Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

An analogue of the Robinson-Schensted-Knuth Algorithm and its application to standard bases

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Abstract. The Schur functions, $s_\lambda(x)$, form a basis for the vector space of symmetric functions. Recently Haglund, Haiman and Loehr derived a combinatorial formula for nonsymmetric Macdonald polynomials, which gives a new decomposition of the Macdonald polynomial into nonsymmetric components. Letting $q = t = 0$ in this identity implies $s_\lambda(x) = \sum_\mu NS_\mu(x)$, where the sum is over all rearrangements μ of the partition λ. We exhibit a bijection involving an analogue of Robinson-Schensted-Knuth Insertion between semi-standard Young tableaux and semi-standard skyline fillings to give a combinatorial proof of the formula. The insertion procedure led us to determine an analogue of the RSK Algorithm for semi-standard skyline fillings. This analogue is used to prove that the non-symmetric Schur functions equal the standard bases of Lascoux and Schützenberger.

Résumé. La fontion Schur forme une base pour l'espace de vecteur des fonctions symétriques. Récemment Haglund, Haiman et Loehr ont dérivé une formule combinatoire pour des polynômes nonsymmetric de Macdonald, qui donne une nouvelle décomposition du polynôme de Macdonald dans les composants nonsymmetric. Laisser $q = t = 0$ dans cette identité implique la fonction de Schur s_λ est la somme des fonctions nonsymmetric de Schur au-dessus de toutes les remises en ordre de la cloison λ. Nous exhibons un bijection impliquant un analogue d'insertion de Robinson-Schensted-Knuth entre de Young tableaux de semi-standard et remplissages d'horizon de semi-standard pour fournir des preuves combinatoires de la formule. Le procédé d'insertion nous a menés à déterminer un analogue de l'algorithme de RSK pour des remplissages d'horizon de semi-finale-standard. Cet analogue est employé pour montrer que les fonctions dissymétriques de Schur égalent les bases standard de Lascoux et de Schützenberger.

1. Introduction

A *symmetric function of degree n* over a commutative ring R (with identity) is a formal power series $f(x) = \sum_\alpha c_\alpha x^\alpha$, where α ranges over all weak compositions of n (of infinite length), $c_\alpha \in R$, x^α stands for the the monomial $x_1^{\alpha_1} x_2^{\alpha_2}...$, and $f(x_{\omega(1)}, x_{\omega(2)}, ...) = f(x_1, x_2, ...)$ for every permutation ω of the positive integers, $\mathbb{P}$. Many different bases for the vector space of symmetric functions are known. One important basis is the Schur functions.

The *Schur function* $s_\lambda = s_\lambda(x)$ of shape λ in variables $x = (x_1, x_2, ...)$ is the formal power series $s_\lambda = \sum_T x^T$, summed over all semi-standard Young tableaux of shape λ. A *semi-standard Young tableau* is formed by first placing the parts of λ into rows of squares, where the i^{th} row has λ_i squares, called *cells*. This diagram, called the Young (or Ferrers) diagram, is drawn in the first quadrant, French style, as in [**3**]. Then each of these cells is assigned a positive integer in such a way that the row entries are weakly increasing and the column entries are strictly increasing. Thus, the values assigned to the cells of λ collectively form the multiset $\{1^{a_1}, 2^{a_2}, ..., n^{a_n}\}$, for some n, where a_i is the number of times i appears in T. Here, $x^T = \prod_{i=1}^n x_i^{a_i}$. See [**10**] for a more detailed discussion of symmetric functions and the Schur functions in particular.

The Macdonald polynomials $\tilde{H}_\mu(x; q, t)$ are a special class of symmetric functions which contain a vast array of information. Macdonald [**8**] introduced them and conjectured that their expansion in terms of Schur

2000 *Mathematics Subject Classification.* Primary 05E05; Secondary 05E10.
Key words and phrases. algebraic combinatorics, symmetric functions, Macdonald polynomials, compositions.

polynomials should have positive coefficients. A combinatorial formula for the Macdonald polynomials was recently proved by Haglund, Haiman, and Loehr [**5**].

Building on this work, Haglund, Haiman, and Loehr [**4**] derive a combinatorial formula for nonsymmetric Macdonald polynomials, which gives a new decomposition of the Macdonald polynomial into nonsymmetric components. The statistics involved in this formula can be used to define nonsymmetric Schur polynomials, NS_λ. Letting $q = t = 0$ in the identity implies $s_\mu(x) = \sum_\lambda NS_\lambda(x)$, where the sum is over all rearrangements λ of the partition μ. (A composition μ of n is called a *rearrangement* of a partition λ if it consists of n parts such that when the parts are arranged in descending order, the i^{th} part equals λ_i, for all i.) We give a bijective proof of this decomposition.

Theorem 1.1. $\sum_{\lambda'} NS_{\lambda'}(x_1, ..., x_n) = s_\lambda(x_1, ..., x_n)$, *where the sum is over all rearrangements* λ' *of* λ.

We exhibit a weight-preserving bijection between semi-standard Young tableaux and semi-standard skyline fillings to prove Theorem 1.1. The bijection involves an insertion procedure similar to Schensted insertion. This procedure is the fundamental operation in an analogue of the Robinson-Schensted-Knuth algorithm.

Theorem 1.2. *There exists a bijection between* $\mathbb{N} -$ *matrices with finite support and pairs* (F, G) *of semi-standard skyline fillings of compositions which rearrange the same partition.*

The Schur functions are alternatively defined as the irreducible characters of the linear group on $\mathbb{C}$. Demazure's "Formule des caractère" [**1**] [**2**] provides an interpolation between a dominant weight corresponding to a partition I and the Schur function of index I. For each permutation μ, he obtains a"partial" character whose interpretation involves the Schubert variety of index μ, best understood through the study of the "standard bases" of these spaces. Considering Young tableaux as words in the free algebra, Lascoux and Schützenberger [**6**] describe the standard bases $\mathfrak{U}(\mu, I)$ using symmetrizing operators on the free algebra which lift the operators used by Demazure. This description provides a recursive algorithm to determine the basis $\mathfrak{U}(\mu, I)$ given the basis $\mathfrak{U}(\lambda, I)$, where $\mu = \sigma_i\lambda$ for some i. The nonsymmetric Schur functions provide a non-recursive combinatorial description of $\mathfrak{U}(\mu, I)$ for arbitrary μ, I.

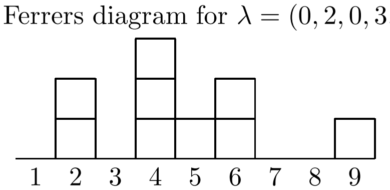
Theorem 1.3. $\mathfrak{U}(\mu, I) = NS_{\mu(I)}$, *where* $\mu(I)$ *denotes the action of* μ *on the parts of* I.

This theorem provides a mapping between the combinatorics of symmetrizing (or string) operators and nonsymmetric Schur functions.

2. Combinatorial description of the nonsymmetric Schur functions

2.1. Semi-standard skyline fillings. Let $\gamma = (\gamma_1, \gamma_2, ..., \gamma_n)$ be a composition of n into n parts, allowing zero as a part. (We will consider compositions of n into arbitrarily many parts in section 2.3.) The *composition Ferrers diagram of* γ is a figure consisting of n *cells* arranged in n columns. The i^{th} column contains γ_i cells, and the number of cells in a column is called the *height* of that column. This is an analogue of the Ferrers diagram of a partition λ, which consists of rows of cells such that the i^{th} row contains λ_i cells.

Example 2.1. The composition Ferrers diagram for $\lambda = (0, 2, 0, 3, 1, 2, 0, 0, 1)$



A *filling,* σ, of a composition Ferrers diagram, λ, is a function $\sigma : \lambda \to \mathbb{Z}_+$, which we picture as an assignment of positive integer entries to the cells of λ. We consider the 0^{th} row to consist of cells numbered from 1 to n in strictly increasing order. Let $\sigma(i)$ denote the entry in the i^{th} square of the composition Young diagram encountered if we read across rows from left to right, beginning at the highest row and working downwards.

To define the nonsymmetric Schur functions, we need the statistics $Des(\sigma)$ and $Inv(\sigma)$. As in [**3**], a *descent* of σ is a pair of entries $\sigma(a) > \sigma(b)$, where the cell a is directly above b. In other words, $b = (i, j)$ and $a = (i + 1, j)$, where the i^{th} coordinate denotes the height of cell b and the j^{th} coordinate denotes the column containing j. Define $Des(\sigma) = \{a \in \lambda : \sigma(a) > \sigma(b)$ is a descent$\}$.

Three cells $a, b, c \in \lambda$ form a *triple of type A* if they are situated as follows,

$$\begin{array}{|c|ccc|c|} \hline a & & \cdots & & c \\ \hline b & & & & \\ \hline \end{array}$$

where a and c are in the same row, possibly the first row, possibly with cells between them, and the column containing a and b has height greater than or equal to the height of the column containing c.

Define for $x, y \in \mathbb{Z}_+$

$$I(x,y) = \begin{cases} 1 & \text{if } x > y \\ 0 & \text{if } x \leq y \end{cases}$$

Let σ be a composition filling and let α, β, δ be the entries of σ in the cells of a type A triple (a, b, c):

$$\begin{array}{|c|ccc|c|} \hline \alpha & & \cdots & & \delta \\ \hline \beta & & & & \\ \hline \end{array}$$

Then the triple (a, b, c) is called an *inversion triple of type A* if and only if $I(\alpha, \delta) + I(\delta, \beta) - I(\alpha, \delta) =1$.

The *reading order* of a filling is an ordering of its cells beginning with the top row and listing the cells from left to right, travelling down, row by row, to the bottom row. Define a filling σ to be *standard* if it is a bijection $\sigma : \mu \cong \{1,...,n\}$. The *standardization* of a composition filling is the unique standard filling ξ such that $\sigma \circ \xi^{-1}$ is weakly increasing, and for each α in the image of σ, the restriction of ξ to $\sigma^{-1}(\{\alpha\})$ is increasing with respect to the reading order. Therefore the triple (a, b, c) is an inversion triple of type A if and only if after standardization, the ordering from smallest to largest of the entries in cells a, b, c induces a counter-clockwise orientation.

Similarly, three cells $a, b, c \in \lambda$ form a *triple of type B* if they are situated as shown below,

$$\begin{array}{cccc|c|} \cline{5-5} & & & & b \\ \hline \multicolumn{1}{|c|}{a} & & \cdots & & c \\ \hline \end{array}$$

Here a and c are in the same row (possibly row 0) and the column containing b and c has greater height than the column containing a.

Let σ be a composition filling and let α, β, δ be the entries of σ in the cells of a type B triple (a, b, c).

$$\begin{array}{cccc|c|} \cline{5-5} & & & & \beta \\ \hline \multicolumn{1}{|c|}{\alpha} & & \cdots & & \delta \\ \hline \end{array}$$

Then the triple (a, b, c) is called an *inversion triple of type B* if and only if $I(\beta, \alpha) + I(\alpha, \delta) - I(\beta, \delta) = 1$. In other words, the triple (a, b, c) is an inversion triple of type B if and only if after standardization, the ordering from smallest to largest of the entries in cells a, b, c induces a clockwise orientation.

Denote by *semi-standard skyline filling* any composition filling F such that $Des(F) = \emptyset$ and every triple is an inversion triple. These conditions arise by setting $q = t = 0$ in the combinatorial formula for the nonsymmetric Macdonald polynomials [**4**].

DEFINITION 2.2. Let λ be a composition of n into n parts, where some of the parts could be equal to zero. The nonsymmetric Schur function $NS_\lambda = NS_\lambda(x)$ in the variables $x = (x_1, x_2, ..., x_n)$ is the formal power series $NS_\lambda(x) = \sum_F x^F$ summed over all semi-standard skyline fillings F of composition λ. Here, $x^F = \prod_{i=1}^n x_i^{\sigma_i}$ is the *weight* of σ. (See Figure 2.1.)

The combinatorial formula for nonsymmetric Macdonald polynomials [**4**] contains an additional "non-attacking" condition. This condition states that for each pair of cells a and b with a to the left of b in the row directly below b, $\sigma(a) \neq \sigma(b)$. (If $\sigma(a) = \sigma(b)$, a and b are called *attacking* cells.)

LEMMA 2.3. *The descent and inversion conditions used to describe the semi-standard skyline fillings guarantee that no two cells of a semi-standard skyline filling are attacking.*

PROOF. Assume there exist two attacking cells a and b with $\sigma(a) = \sigma(b) = \alpha$ to get a contradiction. If the column containing a is taller than or equal to the column containing b, then a lies directly below a cell c which must have $\sigma(c) \leq \alpha$. When the values in these three cells are standardized, c, b, a form a non-inversion triple of type A. If the column containing b is taller than the column containing a, b is directly on top of a cell c which must have $\sigma(c) \geq \alpha$. The cells a, b, c form a type B non-inversion triple. $\square$

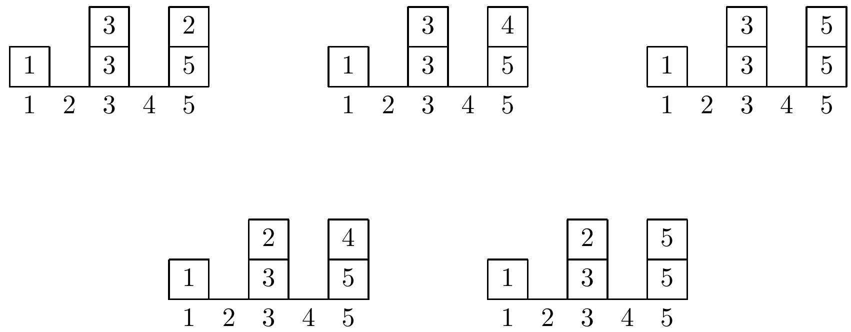
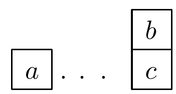


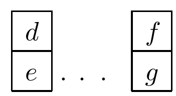
Figure 2.1. $NS_{(1,0,2,0,2)} = x_1x_2x_3^2x_5 + x_1x_3^2x_4x_5 + x_1x_3^2x_5^2 + x_1x_2x_3x_4x_5 + x_1x_2x_3x_5^2$.

Lemma 2.4. *If a, b, c is a type B triple with a and c on the same row and b directly above c, then $\sigma(a) < \sigma(c)$.*

Proof. Let a, b, c be a type B triple situated as pictured below.



To get a contradiction, first assume $\sigma(a) > \sigma(c)$. In the basement row, the column containing a has a value less than the value of the column containing c. So at some intermediate row we have



with $\sigma(d) > \sigma(f)$ and $\sigma(e) \leq \sigma(g)$. We must have $\sigma(d) \leq \sigma(e)$. Therefore, $\sigma(f) < \sigma(d) \leq \sigma(e) \leq \sigma(g)$. But then $\sigma(f) < \sigma(e) \leq \sigma(g)$ and this type B triple f, e, g is not an inversion triple.

Next assume $\sigma(a) = \sigma(c)$. If so, by standardization we may assume that $\sigma(a) < \sigma(c)$. To have an inversion triple, $\sigma(b)$ must be between $\sigma(a)$ and $\sigma(c)$. But then $\sigma(b)$ must equal $\sigma(a)$ and $\sigma(c)$, which implies that a and b are attacking. So $\sigma(a)$ cannot equal $\sigma(c)$.

$\square$

Lemmas 2.3 and 2.4 provide us with several conditions on the cells in our diagram. They will be useful in proving facts about the insertion process.

2.2. A basis for homogeneous polynomials of degree n in n variables. Several other bases for symmetric functions have nonsymmetric analogues. For instance, the nonsmmyetric monomial corresponding to a given composition γ of n into n parts is given by $NM_\gamma = x_1^{\gamma_1}x_2^{\gamma_2}...x_n^{\gamma_n}$. It is clear that the sum over all rearrangements of a given partition μ of the nonsymmetric monomials is equal to the monomial symmetric function m_μ. Every polynomial of degree n in n variables can be written as a sum of nonsymmetric monomials, so the nonsymmetric monomials form a basis for the algebra of homogeneous polynomials of degree n in n variables.

Definition 2.5. The **reverse dominance order** on compositions is defined as follows:
$\mu \leq \gamma \iff \sum_{i=k}^n \mu_i \leq \sum_{i=k}^n \gamma_i$ for $1 \leq i \leq n$.

A semi-standard skyline filling is said to have type μ if it contains μ_i copies of the number i for each i. If γ and μ are compositions of n into n parts, let $NK_{\gamma,\mu}$ denote the number of semi-standard skyline fillings of shape γ and type μ. $NK_{\gamma,\mu}$ is called a *nonsymmetric Kostka number*. The ordinary Kostka numbers are obtained as a sum of nonsymmetric Kostka numbers: $K_{\lambda,\mu} = \sum NK_{\gamma,\mu}$, where the sum is over all rearrangements γ of λ.

Theorem 2.6. *Suppose that γ and μ are both compositions of n into n parts and $NK_{\gamma,\mu} \neq 0$. Then $\gamma \geq \mu$ in the dominance order. Moreover, $NK_{\gamma,\gamma} = 1$.*

Proof. Assume that $NK_{\gamma,\mu} \neq 0$. By definition, there exists a semi-standard skyline filling of shape γ and type μ. Assume that an entry k appears in one of the first $k-1$ columns. Then this column would contain a descent, since there is an entry less than k in the column at a lower position than k, namely the basement entry. Therefore, the parts $k, k+1, ..., n$ all appear in the last $n-k+1$ columns. So $\mu_k + \mu_{k+1} + ... + \mu_n \leq \gamma_k + \gamma_{k+1} + ... + \gamma_n$ for each k, as desired. Moreover, if $\mu = \gamma$, then the i^{th} column must contain only entries with value i, so $NK_{\gamma,\gamma} = 1$. □

Corollary 2.7. *The nonsymmetric Schur functions form a basis for the algebra of homogeneous polynomials of degree n in n variables.*

Proof. Theorem 2 is equivalent to the assertion that the transition matrix from the nonsymmetric Schur functions to the nonsymmetric monomials (with respect to the reverse dominance order) is upper triangular with 1's on the main diagonal. Since this matrix is invertible, the nonsymmetric Schur functions of degree n are a basis for homogeneous polynomials of degree n in n variables. □

2.3. Nonsymmetric Schur functions in infinitely many variables. We may relax the restriction on the number of parts to obtain nonsymmetric Schur functions in infinitely many variables.

Definition 2.8. A **weak composition** of n is an infinite sequence of non-negative integers which sum to n.

Let γ be a weak composition of n. Its composition Ferrers diagram consists of infinitely many columns such that the i^{th} column contains γ_i cells. As above, fill this diagram with positive integers in such a way that there are no descents and every triple is an inversion triple to get a semi-standard skyline filling. Then $NS_\gamma(x) = \sum_F x^F$, where F ranges over all semi-standard skyline fillings of the composition Ferrers diagram of γ.

We may also define the nonsymmetric monomials in infinitely many variables. The nonsmmyetric monomial corresponding to a weak composition γ of n is given by $NM_\gamma = \prod_i x_i^{\gamma_i}$. It is clear that the sum over all rearrangements of a given partition μ of the nonsymmetric monomials is equal to the monomial symmetric function m_μ. Every polynomial can be written as a sum of nonsymmetric monomials, so the nonsymmetric monomials form a basis for all polynomials.

Definition 2.9. Let μ and γ be weak compositions of n. The **reverse dominance order** on weak compositions is defined as follows.

$$\mu \leq \gamma \iff \sum_{i=k}^{\infty} \mu_i \leq \sum_{i=k}^{\infty} \gamma_i \qquad \forall k, \quad k \geq 1$$

.

If γ and μ are weak compositions of n, $NK_{\gamma,\mu}$ denote the number of semi-standard skyline fillings of shape γ and type μ as above. Again, the ordinary Kostka numbers are obtained as a sum of nonsymmetric Kostka numbers. $K_{\lambda,\mu} = \sum NK_{\gamma,\mu}$, where the sum is over all rearrangements γ of λ.

Theorem 2.10. *Suppose that γ and μ are both weak compositions of n and $NK_{\gamma,\mu} \neq 0$. Then $\mu \leq \gamma$ in the dominance order. Moreover, $NK_{\gamma,\gamma} = 1$.*

Proof. Assume that $NK_{\gamma,\mu} \neq 0$. By definition, there exists a semi-standard skyline filling of shape γ and type μ. Assume that an entry k appears in one of the first $k-1$ columns. Then this column would contain a descent, since there is an entry less than k in the column at a lower position than k, namely the basement entry. Therefore, the entries greater than or equal to k all appear after the $(k-1)^{th}$ column. So $\sum_{i=k}^{\infty} \mu_i \leq \sum_{i=k}^{\infty} \gamma_i$ for each k, as desired. Moreover, if $\mu = \gamma$, then the i^{th} column must contain only entries with value i, so $NK_{\gamma,\gamma} = 1$. □

Corollary 2.11. *The nonsymmetric Schur functions form a basis for all polynomials.*

Proof. Theorem 2.10 is equivalent to the assertion that the transition matrix from the nonsymmetric Schur functions to the nonsymmetric monomials (with respect to the reverse dominance order) is upper triangular with 1's on the main diagonal. Since this matrix is invertible, the nonsymmetric Schur functions are a basis for all polynomials. □

3. Proof of Theorem 1.1

We will in fact prove a slightly more general statement

$$\sum_{\lambda'} NS_{\lambda'} = s_\lambda, \tag{3.1}$$

where the sum is over all weak compositions λ' which rearrange λ.

3.1. An analogue of Schensted insertion. Let F be a semi-standard skyline filling of a weak composition γ of n. Then $F = (F_j)$, where F_j is the j^{th} cell when the cells are in reading order, including the cells in the basement. We define the operation $F \leftarrow k$.

Let r be the smallest integer such that $\sigma(F_r) \geq k$ and there is no cell c with $\sigma(c) = k$ in the row directly above the row containing F_r. If there is no cell directly on top of F_r, then place k on top of F_r are the resulting figure is $F \leftarrow k$. Otherwise let a be the cell directly on top of F_r. If $\sigma(a) < k$ then k "bumps" $\sigma(a)$. In other words, k replaces $\sigma(a)$ and we now find the least r' such that $r' > r$ and $\sigma(F_{r'}) \geq a$ and repeat. If $\sigma(a) > k$ then continue to the next r' such that $r' > r$ and $\sigma(F_{r'}) \geq k$ and repeat. This procedure terminates, since there are infinitely many basement entries greater than k.

Lemma 3.1. *When restricted to n-compositions, this procedure terminates.*

Proof. Assume that the procedure does not terminate to get a contradiction. This could only occur if some letter α reaches the last cell in the basement without finding an r such that $\sigma(F_r) \geq \alpha$ and such that the cell b on top of F_r has $\sigma(b) \leq \alpha$. The value α is an entry in the basement, say $\sigma(F_j)$. The letter α which is unplaced could not have been bumped from a cell to the right of F_j in the row above F_j, for otherwise the cell containing α and F_j would be attacking. Since α was not inserted on top of F_j, the entry b on top of F_j must have $\sigma(b) \geq \alpha$. But since F has no descents, $\sigma(b) = \alpha$. So the leftover α must have come from a higher row. Continuing this line of reasoning, we see a column containing the value α at each row until a certain height h at which this column contains an entry strictly smaller than α. If α was bumped from row h, α must have been bumped from a cell to the right of the α^{th} column. However, then α and the α in row $h-1$ of column α would be entries in attacking cells in F. By Lemma 2.3, there are no attacking cells in F. Therefore we have a contradiction. $\square$

The resulting diagram is $F \leftarrow k$.

Proposition 3.1. *If F is a semi-standard skyline filling, then $F \leftarrow k$ is a semi-standard skyline filling.*

Proof. It is clear by construction that $F \leftarrow k$ has no descents. We must prove that every triple is an inversion triple. We argue by contradiction. To get a contradiction, assume $F \leftarrow k$ contains a type A non-inversion triple, a, b, c situated as shown.

$$\begin{array}{ccc} \boxed{a} & \quad & \boxed{b} \\ \boxed{c} & & \end{array}$$

Then we must have $\sigma(a) \leq \sigma(b) \leq \sigma(c)$. In F, we must have had different (possibly empty) entries in these cells. Because the insertion path moves along the reading word and its entries are decreasing, at most one of $\sigma(a)$, $\sigma(b)$, and $\sigma(c)$ is different from its value in F. Examine each cell individually to get a contradiction. For example, assume the cell a in $F \leftarrow k$ contained a different value, $\beta \neq \sigma(a)$, in F. Since $\beta, \sigma(b), \sigma(c)$ was an inversion triple in F, $\sigma(b) < \beta \leq \sigma(c)$. But since $\sigma(b)$ bumped β, $\sigma(a) > \beta$, so $\sigma(a) > \sigma(b)$ contradicts $\sigma(a) \leq \sigma(b)$.

Next assume that $F \leftarrow k$ contains a type B non-inversion triple, a, b, c situated as depicted.

$$\begin{array}{ccc} & & \boxed{b} \\ \boxed{a} & \quad & \boxed{c} \end{array}$$

We must have $\sigma(b) < \sigma(a) \leq \sigma(c)$. Again, only one of the entries is different from its value in F. Examine each cell individually to derive a contradiction.

□

3.2. The bijection Ψ between SSYT and SSSF. Let T be a semi-standard young tableau. We may associate to T the word $col(T)$ which consists of the entries of each column of T, read top to bottom from columns left to right.

Example 3.2. For T as below, $col(T) = 10\ 9\ 8\ 4\ 2\ 1 \cdot 11\ 10\ 7\ 5\ 2 \cdot 10\ 8\ 5\ 3 \cdot 5 \cdot 10$.

10				
9	11			
8	10	10		
4	7	8		
2	5	5		
1	2	3	5	10

Begin with an arbitrary SSYT T and the empty SSSF ϕ with the basement row containing all letters of $\mathbb{Z}_+$. Let k be the rightmost letter in $col(T)$. Insert k into ϕ to get the SSSF $F = \phi \leftarrow k$. Then let k' be the next letter in $col(T)$ reading right to left. Obtain the SSSF $F \leftarrow k'$. Continue in this manner until you have inserted all the letters of $col(T)$. The resulting diagram is the SSSF $\Psi(T)$.

Lemma 3.3. *Let C_i be a column of $col(T)$. The placement of each letter of C_i terminates at a different column, with the smallest letter of C_i terminating at the top of the highest column, the second smallest letter terminating at the top of the second highest column, and so forth. (If there are two columns of the same height, the one farther left is the termination point of the smaller letter.)*

Proof. The first letter of C_i is smaller than or equal to all letters which came before it, so it is placed onto the top of the tallest column. To argue inductively, assume that Lemma 3.3 is true after the first j letters of C_i have been placed. The next letter α is greater than each of the other letters, therefore its inertion path lies below that of the other letters, so the first place it can terminate is on top of the tallest column which has not yet been a termination point for a letter of C_i. The highest entry in this column is greater than or equal to the letter β which has been most recently bumped, so β is placed on top of this column and the proof is complete. □

Proposition 3.2. *The shape of $\Psi(T)$ is a rearrangement of the shape of T.*

Proof. Argue by induction on the number of columns of T. First assume that T contains only one column. The shape of T is 1^n. Then $col(T)$ is a strictly decreasing word. Therefore each letter maps to the bottom row of the semi-standard skyline filling. The resulting shape is an arrangement of zeros and ones, a rearrangement of 1^n.

Next, assume that if T contains j columns then the shape of $\Psi(T)$ is a rearrangement of the shape of T. Let T be an SSYT of shape λ which contains $j+1$ columns. After mapping the first j columns of T, the shape of the resulting figure is a rearrangement of $(\lambda_1 - 1, \lambda_2 - 1, ...)$. By Lemma 3.3, mapping the next column into the shape adds one cell to each existing column, plus possibly several new cells if the $(j+1)^{th}$ column is taller than the j^{th} column. Therefore the resulting shape is a rearrangement of $(\lambda_1, \lambda_2, ...) = \lambda$. □

Proposition 3.3. *The map Ψ is invertible.*

Proof. Consider the set S of cells which are in the top row of some column. Of these, begin with the cell c which is farthest right in the reading order. This was the last cell to be bumped into place. Scan backwards through the reading order to find the next cell, d, such that $\sigma(d) > \sigma(c)$ and d lies directly below an entry less than or equal to c. This entry $\sigma(d)$ bumped $\sigma(c)$. Repeat with $\sigma(d)$. Continue this scanning procedure until there are no cells farther back in the reading order which could have bumped the selected entry, e. This entry is the first letter in $col(T)$.

Choose the next element of S to appear in the backwards reading order. (If there are no other cells in S, create a new set S' consisting of all the cells which are in the top row of some column.) Move backwards from this element through the reading word to determine the initial element whose placement terminated with this particular element. Continue this procedure until the entire word $col(T)$ has been determined. This procedure inverts the map Ψ. □

The map Ψ : SSYT $\rightarrow$ SSSF is a weight-preserving invertible map between semi-standard Young tableaux and semi-standard skyline fillings. In particular, this means that the number of SSYT of shape λ with weight $\prod x_i^{a_i}$ is equal to the number of SSSF with weight $\prod x_i^{a_i}$ whose shape rearranges λ. Thus the coefficient of $\prod x_i^{a_i}$ in $\sum_{\lambda'} NS_{\lambda'}$ is equal to the coefficient of $\prod x_i^{a_i}$ in s_λ. This completes the proof of Equation 3.1.

4. An analogue of the Robinson-Schensted-Knuth Algorithm

The insertion process utilized in the above bijection is reminiscent of Schensted insertion, the fundamental operation of the Robinson-Schensted-Knuth Algorithm.

Theorem 4.1. *(Robinson-Schensted-Knuth* [**9**]*) There exists a bijection between* $\mathbb{N}$*-matrices of finite support and pairs of semi-standard Young tableaux of the same shape.*

We apply the same procedure to arrive at an analogue of the RSK Algorithm for semi-standard skyline fillings. Recall that Theorem 1.2 states that there exists a bijection between $\mathbb{N}$-matrices of finite support and pairs of semi-standard skyline fillings whose shapes are rearrangements of the same partition.

4.1. ρ **:** $\mathbb{N}$**-matrices** $\longrightarrow$ **SSSF** $\times$ **SSSF.** Let $A = (a_{i,j})$ be an $\mathbb{N}$-matrix with finite support. There exists a unique two-line array corresponding to A which is defined by the non-zero entries in A. Beginning at the upper lefthand corner and reading left to right, top to bottom, find the first non-zero entry $a_{i,j}$. Place an i in the top line and a j in the bottom line $a_{i,j}$ times. When this has been done for each non-zero entry, we have the resulting array

$$w_A = \begin{pmatrix} i_1 & i_2 & \dots \\ j_1 & j_2 & \dots \end{pmatrix}$$

Begin with an empty semi-standard skyline filling F. Read the bottom row from right to left, inserting the entries into F according to the map Ψ described above as they are read. Each time an entry from the bottom line is placed, send the entry directly above it into an SSSF G which records the place where a cell is added. If the cell j_k is added to the bottom row of F, the corresponding entry i_k is placed on the bottom row in the i_k^{th} column of G. If there is ambiguity about which column of G an entry is placed on, it is always placed on the leftmost possible column of the same height as the column in F on which its counterpart was placed. In this way the shape of G becomes a rearrangement of the shape of F. When the process is complete, the result is a pair (F, G) of SSSF whose shapes are rearrangements of the same partition.

4.2. The inverse of the map ρ**.** Given (F, G), a pair of semi-standard skyline fillings whose shapes are rearrangements of the same partition μ, let G_{rs} be the highest occurrence of the smallest entry of G. Here G_{rs} is the element of G in row r and column s. Since equal elements of G are inserted bottom to top, it follows that $G_{rs} = i_1$ and $F_{rs'}$ was the last element of F to be bumped into place after inserting j_1. (If s is the i^{th} column of height r in G, then s' is the i^{th} column of height r in F.)

Delete $F_{rs'}$ from F and G_{rs} from G. Scan right to left, bottom to top (backwards through the reading word) starting with the cell directly to the left of $F_{rs'}$ to determine which (if any) cell bumped $F_{rs'}$. If there exists a cell k before $F_{rs'}$ in the reading word such that $\sigma(k) > \sigma(F_{rs'})$ and the cell directly on top of k has value less than or equal to $\sigma(F_{rs'})$, this k bumped $F_{rs'}$. Replace $\sigma(k)$ by $\sigma(F_{rs'})$ and repeat the procedure with $\sigma(k)$. Continue working backward through the reading word until there are no more letters. The remaining entry is the letter i_1.

Next find the highest occurrence of the smallest entry j_2 of G. Repeat the procedure to find i_2. Continue until there are no more entries in F and G. Then all of the i and j values of the array w_A have been determined, and the process is inverted.

5. The standard bases of Lascoux and Schützenberger

The Schubert polynomials were introduced by Lascoux and Schützenberger [**7**] as a combinatorial tool for attacking problems in algebraic geometry. The Schubert polynomials can be described as a sum of *standard bases*, $\mathfrak{U}(\mu, I)$, where μ is a permutation and I is a partition. Lascoux and Schützenberger [**6**] define an action of the symmetric group on the free algebra and this action is used to build the standard bases inductively.

5.1. Constructing the standard bases. Each permutation in the symmetric group can be decomposed into a series of elementary transpositions, so it is enough to define the action for a simple transposition, σ_i, which permutes i and $i+1$. The operator $\overline{\pi}_i = \overline{\pi}_{\sigma_i}$ is

$$f \longrightarrow (f^{\sigma_i} - f)/(1 - x_i/x_{i+1}) = f\overline{\pi}_i,$$

where f^{σ_i} denotes the transposition action of σ_i on the indices of the variables in f.

The operators $\overline{\pi}_i$ satisfy the Coxeter relations $\overline{\pi}_i\overline{\pi}_{i+1}\overline{\pi}_i = \overline{\pi}_{i+1}\overline{\pi}_i\overline{\pi}_{i+1}$ and $\overline{\pi}_i\overline{\pi}_j = \overline{\pi}_j\overline{\pi}_i$ for $\|j-i\| > 1$. We can lift the operator $\overline{\pi}_i$ into an operator θ_i on the free algebra by the following process. Given i and a word w in the alphabet X, let m be the number of times the letter x_{i+1} occurs in w and let $m+k$ be the number of times the letter x_i occurs in w. Then if $k \geq 0$, w and w^{σ_i} differ by the exchange of a subword x_i^k into x_{i+1}^k. If $k < 0$, then w and w^{σ_i} differ by the exchange of x_{i+1}^{-k} into x_i^{-k}.

When $k \geq 0$, define $w\theta_i$ to be the sum of all words in which the subword x_i^k has been changed respectively into $x_i^{k-1}x_{i+1}, x_i^{k-2}x_{i+1}^2, ..., x_{i+1}^k$. When $k < 0$, define $w\theta_i$ to be $-(w^{\sigma_i})\theta_i$. (This second case will not be needed in this paper.)

Every partition $I = (I_1, I_2, ...)$ has a corresponding *dominant monomial* $x^I = (x_{I_1}...x_2x_1)(x_{I_2}...x_2x_1)...$, which equals the weight of the *super tableau*, which is the SSYT with is in the i^{th} row. We take the following theorem to be the definition of the *standard basis* $\mathfrak{U}(\mu, I)$ associated to the pair μ, I (where μ is a permutation and I is a partition).

THEOREM 5.1. *(Lascoux-Schützenberger* [**6**]*) Let x^I be a dominant monomial and $\sigma_i\sigma_j...\sigma_k$ be any reduced decomposition of a permutation μ. Then $\mathfrak{U}(\mu, I) = x^I\theta_i\theta_j...\theta_k$.*

Theorem 5.1 provides an inductive method for constructing the standard basis $\mathfrak{U}(\mu, I)$. Begin with $\mathfrak{U}(id, I)$ and apply θ_i to determine $\mathfrak{U}(\sigma_i, I)$. Then apply θ_j to $\mathfrak{U}(\sigma_i, I)$ to get $\mathfrak{U}(\sigma_i\sigma_j, I)$. Continue this process until the desired standard basis is obtained. Figure 5.1 depicts all the standard bases for the partition $(2, 1)$.

5.2. A non-inductive construction of the standard bases. The standard bases with partition λ can be considered as a decomposition of the Schur function s_λ. For any partition λ of n, we have

$$\sum_{\sigma \in S_n} \mathfrak{U}(\sigma, \lambda) = s_\lambda.$$

Since the nonsymmetric Schur functions are also a decomposition of the Schur functions, it is natural to determine their relationship to the standard bases. Theorem 1.3 states that $\mathfrak{U}(\mu, I) = NS_{\mu(I)}$, where $\mu(I)$ denotes the action of μ on the parts of I. To prove Theorem 1.3, we need a few lemmas.

LEMMA 5.2. *Let col(T) be the column reading word of T. Label the occurrences of the entry α in col(T) in increasing order starting from the right. Then $i < j \Rightarrow$ the i^{th} occurrence of α (denoted α_i) in $\Psi(T)$ is in a lower row than the j^{th} occurrence of α, denoted α_j.*

PROOF. Consider the step during which α_j is being placed. At this step, the α_i is already placed in some row of the partial semi-standard skyline filling. If α_j reaches a cell a with $\sigma(a) = \alpha$ without being placed, the cell b on top of α must have $\sigma(b) < \alpha$. Therefore, α_j will bump $\sigma(b)$ and be placed on top of a. Therefore, α_j will always remain in a higher row than α_i. □

LEMMA 5.3. *Given an arbitrary semi-standard skyline filling F with row entries $R_1, R_2, ...R_k$, where $k = max_i\{\gamma_i\}$, F is the only SSSF with these row entries.*

PROOF. Given the row entries $R_1, R_2, ..., R_k$, map them into a semi-standard skyline filling as follows. Let α_1 be the largest entry in R_1. Place α_1 as far left as possible in the first row of an empty SSSF. Next place the second largest entry of R_1 as far left as possible in the first row of the SSSF. Continue placing the elements of R_1 in this manner. Next, choose the largest entry of R_2. Place it as far left as possible in the second row of the partially constructed SSSF. Continue this procedure until the smallest entry of R_2 has been placed. Do this for each of the k rows. Once R_k has been placed, the resulting figure is indeed a semi-standard skyline filling, and the only SSSF with row entries $R_1, R_2, ..., R_k$. □

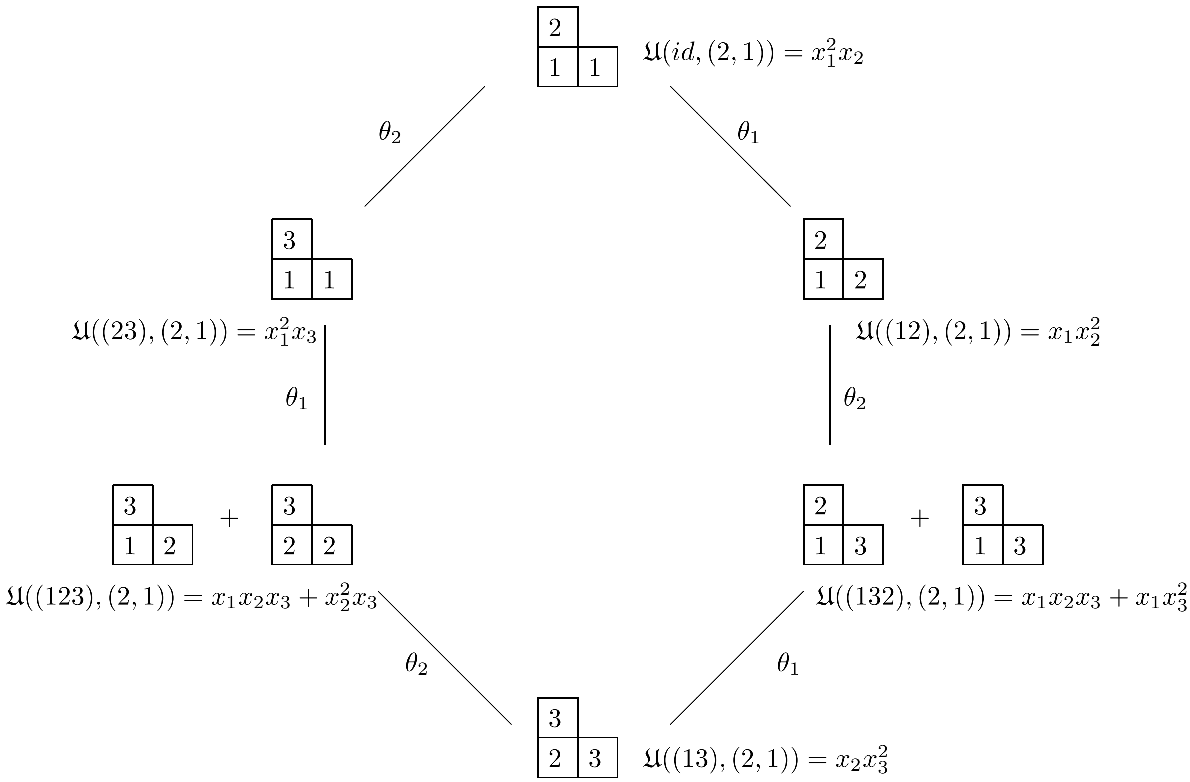


Figure 5.1. The standard bases for the partition $(2,1)$ and all permutations in S_3.

Let $\tilde{\theta_i}$ be the action of θ_i on an individual semi-standard Young tableau. This action is described by a matching procedure. Let $col(T)$ be the column word of T and let $(i+1)_1$ be the leftmost occurrence of $i+1$ in $col(T)$. Match $(i+1)_1$ with the leftmost occurrence of i which lies to the right of $(i+1)_1$ in $col(T)$. If there is no such i, the matching procedure is complete. Otherwise, continue with the next $i+1$ until there are no more occurrences of $i+1$.

When the matching procedure is complete, send the rightmost occurrence of i to $i+1$. The resulting word is $\tilde{\theta_i}(T) = T'$. If $T \in \mathfrak{U}(\mu, I)$ then either $\tilde{\theta_i}(T) \in \mathfrak{U}(\mu, I)$ or $\tilde{\theta_i}(T) \in \mathfrak{U}(\sigma_i\mu, I)$.

Lemma 5.4. *There exists a map Θ_i :SSSF$\longrightarrow$SSSF such that for $F \in NS_\mu$, either $\Theta_i(F) \in NS_\mu$ or $\Theta_i(F) \in NS_{\sigma_i\mu}$ and the following diagram commutes.*

$$\begin{tikzcd} T \arrow[r, "\tilde{\theta_i}"] \arrow[d, "\Psi"] & T' \arrow[d, "\Psi"] \\ F \arrow[r, "\Theta_i"] & F' \end{tikzcd}$$

Proof. Let F be an arbitrary semi-standard skyline filling and let $leftread(F)$ be the reading word obtained by reading F right to left, top to bottom, keeping track of the rows. Find the first entry a of this word such that $\sigma(a) = i+1$. Match this entry $i+1$ to the first $\sigma(b)$ which lies to the right of a in $leftread(F)$ such that $\sigma(b) = i$. If there is no such b, $\sigma(a)$ is unmatched and the matching process is complete. Continue this matching until an unmatched $i+1$ is reached.

Pick the rightmost unmatched i. Change it to $i+1$. (If there is none, then $\Theta_i(F) = F$.) The result is a collection of rows which differ from $leftread(F)$ in precisely one entry. Lemma 5.3 provides a procedure for mapping this collection of rows to a unique SSSF. This SSSF is $\Theta_i(F) = F'$, and $\Theta_i(\Psi(T)) = \Psi(\tilde{\theta_i}(T))$. So the diagram commutes.

Assume $F \in NS_\mu$. When $\Theta_i(F) = F$, $\Theta_i(F) \in NS_\mu$. We must show that in the case where an unmatched i is mapped to $i+1$, the resulting semi-standard skyline filling is either in NS_μ or in $NS_{\sigma_i\mu}$. But the map Θ_i shifts the highest unmatched i in F. If there are no occurrences of the letter i in the row directly below the shifted i, this $i+1$ is mapped to the same position, as are all the cells above it. So the shape of the diagram remains the same. Otherwise, there is a column consisting only of is and a column consisting only of $(i+1)$s below the first unmatched i. Sending this i to $i+1$ moves it into the $(i+1)^{th}$ column and therefore permutes the i^{th} and $(i+1)^{th}$ column, resulting in the shape $\sigma_i\mu$. So our proof is complete.

□

5.3. Proof of Theorem 1.3. We fix a partition I and argue by induction on the length of the permutation μ in $\mathfrak{U}(\mu, I)$. Let μ be the identity. Then $\mathfrak{U}(\mu, I)$ is the dominant monomial. Consider I as a composition of n into n parts by adding zeros to the right if necessary. Each entry a in I_1 must have $\sigma(a) = 1$, for otherwise there would be a descent. If the second column contained an entry b such that $\sigma(b) = 1$, this cell and the cell in the row directly below of the first column would be attacking. Continuing in this manner, we see that the i^{th} column must have $\sigma(c) = i$ for each cell c. Therefore, the $NS_I = \mathfrak{U}(\mu, I)$.

Next assume that $\mathfrak{U}(\mu, I) = NS_{\mu(I)}$, where $\mu(I)$ is the permutation μ applied to the columns of I when I is considered as a composition of n into n parts. The monomials in $\mathfrak{U}(\sigma_i\mu, I)$ are the monomials of $\mathfrak{U}(\mu, I)$ whose image under (possibly multiple applications of) $\tilde{\theta}_i$ is not a monomial of $\mathfrak{U}(\mu, I)$. Pick some such monomial, represented by the SSYT T. By Lemma 5.4 $\Psi(\tilde{\theta}_i(T)) = \Theta_i(\Psi(T))$. Since $\Psi(T) \in NS_\mu$, $\Theta_i(\Psi(T)) \in NS_\mu$ or $\Theta_i(\Psi(T)) \in NS_{\sigma_i\mu}$. If $\Theta_i(\Psi(T)) \in NS_\mu$, then $\tilde{\theta}_i(T) \in \mathfrak{U}(\mu, I)$ by assumption. But this is a contradiction, so $\Theta_i(\Psi(T)) \in NS_{\sigma_i\mu}$. Therefore, $\mathfrak{U}(\sigma_i\mu, I) \subseteq NS_{\sigma_i\mu}$. If F is a monomial in $NS_{\sigma_i\mu}$, one can determine an element of NS_μ which, after possibly multiple applications of Θ_i maps to F. Therefore $NS_{\sigma_i\mu} \subseteq \mathfrak{U}(\mu, I)$. So $NS_{\sigma_i\mu} = \mathfrak{U}(\mu, I)$.

6. Applications of Theorems 1.2 and 1.3

The analogue of the Robinson-Schensted-Knuth algorithm (Theorem 1.2) can be used to extend results about plane partitions and permutation enumeration. The non-inductive description of standard bases provided in Theorem 1.3 facilitates our understanding of the representation theory of Schubert polynomials and nonsymmetric Schur functions.

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