},
\]
and for \( x \in X_\nu \),
\[
L_\nu(x) = \begin{cases} U(|\nu(x)|, \mathbb{K}|_{2|x|}) & \text{if } |x| \text{ is odd,} \\ \text{GL}(\nu(x), \mathbb{K}|_{2|x|}) & \text{if } |x| \text{ is even.} \end{cases}
\]
Then \( L_\nu \) is a Levi subgroup of \( U_n = U(n, \mathbb{K}_2) \) (though not uniquely determined by \( \nu \)). The Weyl group
\[
W_\nu = \bigoplus_{x \in X_\nu} S_{|\nu(x)|},
\]
of \( L_\nu \) has conjugacy classes indexed by \( \mathcal{P}_s^{\nu} = \{ \gamma \in P^X | \gamma_s = \nu_s \} \),
and the size of the conjugacy class \( c_\gamma \) is
\[
|c_\gamma| = \frac{|W_\gamma|}{z_\gamma}, \quad \text{where } z_\gamma = \prod_{x \in X} z_{\gamma(x)} \text{ and } z_{\gamma} = \prod_{i=1}^{\ell(\gamma)} i^{m_i} m_i!,
\]
for \( \gamma = (1^{m_1} 2^{m_2} \cdots) \in \mathcal{P} \).

For every \( \nu = (\nu_1, \nu_2, \ldots, \nu_t) \in P_n \) there exists a maximal torus (unique up to isomorphism) \( T_\nu \) of \( U_n \) such that
\[
T_\nu \cong M_{\nu_1} \times M_{\nu_2} \times \cdots \times M_{\nu_t}.
\]
For every \( \gamma \in \mathcal{P}_s^\mu \), there exists a maximal torus (unique up to isomorphism) \( T_\gamma \subseteq L_\nu \) such that
\[
T_\gamma = \bigoplus_{x \in X_\nu} T_\gamma(x), \quad \text{where } T_\gamma(x) \cong M_{|x| \gamma(x)_1} \times \cdots \times M_{|x| \gamma(x)_\ell}.
\]
Note that as a maximal torus of \( U_n \), the torus \( T_\gamma \cong T_{\gamma}(1) \).

### 2.4. Conjugacy classes and Jordan decomposition

**Proposition 2.1.** The conjugacy classes \( c_\mu \) of \( U_n \) are indexed by \( \mu \in P_n^\Phi \).

For \( r \in \mathbb{Z}_{\geq 0} \), let \( \psi_r(x) = \prod_{i=1}^r (1 - x^i) \).

**Proposition 2.2 (Wall).** Let \( g \in c_\mu \). The order \( a_\mu \) of the centralizer \( g \) in \( U_n \) is
\[
a_\mu = (-1)^{|\mu|} \prod_{f \in \Phi} a_{\mu(f)}((-q)^d(f)), \quad \text{where } a_{\mu(g)}(x) = x^{|\mu| + 2n(\mu)} \prod_{f \in \Phi} \psi_{m_f}(x^{-1}),
\]
for \( \mu = (1^{m_1} 2^{m_2} 3^{m_3} \cdots) \in \mathcal{P} \).

For \( \mu \in P_n^\Phi \), let \( L_\mu \) be as in (2.10). Note that \( |L_\mu| = a_{\mu_s} \).

**Lemma 2.1.** Suppose \( g \in c_\mu \) with Jordan decomposition \( g = su \). Then
(a) \( s \in c_{\mu_s} \) and \( u \in c_{\mu_u} \), where \( \mu_s \) and \( \mu_u \) are as in (2.8) and (2.9),
(b) the centralizer \( C_{U_n}(s) \) of \( s \) in \( U_n \) is isomorphic to \( L_\mu \).
3. The Ennola Conjecture

Let $X = \{X_1, X_2, \ldots\}$ be an infinite set of variables and let $\Lambda(X)$ be the graded $\mathbb{C}$-algebra of symmetric functions in the variables $\{X_1, X_2, \ldots\}$. Define the power-sum symmetric function, $p_{\nu}(X)$, and the Schur function, $s_{\lambda}(X)$, for $\nu, \lambda \in \mathcal{P}$, as they are in [14, Chapter I].

The irreducible characters $\omega_{\lambda}$ of $S_n$ are indexed by $\lambda \in \mathcal{P}_n$, as in [14, Chapter I]. Let $\omega_{\lambda}(\nu)$ be the value of $\omega_{\lambda}$ on a permutation with cycle type $\nu$. The relationship between $p_{\nu}(X)$ and $s_{\lambda}(X)$ is given by

$$s_{\lambda}(X) = \sum_{\nu \in \mathcal{P}_{\lambda}} \omega_{\lambda}(\nu) z_{\nu}^{-1} p_{\nu}(X), \quad \text{where} \quad z_{\nu} = \prod_{i \geq 1} t^{m_i} \nu_i!$$

is the order of the centralizer in $S_n$ of the conjugacy class corresponding to $\nu = (1^{m_1} 2^{m_2} \cdots) \in \mathcal{P}$. Let $t \in \mathbb{C}$. For $\mu \in \mathcal{P}$, let the Hall-Littlewood symmetric function $P_\mu(X;t)$ be as it is defined in [14].

For $\nu, \mu \in \mathcal{P}_n$, the classical Green function $Q_{\nu}^\mu(t)$ is given by

$$p_{\nu}(X) = \sum_{\mu \in \mathcal{P}_n} Q_{\nu}^\mu(t^{-1}) t^{n(\mu)} P_\mu(X;t).$$

The $p_{\nu}(X)$, $s_{\lambda}(X)$, and $P_\mu(X;t)$, are all bases of $\Lambda(X)$ as a $\mathbb{C}$-algebra.

For every $f \in \Phi$, fix a set of independent variables $X^{(f)} = \{X_1^{(f)}, X_2^{(f)}, \ldots\}$, and for any symmetric function $h$, let $h(f) = h(X^{(f)})$ denote the symmetric function in the variables $X^{(f)}$. Let

$$\Lambda = \mathbb{C}\text{-span}\{P_{\mu} \mid \mu \in \mathcal{P}^\Phi\}, \quad \text{where} \quad P_{\mu} = (-q)^{-n(\mu)} \prod_{f \in \Phi} P_{\mu(f)}(f; (-q)^{-d(f)}).$$

Then

$$\Lambda = \bigoplus_{n \geq 0} \Lambda_n, \quad \text{where} \quad \Lambda_n = \mathbb{C}\text{-span}\{P_{\mu} \mid \|\mu\| = n\},$$

makes $\Lambda$ a graded $\mathbb{C}$-algebra. Define a Hermitian inner product on $\Lambda$ by

$$\langle P_{\mu}, P_{\nu} \rangle = c_{\mu}^{-1} \delta_{\mu\nu}. $$

For each $\varphi \in \Theta$ let $Y^{(\varphi)} = \{Y_1^{(\varphi)}, Y_2^{(\varphi)}, \ldots\}$ be an infinite variable set, and for a symmetric function $h$, let $h(\varphi) = h(Y^{(\varphi)})$. Relate symmetric functions in the $X$ variables to symmetric functions in the $Y$ variables via the transform

$$p_{\nu}(\varphi) = (-1)^{n(\nu)} \sum_{x \in \mathcal{M}_{\nu}} \xi(x) p_{n(\nu)/|d(f_{\varphi})|}(f_{\varphi}),$$

where $\varphi \in \Theta$, $\xi \in \varphi$, and $f_{\varphi} \in \Phi$ satisfies $f_{\varphi}(x) = 0$.

Then

$$\Lambda = \mathbb{C}\text{-span}\{s_{\lambda} \mid \lambda \in \mathcal{P}^\Theta\}, \quad \text{where} \quad s_{\lambda} = \prod_{\varphi \in \Theta} s_{\lambda(\varphi)}(\varphi).$$

Let $C_n$ denote the set of complex-valued class functions of the group $U_n$, and for $\|\mu\| = n$, let $\pi_{\mu} : U_n \to \mathbb{C}$ be the class function which is 1 on $c_{\mu}$ and 0 elsewhere. Then the $\pi_{\mu}$ form a $\mathbb{C}$-basis for $C_n$. By Proposition 2.2, the usual inner product on class functions of finite groups, $(\cdot, \cdot) : C_n \times C_n \to \mathbb{C}$, satisfies

$$\langle \pi_{\mu}, \pi_{\lambda} \rangle = a_{\mu}^{-1} \delta_{\mu\lambda}. $$

For $\alpha_{i} \in C_n$, Ennola [5] defined a product $\alpha_1 \star \alpha_2 \in C_{n_1+n_2}$, which takes the following value on the conjugacy class $c_{\lambda}$:

$$\alpha_1 \star \alpha_2(c_{\lambda}) = \sum_{\|\mu\| = n_1} g_{\mu_1, \mu_2}(\nu_1) \alpha_1(c_{\mu_1}) \alpha_2(c_{\mu_2}),$$

where $g_{\mu_1, \mu_2}$ is the product of Hall polynomials (see [14, Chapter II])

$$g_{\mu_1, \mu_2} = \prod_{f \in \Phi} g_{\mu_1(f), \mu_2(f)}((-q)^{d(f)}).$$
Extend the inner product to $C = \bigoplus_{n \geq 0} C_n$, by requiring the components $C_n$ and $C_m$ to be orthogonal for $n \neq m$. This gives $C$ a graded $\mathbb{C}$-algebra structure. The characteristic map is

$$\text{ch}: \ C \longrightarrow \Lambda$$

$$\pi_\mu \mapsto P_\mu, \ \text{for } \mu \in P^\Phi.$$

**Proposition 3.1.** Let multiplication in the character ring $C$ of $U_n$ be given by $\star$. Then the characteristic map $\text{ch}: C \longrightarrow \Lambda$ is an isometric isomorphism of graded $\mathbb{C}$-algebras.

Following the work of Green [7] on the general linear group, Ennola was able to obtain the following result. We may follow the proof in Macdonald [14, IV.4] on the general linear group case, making the appropriate changes.

**Proposition 3.2** (Ennola). The set $\{ s_\lambda \ | \ \lambda \in P^\Phi \}$ is an orthonormal basis for $\Lambda$.

Now let $\chi^\lambda \in R$ be class functions so that $\chi^\lambda(1) > 0$ and $\text{ch}(\chi^\lambda) = \pm s_\lambda$. Ennola conjectured that $\{ \chi^\lambda \ | \ \lambda \in P^\Phi \}$ is the set of irreducible characters of $U_n$. He pointed out that if one could show that the product $\star$ takes virtual characters to virtual characters, then the conjecture would follow. There is no known direct proof of this fact, however. Significant progress on Ennola’s conjecture was only made after the work of Deligne and Lusztig [2].

### 3.2. Deligne-Lusztig Induction

Let $T_\nu \cong M_{\nu_1} \times \cdots \times M_{\nu_k}$ be a maximal torus of $U_n$. If $t \in T_\nu$, then $t$ is conjugate to

$$J_{(1^\nu_1)}(f_1) \oplus \cdots \oplus J_{(1^\nu_k)}(f_k), \ \text{where } f_i \in \Phi, m_id(f_i) = \nu_i.$$

Define $\gamma_t \in P^\Phi$ by

$$\gamma_t(f) \quad \text{has parts } \{ m_i \ | \ f_i = f \}.$$

Note that $(\gamma_t)_u(t - 1) = \nu$, but in general $t \notin c_{\gamma_t}$.

For $\mu \in P^\Phi$, let $L_\mu, \gamma \in P^\Phi$ and $T_\gamma$ be as in Section 2.3. Let $\theta$ be a character of $T_\nu$. The Deligne-Lusztig character $R_\nu(\theta) = R_{T_\nu}(\theta)$ is the virtual character of $U_n$ given by

$$(R_\nu(\theta))(g) = \sum_{t \in \gamma_{\nu,\mu}} \theta(t)Q^L_{T_\gamma}(u),$$

where $g \in c_\mu$ has Jordan decomposition $g = su$ (thus, by Lemma 2.1 $C_{U_n}(s) \cong L_\mu$), and $Q^L_{T_\gamma}(u)$ is a Green function for the unitary group (see, for example, [1]).

Deligne and Lusztig proved that the $R_\nu(\theta)$ span the class functions of $U_n$,

$$C_n = \mathbb{C} \text{-span}\{ R_\nu(\theta) \ | \ \nu \in P_n, \theta \in \text{Hom}(T_\nu, \mathbb{C}^\times) \},$$

so we may define Deligne-Lusztig induction by

$$R_{U_{m+n}}^{U_m \oplus U_n} : C_m \otimes C_n \longrightarrow C_{m+n}$$

$$R_{\alpha}^{U_m}(\theta_\alpha) \otimes R_{\beta}^{U_n}(\theta_\beta) \mapsto R_{U_{m+n}}^{U_m \oplus U_n}(\theta_\alpha \otimes \theta_\beta),$$

for $\alpha \in P_m, \beta \in P_n, \theta_\alpha \in \text{Hom}(T_\alpha, \mathbb{C})$, and $\theta_\beta \in \text{Hom}(T_\beta, \mathbb{C})$.

Let $\Lambda$ and $C$ be as in Section 3.1, except we now give $C$ a graded $\mathbb{C}$-algebra structure using Deligne-Lusztig induction. That is, we define a multiplication $\circ$ on $C$ by

$$\chi \circ \eta = R_{U_{m+n}}^{U_m \oplus U_n}(\chi \otimes \eta), \ \text{for } \chi \in C_m \text{ and } \eta \in C_n.$$

We recall the characteristic map defined in Section 3.1,

$$\text{ch}: \ C \longrightarrow \Lambda$$

$$\pi_\mu \mapsto P_\mu, \ \text{for } \mu \in P^\Phi.$$

It is immediate that $\text{ch}$ is an isometric isomorphism of vector spaces, but it is not yet clear if $\text{ch}$ is also a ring homomorphism when $C$ has multiplication given by Deligne-Lusztig induction.
3.3. The Ennola conjecture. In this section we summarize the remaining steps that are necessary to obtain the proof of the Ennola conjecture. First, we must compute $\text{ch}(U)$, where $U$ is an irreducible character of $G$. These Green functions turn out to be those of the general linear group, except with $q$ replaced by $-q$, which is the essence of Ennola’s original idea. This fact was proven by Hotta and Springer [9] for the case that $p = \text{char}(\mathbb{F}_q)$ is large compared to $n$, and was finally proven in full generality by Kawanaka [11].

**Theorem 3.1 (Hotta-Springer, Kawanaka).** The Green functions for the unitary group are given by

$$Q^n_\gamma(-q) = \prod_{f \in \Phi_n} Q^n_{\gamma(f)}((-q)^d(f)),$$

and $Q^n_\gamma(q)$ is the classical Green function as in (3.2).

For $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}$ and $\theta = \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_\ell$ a character of $T_\nu$, define

$$p_{\mu, \phi} = \prod_{\varphi \in \Theta} p_{\mu, \phi}(\varphi), \quad \text{where} \quad \mu, \phi \in \mathcal{P}.$$

From Theorem 3.1 and the machinery of the characteristic map, we obtain the following.

**Theorem 3.2.** Let $\nu = (\nu_1, \nu_2, \ldots, \nu_\ell) \in \mathcal{P}$, $\theta = \theta_1 \otimes \theta_2 \otimes \cdots \otimes \theta_\ell$ be a character of $T_\nu$, and $\nu = \nu_{\ell \theta} \in \mathcal{P}$. Then

$$\text{ch}(R_{\nu}(\theta)) = (-1)^{|\nu|+\ell(\nu)} p_{\nu}.$$

**Corollary 3.1.** Let multiplication in the character ring $C$ of $U_n$ be given by $\circ$. Then the characteristic map $\text{ch} : C \to \Lambda$ is an isometric isomorphism of graded $C$-algebras.

An immediate consequence is that the graded multiplication that Ennola originally defined on $C$ is exactly Deligne-Lusztig induction, or

**Corollary 3.2.** Let $\chi \in C_m$ and $\eta \in C_n$. Then

$$\chi \circ \eta = \chi \ast \eta.$$

We therefore have the advantage of taking either definition when convenience demands.

For $\lambda \in \mathcal{P}$, let $L_\lambda$, $W_\lambda$, and $T_\lambda$, $\gamma \in \mathcal{P}_\lambda$, be as in Section 2.3.

Note that the combinatorics of $\gamma$ almost specifies character $\theta_\gamma$ of $T_\lambda$ in the sense that

$$\theta_\gamma(T_\lambda(\varphi)) = \theta(\tau(\varphi)), \quad \text{for some} \quad \tau(\varphi) \in \varphi.$$

In fact, we may define

$$R_\gamma = R^{T_\lambda}_{T_\gamma}(\theta_\gamma) = \text{ch}^{-1}((-1)^{|\gamma|+\ell(\gamma)} p_\gamma),$$

where $\theta_\gamma$ is any choice of the $\theta_\gamma$’s.

For every $\lambda \in \mathcal{P}$ there exists a character $\omega^\lambda$ of $W_\lambda$ defined by

$$\omega^\lambda(\gamma) = \prod_{\varphi \in \Theta} \omega^\lambda(\gamma) $$

where $\omega^\lambda(\gamma)$ is the value of $\omega^\lambda$ on the conjugacy class $c_\gamma$ corresponding to $\gamma \in \mathcal{P}_\lambda$.

In [13], Lusztig and Srinivasan decomposed the irreducible characters of $U_n$ as linear combinations of Deligne-Lusztig characters, as follows.

**Theorem 3.3 (Lusztig-Srinivasan).** Let $\lambda \in \mathcal{P}_n$. Then there exists $\tau(\lambda) \in \mathbb{Z}_{\geq 0}$ such that the class function

$$R(\lambda) = (-1)^{\tau(\lambda)(n/2) + \sum_{\varphi \in \Theta} |\lambda(\varphi)| + |\lambda(\varphi)|/2} \sum_{\gamma \in \mathcal{P}_\lambda} \frac{\omega^\lambda(\gamma) \gamma}{z_{\gamma}}$$

is an irreducible character of $U_n$ ($z_{\gamma}$ is as in (2.14)).

Finally, we obtain the Ennola Conjecture.
Corollary 3.3 (Ennola Conjecture). For \( \lambda \in \mathcal{P}_\Theta \), there exists \( \tau(\lambda) \in \mathbb{Z}_{\geq 0} \) such that

\[
\left\{ \mathrm{ch}^{-1} \left( (-1)^{\tau(\lambda)} s_\lambda \right) \mid \lambda \in \mathcal{P}_n^\Theta \right\}
\]

is the set of irreducible characters of \( U_n \).

Proof. By Theorem 3.3 and Theorem 3.2,

\[
\mathrm{ch}(R(\lambda)) = (-1)^{\tau(\lambda)+n/2+\sum_{\varphi \in \Theta} |\lambda(\varphi)|+|\varphi||\lambda(\varphi)||/2} \sum_{\gamma \in \mathcal{P}_\Theta} \frac{\omega^\lambda(\gamma)}{z_\gamma} (-1)^{n-\ell(\gamma)} p_\gamma
\]

Note that the sign character \( \omega^\lambda \) of \( W_\lambda \) acts by

\[
\omega^\lambda(\gamma) = (-1)^{\sum_{\varphi \in \Theta} |\gamma(\varphi)|-\ell(\gamma)}
\]

and that \( \omega^\lambda \otimes \omega^\lambda = \omega^\lambda' \), so since \( \gamma \in \mathcal{P}_\lambda \),

\[
\mathrm{ch}(R(\lambda)) = (-1)^{\tau(\lambda)+n/2+n+\sum_{\varphi \in \Theta} |\varphi||\lambda(\varphi)||/2} \sum_{\gamma \in \mathcal{P}_\lambda} \frac{(\omega^\lambda \otimes \omega^\lambda')(\gamma)}{z_\gamma} p_\gamma
\]

and by applying (3.1) to a product over \( \Theta \),

\[
= (-1)^{\tau(\lambda)+n/2+n+\sum_{\varphi \in \Theta} |\varphi||\lambda(\varphi)||/2} s_\lambda'.
\]

Remark. There are at least two natural ways to index the irreducible characters of \( U_n \) by \( \Theta \)-partitions: Theorem 3.3 gives a natural indexing by \( \Theta \)-partitions, but Corollary 3.3 indicates that the conjugate choice is equally natural. Since we like to think of Schur functions as irreducible characters, we have chosen the latter indexing. However, several references, including Ennola [5], Ohmori [15], and Henderson [8], make use of the former.

4. Characters degrees

In this section, we calculate the degrees of the irreducible characters of the finite unitary group and find several character degree sums.

Let \( \lambda \in \mathcal{P}_\Theta \), and suppose \( \Box \in \lambda \) is in position \((i, j)\) in \( \lambda(\varphi) \) for some \( \varphi \in \Theta \). The hook length \( h(\Box) \) of \( \Box \) is

\[
h(\Box) = |\varphi|h(\Box), \quad \text{where} \quad h(\Box) = \lambda(\varphi)_i - \lambda(\varphi)_j - i - j + 1,
\]

is the usual hook length for partitions.

For \( \lambda \in \mathcal{P}_\Theta \), let

\[
y^\lambda = \mathrm{ch}^{-1}(s_\lambda).
\]

Adapting the computations in [14, IV.6], we obtain the following result.

Theorem 4.1. Let \( \lambda \in \mathcal{P}_\Theta \) and let 1 be the identity in \( U_{||\lambda||} \). Then

\[
y^\lambda(1) = (-1)^{\tau(\lambda)} q^{n(\lambda)} \prod_{\Box \in \lambda} (q^{h(\Box)} - (-1)^{h(\Box)})^{-1},
\]

where \( \tau(\lambda) = ||\lambda||(||\lambda||+3)/2+n(\lambda) \equiv ||\lambda||/2+n(\lambda) \pmod{2} \). So for each \( \lambda \), we have \( \chi^\lambda = (-1)^{\tau(\lambda)} y^\lambda \).

The following result follows from Theorem 4.1 and the Littlewood-Richardson rule.
Corollary 4.1. Let $\mu, \nu \in \mathcal{P}^\Theta$. Then $\chi^\mu \circ \chi^\nu$ is a character if and only if every $\lambda \in \mathcal{P}^\Theta$ such that $c_{\lambda, \mu, \nu} > 0$ satisfies
\[
n(\mu) + n(\nu) \equiv n(\lambda) + ||\mu|| ||\nu|| \pmod{2}.
\]

Following a similar approach to [14, IV.6, Example 5], we consider the coefficient of $t^m$ in the series
\[
S = \sum_{\lambda \in \mathcal{P}^m} (-1)^n(\lambda) + ||\lambda|| \delta(s_\lambda) ||t||^{|\lambda|},
\]
where $n(\lambda)$ is as in (2.7), and we obtain the following result.

Theorem 4.2. The sum of the degrees of the complex irreducible characters of $U_m$ is given by
\[
\sum_{||\lambda||=m} \chi^\lambda(1) = (q+1)q^2(q^3+1)q^4(q^5+1)\cdots(q^m + \frac{(1-(-1)^m)}{2}).
\]

Write $f_{U_m}(q) = \sum_{||\lambda||=m} \chi^\lambda(1)$. The polynomial $f_{G_m}(q)$ expressing the sum of the degrees of the complex irreducible characters of $G_m = GL(m, \mathbb{F}_q)$, was computed in [6] for odd $q$ and in [12] and Example 6 of [14, IV.6] for general $q$. From these results we see that
\[
f_{U_m}(q) = (-1)^{m(m+1)/2}f_{G_m}(-q),
\]

another example of Ennola duality.

Gow [6] and Klyachko [12] proved that the sum of the degrees of the complex irreducible characters of $G_n$ is equal to the number of symmetric matrices in $G_n$. We obtain the same result for $U_n$ by applying Theorem 4.2 and a counting argument.

Corollary 4.2. The sum of the degrees of the complex irreducible characters of $U(n, \mathbb{F}_{q^2})$ is equal to the number of symmetric matrices in $U(n, \mathbb{F}_{q^2})$.

A $\Theta$-partition $\lambda$ is even if every part of $\lambda(\varphi)$ is even for every $\varphi \in \Theta$.

Theorem 4.3. The sum of the degrees of the complex irreducible characters of $U_{2m}$ corresponding to $\lambda$ such that $\lambda'$ is even is given by
\[
\sum_{||\lambda||=2m, \lambda' \text{ even}} \chi^\lambda(1) = (q+1)q^2(q^3+1)\cdots q^{2m-2}(q^{2m-1}+1) = \frac{|U(2m, \mathbb{F}_{q^2})|}{|Sp(2m, \mathbb{F}_q)|}.
\]

Write $g_{U_m}(q) = \sum_{||\lambda||=2m, \lambda' \text{ even}} \chi^\lambda(1)$, and let $g_{G_m}(q)$ denote the corresponding sum for $G_m$. The polynomial $g_{G_m}(q)$ was calculated in Example 7 of [14, IV.6], and similar to the previous example, we see that we have
\[
g_{U_m}(q) = (-1)^m g_{G_m}(-q).
\]

In the case that $q$ is odd, Proposition 4.3 follows from the following stronger result obtained by Henderson [8]. Let $Sp_{2n} = Sp(2n, \mathbb{F}_q)$ be the symplectic group over the finite field $\mathbb{F}_q$.

Theorem 4.4 (Henderson). Let $q$ be odd. The decomposition of $\text{Ind}^{U_{2n}}_{Sp_{2n}}(1)$ into irreducibles is given by
\[
\text{Ind}_{Sp_{2n}}^{U_{2n}}(1) = \sum_{||\lambda||=2m, \lambda' \text{ even}} \chi^\lambda.
\]

The fact that Proposition 4.3 holds for all $q$ suggests that Theorem 4.4 should as well.

5. A Deligne-Lusztig model

A model of a finite group $G$ is a representation $\rho$, which is a direct sum of representations induced from one-dimensional representations of subgroups of $G$, such that every irreducible representation of $G$ appears as a component with multiplicity 1 in the decomposition of $\rho$.

Klyachko [12] and Inglis and Saxl [10] obtained a model for $GL(n, \mathbb{F}_q)$, where the induced representations can be written as a Harish-Chandra product of Gelfand-Graev characters and the permutation character of the finite symplectic group.

In this section we show that the same result is true for the finite unitary group, except the Harish-Chandra product is replaced by Deligne-Lusztig induction. The result is therefore not a model for $U(n, \mathbb{F}_q)$. 

\[
\text{Ind}_{Sp_{2n}}^{U_{2n}}(1) = \sum_{||\lambda||=2m, \lambda' \text{ even}} \chi^\lambda.
\]
in the finite group character induction sense, but rather from the Deligne-Lusztig point of view.

Let $U'_n = \text{GL}(n, \mathbb{K})^F$ as in (2.3), and let

$$B_\prec = \{ u \in U'_n \mid u \text{ unipotent and uppertriangular} \} \subseteq U'_n.$$  

Fix a nontrivial character $\psi : \mathbb{K}_2^+ \to \mathbb{C}^\times$ of the additive group of the field $\mathbb{K}_2$ such that the restriction to the subgroup $\{ x \in \mathbb{K}_2 \mid x^2 + x = 0 \}$ is also nontrivial. The map $\psi(n) : B_\prec \to \mathbb{C}$ given by

$$\psi(n)(u) = \psi(u_{12} + \cdots + u_{\lfloor n/2 \rfloor - 1, \lfloor n/2 \rfloor} + u_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor + 1}),$$

for $u = (u_{ij}) \in B_\prec$, is a linear character of $B_\prec$. Then

$$\Gamma(n) = \text{Ind}_{B_\prec}^{U'_n}(\psi(n))$$

is the Gelfand-Graev character of $U'_n$. Let $\Gamma(n)$ be the corresponding Gelfand-Graev character of $U_n = \text{GL}(n, \mathbb{K})^F$. For $\lambda \in \mathcal{P}^\Theta$, define

$$\text{ht}(\lambda) = \max\{ \ell(\lambda(\varphi)) \mid \varphi \in \Theta \}.$$  

The following appears in Section 5.2 of [15], and we can also give a proof using the characteristic map.

**Theorem 5.1.** The decomposition of $\Gamma(m)$ into irreducibles is given by

$$\Gamma(m) = \sum_{\lambda \in \mathcal{P}^\Theta_{\text{ht}(\lambda) = 1}} \chi^{\lambda}.$$  

For a partition $\lambda$, let $o(\lambda)$ denote the number of odd parts of $\lambda$, and for $\lambda \in \mathcal{P}^\Theta$, let $o(\lambda) = \sum_{\varphi \in \Theta} |\varphi| o(\lambda(\varphi))$.

**Theorem 5.2.** Let $q$ be odd. For each $r$ such that $0 \leq 2r \leq m$,

$$\Gamma_{m-2r} \circ \text{Ind}_{\text{Sp}_2}^{U'_2} (1) = \sum_{o(\lambda') = m-2r} \chi^{\lambda}.$$  

Furthermore,

$$\sum_{0 \leq 2r \leq m} \Gamma_{m-2r} \circ \text{Ind}_{\text{Sp}_2}^{U'_2} (1) = \sum_{||\lambda|| = m} \chi^{\lambda}.$$  

**Proof.** Suppose $\mu, \nu \in \mathcal{P}^\Theta$, such that $\text{ht}(\mu) = 1$ and $\nu'$ is even. From the characteristic map, Corollary 3.3, and Pieri’s formula [14, I.5.16],

$$\chi^{\mu} \circ \chi^{\nu} = (-1)^{\tau(\mu) + \tau(\nu)} \sum_{\lambda} \chi^{\lambda},$$

where the sum is taken over all $\lambda$ such that for every $\varphi \in \Theta$, $\lambda(\varphi) - \nu(\varphi)$ is a horizontal $|\mu(\varphi)|$-strip.

We now use Corollary 4.1 to show that $\chi^{\mu} \circ \chi^{\nu}$ is a character. As $\lambda(\varphi) - \nu(\varphi)$ is a horizontal $|\mu(\varphi)|$-strip, the part $\lambda(\varphi)'_i$ is either $\nu(\varphi)'_i$ or $\nu(\varphi)'_i + 1$ for every $i = 1, 2, \ldots, \ell(\lambda(\varphi))$. By assumption, $\nu'$ is even, so $\nu(\varphi)'_i$ is even for every $\varphi \in \Theta$, and so

$$\left(\frac{\nu(\varphi)'_i + 1}{2}\right) = \nu(\varphi)'_i + \left(\frac{\nu(\varphi)'_i}{2}\right) \equiv \left(\frac{\nu(\varphi)'_i}{2}\right) \pmod{2}.$$  

Thus, $n(\lambda(\varphi)) = \sum_i \left(\frac{\lambda(\varphi)'_i}{2}\right) \equiv n(\nu(\varphi)) \pmod{2}$. The assumption $\text{ht}(\mu) = 1$ implies $n(\mu(\varphi)) = 0$, and since $||\nu||$ is even, $n(\mu) + n(\nu) \equiv n(\lambda) + ||\mu|| ||\nu|| \pmod{2}$.

By Corollary 4.1, $\chi^{\mu} \circ \chi^{\nu}$ is a character.

Use the decompositions of Theorem 4.4 and Theorem 5.1 in the product (5.1) to observe that the irreducible characters $\chi^{\lambda}$ in the decomposition of $\Gamma_{m-2r} \circ \text{Ind}_{\text{Sp}_2}^{U'_2} (1)$ are indexed by $\lambda \in \mathcal{P}^\Theta_m$ such that for every $\varphi$, $\lambda(\varphi) - \nu(\varphi)$ is a horizontal $|\mu(\varphi)|$-strip, where $||\mu|| = m - 2r$, for some $\nu(\varphi)$ such that $\nu(\varphi)'$ is even. Then the number of odd parts of $\lambda(\varphi)'$ is exactly $|\mu(\varphi)|$, and so the $\lambda$ in the decomposition must satisfy $\sum_{\varphi \in \Theta} |\varphi| o(\lambda(\varphi)) = ||\mu|| = m - 2r$. □
ON THE CHARACTERISTIC MAP OF FINITE UNITARY GROUPS

References


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