Formal Power Series and Algebraic Combinatorics
Séries Formelles et Combinatoire Algébrique
San Diego, California 2006

Representation theories of some towers of algebras related to the symmetric groups and their Hecke algebras

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Abstract. We study the representation theory of three towers of algebras which are related to the symmetric groups and their Hecke algebras. The first one is constructed as the algebras generated simultaneously by the elementary transpositions and the elementary sorting operators acting on permutations. The two others are the monoid algebras of nondecreasing functions and nondecreasing parking functions. For these three towers, we describe the structure of simple and indecomposable projective modules, together with the Cartan map. The Grothendieck algebras and coalgebras given respectively by the induction product and the restriction coproduct are also given explicitly. This yields some new interpretations of the classical bases of quasi-symmetric and noncommutative symmetric functions as well as some new bases.

Résumé. Nous étudions la théorie des représentations de trois tours d'algèbres liées aux groupes symétriques et à leurs algèbres de Hecke. La première est formée des algèbres engendrées par les transpositions élémentaires ainsi que les opérateurs de tris élémentaires agissant sur les permutations. Les deux autres sont formées des algèbres des monoïdes des fonctions croissantes et des fonctions de parking croissantes. Pour ces trois tours, nous donnons la structure des modules simples et projectifs indécomposables ainsi que l'application de Cartan. Nous calculons également explicitement les algèbres et cogèbres de Grothendieck pour le produit d'induction et le coproduit de restriction. Il en découle de nouvelles interprétations de bases connues des fonctions quasi-symétriques et symétriques noncommutatives ainsi que des nouvelles bases.

Contents

1. Introduction

Given an *inductive tower of algebras*, that is a sequence of algebras

$$A_0 \hookrightarrow A_1 \hookrightarrow \cdots \hookrightarrow A_n \hookrightarrow \cdots\,, \tag{1}$$

with embeddings $A_m \otimes A_n \hookrightarrow A_{m+n}$ satisfying an appropriate associativity condition, one can introduce two *Grothendieck rings*

$$\mathcal{G}(A) := \bigoplus_{n\geq 0} G_0(A_n) \qquad \text{and} \qquad \mathcal{K}(A) := \bigoplus_{n\geq 0} K_0(A_n)\,, \tag{2}$$

1991 *Mathematics Subject Classification.* Primary 16G99; Secondary 05E05.
Key words and phrases. Representation theory, towers of algebras, Grothendieck groups, symmetric groups, Hecke algebras, Quasi-symmetric and Noncommutative symmetric functions.

where $G_0(A)$ and $K_0(A)$ are the (complexified) Grothendieck groups of the categories of finite-dimensional A-modules and projective A-modules respectively, with the multiplication of the classes of an A_m-module M and an A_n-module N defined by the induction product

$$[M] \cdot [N] = [M \widehat{\otimes} N] = [M \otimes N \uparrow_{A_m \otimes A_n}^{A_{m+n}}] \,. \tag{3}$$

If A_{m+n} is a projective $A_m \otimes A_n$ modules, one can define a coproduct on these rings by means of restriction of representations, turning these into coalgebras. Under favorable circumstances the product and the coproduct are compatible turning these into mutually dual Hopf algebras.

The basic example of this situation is the character ring of the symmetric groups (over $\mathbb{C}$), due to Frobenius. Here the $A_n := \mathbb{C}[\mathfrak{S}_n]$ are semi-simple algebras, so that

$$G_0(A_n) = K_0(A_n) = R(A_n) \,, \tag{4}$$

where $R(A_n)$ denotes the vector space spanned by isomorphism classes of indecomposable modules which, in this case, are all simple and projective. The irreducible representations $[\lambda]$ of A_n are parametrized by partitions λ of n, and the Grothendieck ring is isomorphic to the algebra Sym of symmetric functions under the correspondence $[\lambda] \leftrightarrow s_\lambda$, where s_λ denotes the Schur function associated with λ. Other known examples with towers of group algebras over the complex numbers $A_n := \mathbb{C}[G_n]$ include the cases of wreath products $G_n := \Gamma \wr \mathfrak{S}_n$ (Specht), finite linear groups $G_n := GL(n, \mathbb{F}_q)$ (Green), *etc.*, all related to symmetric functions (see [**11, 16**]).

Examples involving non-semisimple specializations of Hecke algebras have also been worked out. Finite Hecke algebras of type A at roots of unity ($A_n = H_n(\zeta)$, $\zeta^r = 1$) yield quotients and subalgebras of Sym [**10**]. The Ariki-Koike algebras at roots of unity give rise to level r Fock spaces of affine Lie algebras of type A [**2**]. The 0-Hecke algebras $A_n = \mathrm{H}_n(0)$ correspond to the pair Quasi-symmetric functions / Noncommutative symmetric functions, $\mathcal{G} = \mathrm{QSym}$, $\mathcal{K} = \mathrm{NCSF}$ [**9**]. Affine Hecke algebras at roots of unity lead to $U(\widehat{sl}_r)$ and $U(\widehat{sl}_r)^*$ [**1**], and the case of affine Hecke generic algebras can be reduced to a subcategory admitting as Grothendieck rings $U(\widehat{gl}_\infty)$ and $U(\widehat{gl}_\infty)^*$ [**1**]. Further interesting examples are the tower of 0-Hecke-Clifford algebras [**13, 3**] giving rise to the peak algebras [**15**], and a degenerated version of the Ariki-Koike algebras [**7**] giving rise to a colored version of QSym and NCSF.

The goal of this article is to study the representation theories of several towers of algebras which are related to the symmetric groups and their Hecke algebras $\mathrm{H}_n(q)$. We describe their representation theory and the Grothendieck algebras and coalgebras arising from them. Here is the structure of the paper together with the main results.

In Section 3, we introduce the main object of this paper, namely a new tower of algebras denoted $\mathrm{H}\mathfrak{S}_n$. Each $\mathrm{H}\mathfrak{S}_n$ is constructed as the algebra generated by both elementary transpositions and elementary sorting operators acting on permutations of $\{1, \dots, n\}$. We show that this algebra is better understood as the algebra of antisymmetry preserving operators; this allows us to compute its dimension and give an explicit basis. Then, we construct the projective and simple modules and compute their restrictions and inductions. This gives rise to a new interpretation of some bases of quasi-symmetric and noncommutative symmetric functions in representation theory. The Cartan matrix suggests a link between $\mathrm{H}\mathfrak{S}_n$ and the incidence algebra of the boolean lattice. We actually show that these algebra are Morita equivalent. We conclude this section by discussing some links with a certain central specialization of the affine Hecke algebra.

In Sections 4 and 5 we turn to the study of two other towers, namely the towers of the monoids algebras of nondecreasing functions and of nondecreasing parking functions. In both cases, we give the structure of projective and simple modules, the cartan matrices, and the induction and restrictions rules. We also show that the algebra of nondecreasing parking functions is isomorphic to the incidence algebra of some lattice. Finally, we prove that those two algebras are the respective quotients of $\mathrm{H}\mathfrak{S}_n$ and $\mathrm{H}_n(0)$, through their representations on exterior powers. The following diagram summarizes the relations between all the mentioned towers of algebras:

$$\begin{tikzcd}
\mathrm{H}_n(-1) \arrow[rrrr, hook, bend left] \arrow[d] & \mathrm{H}_n(0) \arrow[rrr, hook, bend left] \arrow[d] & \mathrm{H}_n(1) = \mathbb{C}[\mathfrak{S}_n] \arrow[rr, hook, bend left] \arrow[d] & \mathrm{H}_n(q) \arrow[r, hook] \arrow[d] & H\mathfrak{S}_n \arrow[d] \\
\text{Temperley-Lieb}_n \arrow[rrrr, hook, bend right] & \mathbb{C}[\mathrm{NDPF}_n] \arrow[rrr, hook, bend right] & \mathbb{C}[\mathfrak{S}_n] \hookrightarrow \bigwedge^{\cdot} \mathbb{C}^n \arrow[rr, hook, bend right] & \mathrm{H}_n(q) \hookrightarrow \bigwedge^{\cdot} \mathbb{C}^n \arrow[r, hook] & \mathbb{C}[\mathrm{NDF}_n]
\end{tikzcd} \tag{5}$$

This paper mostly reports on a computation driven research using the package `MuPAD-Combinat` by the authors of the present paper [**8**]. This package is designed for the computer algebra system `MuPAD` and is freely available from `http://mupad-combinat.sf.net/`. Among other things, it allows to automatically compute the dimensions of simple and indecomposable projective modules together with the Cartan invariants matrix of a finite dimensional algebra, knowing its multiplication table.

2. Background

2.1. Compositions and sets. Let n be a fixed integer. Recall that each subset S of $\{1,\dots,n-1\}$ can be uniquely identified with a p-tuple $K := (k_1,\dots,k_p)$ of positive integers of sum n:

$$S = \{i_1 < i_2 < \cdots < i_p\} \longmapsto \mathrm{C}(S) := (i_1, i_2 - i_1, i_3 - i_2, \dots, n - i_p)\,. \tag{6}$$

We say that K is a *composition of* n and we write it by $K \vDash n$. The converse bijection, sending a composition to its *descent set*, is given by:

$$K = (k_1,\dots,k_p) \longmapsto \mathrm{Des}(K) = \{k_1 + \cdots + k_j,\ j = 1,\dots,p-1\}\,. \tag{7}$$

The number p is called the *length* of K and is denoted by $\ell(K)$.

The notions of complementary of a set S^c and of inclusion of sets can be transfered to compositions, leading to the complementary of a composition K^c and to the refinement order on compositions: we say that I is *finer* than J, and write $I \succeq J$, if and only if $\mathrm{Des}(I) \supseteq \mathrm{Des}(J)$.

2.2. Symmetric groups and Hecke algebras. Take $n \in \mathbb{N}$ and let $\mathfrak{S}_n$ be the n-th symmetric group. It is well known that it is generated by the $n-1$ elementary transpositions σ_i which exchange i and $i+1$, with the relations

$$\begin{aligned} \sigma_i^2 &= 1 && (1 \le i \le n-1)\,,\\ \sigma_i\sigma_j &= \sigma_j\sigma_i && (|i-j| \ge 2)\,,\\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} && (1 \le i \le n-2)\,. \end{aligned} \tag{8}$$

The last two relations are called the *braids relations*. A *reduced word* for a permutation μ is a decomposition $\mu = \sigma_{i_1}\cdots\sigma_{i_k}$ of minimal length. When denoting permutations we also use the *word notation*, where μ is denoted by the word $\mu_1\mu_2\cdots\mu_n := \mu(1)\mu(2)\cdots\mu(n)$. For a permutation μ, the set $\{i,\, \mu_i > \mu_{i+1}\}$ of its *descents* is denoted $\mathrm{Des}(\mu)$. The descents of the inverse of μ are called the *recoils of* μ and their set is denoted $\mathrm{Rec}(\mu)$. For a composition I, we denote by $\mathfrak{S}_I := \mathfrak{S}_{i_1} \times \cdots \times \mathfrak{S}_{i_p}$ the *standard Young subgroup* of $\mathfrak{S}_n$, which is generated by the elementary transpositions σ_i where $i \notin \mathrm{Des}(I)$.

Recall that the (Iwahori-) *Hecke algebra* $\mathrm{H}_n(q)$ of type A_{n-1} is the $\mathbb{C}$-algebra generated by elements T_i for $i < n$ with the braids relations together with the quadratic relations:

$$T_i^2 = (q-1)T_i + q\,, \tag{9}$$

where q is a complex number.

The 0-Hecke algebra is obtained by setting $q = 0$ in these relations. Then, the first relation becomes $T_i^2 = -T_i$ [**12, 9**]. In this paper, we prefer to use another set of generators $(\pi_i)_{i=1\dots n-1}$ defined by $\pi_i := T_i + 1$. They also satisfy the braids relations together with the quadratic relations $\pi_i^2 = \pi_i$.

Let $\sigma =: \sigma_{i_1}\cdots\sigma_{i_p}$ be a reduced word for a permutation $\sigma \in \mathfrak{S}_n$. The defining relations of $\mathrm{H}_n(q)$ ensures that the element $T_\sigma := T_{i_1}\cdots T_{i_p}$ (resp.: $\pi_\sigma := \pi_{i_1}\cdots\pi_{i_p}$) is independent of the chosen reduced word for σ. Moreover, the well-defined family $(T_\sigma)_{\sigma\in\mathfrak{S}_n}$ (resp.: $(\pi_\sigma)_{\sigma\in\mathfrak{S}_n}$) is a basis of the Hecke algebra, which is consequently of dimension $n!$.

2.3. Representation theory. In this paper, we mostly consider *right* modules over algebras. Consequently the composition of two endomorphisms f and g is denoted by $fg = g\circ f$ and their action on a vector v is written $v \cdot f$. Thus $g \circ f(v) = g(f(v))$ is denoted $v \cdot fg = (v\cdot f)\cdot g$.

It is known that $\mathrm{H}_n(0)$ has 2^{n-1} simple modules, all one-dimensional, and naturally labelled by compositions I of n [**12**]: following the notation of [**9**], let η_I be the generator of the simple $\mathrm{H}_n(0)$-module S_I associated with I in the left regular representation. It satisfies

$$\eta_I \cdot T_i := \begin{cases} -\eta_I & \text{if } i \in \mathrm{Des}(I),\\ 0 & \text{otherwise,}\end{cases} \qquad \text{or equivalently} \qquad \eta_I \cdot \pi_i := \begin{cases} 0 & \text{if } i \in \mathrm{Des}(I),\\ \eta_I & \text{otherwise.}\end{cases} \tag{10}$$

The bases of the indecomposable projective modules P_I associated to the simple module S_I of $\mathrm{H}_n(0)$ are indexed by the permutations σ whose descents composition is I.

The Grothendieck rings of $\mathrm{H}_n(0)$ are naturally isomorphic to the dual pair of Hopf algebras of quasi-symmetric functions QSym of Gessel [**6**] and of noncommutative symmetric functions NCSF [**5**] (see [**9**]). The reader who is not familiar with those should refer to these papers, as we will only recall the required notations here.

The Hopf algebra QSym of quasi-symmetric functions has two remarkable bases, namely the *monomial basis* $(M_I)_I$ and the *fundamental basis* (also called *quasi-ribbon*) $(F_I)_I$. They are related by

$$F_I = \sum_{I \succeq J} M_J \qquad \text{or equivalently} \qquad M_I = \sum_{I \succeq J} (-1)^{\ell(I)-\ell(J)} F_J\,. \tag{11}$$

The characteristic map $S_I \mapsto F_I$ which sends the simple $\mathrm{H}_n(0)$ module S_I to its corresponding fundamental function F_I also sends the induction product to the product of QSym and the restriction coproduct to the coproduct of QSym.

The Hopf algebra NCSF of noncommutative symmetric functions [**5**] is a noncommutative analogue of the algebra of symmetric functions [**11**]. It has for multiplicative bases the analogues $(\Lambda^I)_I$ of the elementary symmetric functions $(e_\lambda)_\lambda$ and as well as the analogues $(S^I)_I$ of the complete symmetric functions $(h_\lambda)_\lambda$. The relevant basis in the representation theory of $\mathrm{H}_n(0)$ is the basis of so called *ribbon Schur functions* $(R_I)_I$ which is an analogue of skew Schur functions of ribbon shape. It is related to $(\Lambda_I)_I$ and $(S_I)_I$ by

$$S_I = \sum_{I \succeq J} R_J \qquad \text{and} \qquad \Lambda_I = \sum_{I \succeq J} R_{J^c}\,. \tag{12}$$

Their interpretation in representation theory goes as follows. The complete function S^n is the characteristic of the trivial module $S_n \approx P_n$, the elementary function Λ^n being the characteristic of the sign module $S_{1^n} \approx P_{1^n}$. An arbitrary indecomposable projective module P_I has R_I for characteristic. Once again the map $P_I \mapsto R_I$ is an isomorphism of Hopf algebras.

Recall that S_J is the semi-simple module associated to P_I, giving rise to the duality between $\mathcal{G}$ and $\mathcal{K}$:

$$S_I = P_J/\mathrm{rad}(P_J) \qquad \text{and} \qquad \langle P_I\ ,\ S_J\rangle = \delta_{I,J} \tag{13}$$

This translates into QSym and NCSF by setting that $(F_I)_I$ and $(R_I)_I$ are dual bases, or equivalently that $(M_I)_I$ and $(S^I)_I$ are dual bases.

3. The algebra $\mathrm{H}\mathfrak{S}_n$

The algebra of the symmetric group $\mathbb{C}[\mathfrak{S}_n]$ and the 0-Hecke algebra $\mathrm{H}_n(0)$ can be realized simultaneously as operator algebras by identifying the underlying vector spaces of their right regular representations.

Namely, consider the plain *vector space* $\mathbb{C}\mathfrak{S}_n$ (distinguished from the *group algebra* which is denoted by $\mathbb{C}[\mathfrak{S}_n]$). On the first hand, the algebra $\mathbb{C}[\mathfrak{S}_n]$ acts naturally on $\mathbb{C}\mathfrak{S}_n$ by multiplication on the right (action on positions). That is, a transposition σ_i acts on a permutation $\mu := (\mu_1, \dots, \mu_n)$ by permuting μ_i and μ_{i+1}: $\mu \cdot \sigma_i = \mu\sigma_i$.

On the other hand, the 0-Hecke algebra $\mathrm{H}_n(0)$ acts on the right on $\mathbb{C}\mathfrak{S}_n$ by decreasing sort. That is, π_i acts on the right on μ by:

$$\mu \cdot \pi_i = \begin{cases} \mu & \text{if } \mu_i > \mu_{i+1}, \\ \mu\sigma_i & \text{otherwise.} \end{cases} \tag{14}$$

DEFINITION 1. For each n, the algebra $\mathrm{H}\mathfrak{S}_n$ is the subalgebra of $\mathrm{End}(\mathbb{C}\mathfrak{S}_n)$ generated by both sets of operators $\sigma_1, \dots, \sigma_{n-1}, \pi_1, \dots, \pi_{n-1}$.

By construction, the algebra $\mathrm{H}\mathfrak{S}_n$ contains both $\mathbb{C}[\mathfrak{S}_n]$ and $\mathrm{H}_n(0)$. In fact, it contains simultaneously all the Hecke algebras $\mathrm{H}_n(q)$ for all values of q; each one can be realized by taking the subalgebra generated by the operators:

$$T_i := (q-1)(1-\pi_i) + q\sigma_i, \qquad \text{for } i = 1, \dots, n-1\,. \tag{15}$$

The natural embedding of $\mathbb{C}\mathfrak{S}_n \otimes \mathbb{C}\mathfrak{S}_m$ in $\mathbb{C}\mathfrak{S}_{n+m}$ makes $(\mathrm{H}\mathfrak{S}_n)_{n\in\mathbb{N}}$ into a tower of algebras, which contains the similar towers of algebras $(\mathbb{C}[\mathfrak{S}_n])_{n\in\mathbb{N}}$ and $(\mathrm{H}_n(q))_{n\in\mathbb{N}}$.

3.1. Basic properties of $\mathrm{H}\mathfrak{S}_n$. Let $\overline{\pi}_i$ be the *increasing sort operator* on $\mathbb{C}\mathfrak{S}_n$. Namely: $\overline{\pi}_i$ acts on the right on μ by:

$$\mu \cdot \overline{\pi}_i = \begin{cases} \mu & \text{if } \mu_i < \mu_{i+1}, \\ \mu\sigma_i & \text{otherwise.} \end{cases} \tag{16}$$

Since $\pi_i + \overline{\pi}_i$ is a symmetrizing operator, we have the identity:

$$\pi_i + \overline{\pi}_i = 1 + \sigma_i \ . \tag{17}$$

It follows that the operator $\overline{\pi}_i$ also belongs to $\mathrm{H}\mathfrak{S}_n$.

The following identities are also easily checked:

$$\begin{aligned} \sigma_i\pi_i = \pi_i \ , &\quad \sigma_i\overline{\pi}_i = \overline{\pi}_i \ , \\ \overline{\pi}_i\pi_i = \pi_i \ , &\quad \pi_i\overline{\pi}_i = \overline{\pi}_i \ , \\ \pi_i\sigma_i = \overline{\pi}_i \ , &\quad \overline{\pi}_i\sigma_i = \pi_i \ . \end{aligned} \tag{18}$$

A computer exploration suggests that the dimension of $\mathrm{H}\mathfrak{S}_n$ is given by the following sequence (sequence A000275 of the encyclopedia of integer sequences [**14**]):

$$1, 1, 3, 19, 211, 3651, 90921, 3081513, 136407699, 7642177651, 528579161353, 44237263696473, \dots$$

These are the numbers h_n of pairs (σ, τ) of permutations such that $\mathrm{Des}(\sigma) \cap \mathrm{Des}(\tau) = \emptyset$. Together with Equation (18), this leads to state the following

THEOREM 3.1. *A vector space basis of* $\mathrm{H}\mathfrak{S}_n$ *is given by the family of operators*

$$B_n := \left\{ \sigma\pi_\tau \mid \mathrm{Des}(\sigma) \cap \mathrm{Des}(\tau^{-1}) = \emptyset \right\} \ . \tag{19}$$

One approach to prove this theorem would be to find a presentation of the algebra. The following relations are easily proved to hold in $\mathrm{H}\mathfrak{S}_n$:

$$\begin{aligned} \pi_{i+1}\sigma_i &= \pi_{i+1}\pi_i + \sigma_i\sigma_{i+1}\pi_i\pi_{i+1} - \pi_i\pi_{i+1}\pi_i \ , \\ \pi_i\sigma_{i+1} &= \pi_i\pi_{i+1} + \sigma_{i+1}\sigma_i\pi_{i+1}\pi_i - \pi_i\pi_{i+1}\pi_i \ , \\ \sigma_1\pi_2\sigma_1 &= \sigma_2\pi_1\sigma_2 \ , \end{aligned} \tag{20}$$

and we conjecture that they generate all relations.

CONJECTURE 1. A presentation of $\mathrm{H}\mathfrak{S}_n$ is given by the defining relations of $\mathbb{C}[\mathfrak{S}_n]$ and $\mathrm{H}_n(0)$ together with the relations $\sigma_i\pi_i = \pi_i$ and of Equations (20).

Using those relations as rewriting rules yields a straightening algorithm which rewrites any expression in the σ_i's and π_i's into a linear combination of the $\sigma\pi_\tau$. This algorithm seems, in practice and with an appropriate strategy, to always terminate. However we have no proof of this fact; moreover this algorithm is not efficient, due to the explosion of the number and length of words in intermediate results.

This is a standard phenomenon with such algebras. Their properties often become clearer when considering their concrete representations (typically as operator algebras) rather than their abstract presentation. Here, theorem 3.1 as well as the representation theory of $\mathrm{H}\mathfrak{S}_n$ follow from its upcoming structural characterization as the algebra of operators preserving certain anti-symmetries.

3.2. $\mathrm{H}\mathfrak{S}_n$ as algebra of antisymmetry-preserving operators. Let $\overline{\sigma}_i$ be the *right operator* in $\mathrm{End}(\mathbb{C}\mathfrak{S}_n)$ describing the action of s_i by multiplication *on the left* (action on values), namely $\overline{\sigma}_i$ is defined by

$$\sigma \cdot \overline{\sigma}_i := \sigma_i\sigma \ . \tag{21}$$

A vector v in $\mathbb{C}\mathfrak{S}_n$ is *left i-symmetric* (resp. *antisymmetric*) if $v \cdot \overline{\sigma}_i = v$ (resp. $v \cdot \overline{\sigma}_i = -v$). The subspace of left i-symmetric (resp. antisymmetric) vectors can be alternatively described as the image (resp. kernel) of the idempotent operator $\frac{1}{2}(1 + \overline{\sigma}_i)$, or as the kernel (resp. image) of the idempotent operator $\frac{1}{2}(1 - \overline{\sigma}_i)$.

THEOREM 3.2. $\mathrm{H}\mathfrak{S}_n$ *is the subspace of* $\mathrm{End}(\mathbb{C}\mathfrak{S}_n)$ *defined by the* $n-1$ idempotent sandwich equations*:*

$$\frac{1}{2}(1 - \overline{\sigma}_i) f \frac{1}{2}(1 + \overline{\sigma}_i) = 0, \qquad \textit{for } i = 1, \dots, n-1 \ . \tag{22}$$

In other words, $\mathrm{H}\mathfrak{S}_n$ *is the subalgebra of those operators in* $\mathrm{End}(\mathbb{C}\mathfrak{S}_n)$ *which preserve left anti-symmetries.*

Note that, $\overline{\sigma}_i$ being self-adjoint, the adjoint algebra of $\mathrm{H}\mathfrak{S}_n$ satisfies the equations:

$$\frac{1}{2}(1+\overline{\sigma}_i)f\frac{1}{2}(1-\overline{\sigma}_i) = 0; \tag{23}$$

thus, it is the subalgebra of those operators in $\mathrm{End}(\mathbb{C}\mathfrak{S}_n)$ which preserve left symmetries. The symmetric group algebra has a similar description as the subalgebra of those operators in $\mathrm{End}(\mathbb{C}\mathfrak{S}_n)$ which preserve both left symmetries and antisymmetries.

PROOF. The proof of theorem 3.2 proceeds as follow. We first exhibit a triangularity property of the operators in B_n; this proves that they are linearly independent, so that $\dim \mathrm{H}\mathfrak{S}_n \geq h_n$. Let $<$ be any linear extension of the right permutahedron order. Given an endomorphism f of $\mathbb{C}\mathfrak{S}_n$, we order the rows and columns of its matrix $M := [f_{\mu\nu}]$ accordingly to $<$, and denote by $\mathrm{init}(f) := \min\{\mu, \exists\nu, f_{\mu\nu} \neq 0\}$ the index of the first non zero row of M.

LEMMA 3.1. *(a) Let $f := \sigma\pi_\tau$ in B_n. Then,* $\mathrm{init}(f) = \tau$, *and*

$$f_{\tau\nu} = \begin{cases} 1 & \text{if } \nu \in \mathfrak{S}_{\mathrm{Des}(\tau^{-1})}\sigma^{-1} \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

(b) The family B_n is free.

Then, we note that $\mathrm{H}\mathfrak{S}_n$ preserves all antisymmetries, because its generators σ_i and π_i do. It follows that $\mathrm{H}\mathfrak{S}_n$ satisfies the sandwich equations. We conclude by giving an explicit description of the sandwich equations. Given an endomorphism f of $\mathbb{C}\mathfrak{S}_n$, denote by $(f_{\mu,\nu})_{\mu,\nu}$ the coefficients of its matrix in the natural permutation basis. Given two permutations μ,ν, and an integer i in $\{1,\dots,n-1\}$, let $R_{\mu,\nu,i}$ be the linear form:

$$R_{\mu,\nu,i} : \begin{cases} \mathrm{End}(\mathbb{C}\mathfrak{S}_n) & \mapsto \mathbb{C} \\ f & \mapsto f_{\mu,\nu} + f_{s_i\mu,\nu} - f_{\mu,s_i\nu} + f_{s_i\mu,s_i\nu} \end{cases} \tag{25}$$

Given a pair of permutations μ,ν having at least one descent in common, set $R_{\mu,\nu} = R_{\mu,\nu,i}$, where i is the smallest common descent of μ and ν (the choice of the common descent i is, in fact, irrelevant). Finally, let $R_n := \{R_{\mu,\nu},\ \mathrm{Des}(\mu)\cap\mathrm{Des}(\nu) \neq \emptyset\}$.

LEMMA 3.2. *(a) If an operator f in $\mathrm{End}\mathbb{C}\mathfrak{S}_n$ preserves i-antisymmetries, then $R_{\mu,\nu,i}(f) = 0$ for any permutations μ and ν.*
(b) The $n!^2 - h_n$ linear relations in R_n are linearly independent.

Theorems 3.1 and 3.2 follow. □

3.3. The representation theory of $\mathrm{H}\mathfrak{S}_n$.

3.3.1. *Projective modules of* $\mathrm{H}\mathfrak{S}_n$. Recall that $\mathrm{H}\mathfrak{S}_n$ is the algebra of operators preserving left antisymmetries. Thus, given $S \subset \{1,\dots,n-1\}$, it is natural to introduce the $\mathrm{H}\mathfrak{S}_n$-submodule $\bigcap_{i\in S}\ker(1+\overline{\sigma}_i)$ of the vectors in $\mathbb{C}\mathfrak{S}_n$ which are i-antisymmetric for all $i \in S$. For the ease of notations, it turns out to be better to index this module by the composition associated to the *complementary set*; thus we define

$$P_I := \bigcap_{i\notin\mathrm{Des}(I)} \ker(1+\overline{\sigma}_i)\,. \tag{26}$$

The goal of this section is to prove that the family of modules $(P_I)_{I\vDash n}$ forms a complete set of representatives of the indecomposable projective modules of $\mathrm{H}\mathfrak{S}_n$.

The simplest element of P_I is:

$$v_I := \sum_{\nu\in\mathfrak{S}_I} (-1)^{l(\nu)}\nu, \tag{27}$$

One easily shows that

LEMMA 3.3. *v_I generates P_I as an $\mathrm{H}\mathfrak{S}_n$-module.*

Given a permutation σ, let $v_\sigma := v_{\mathrm{Rec}(\sigma)}\sigma$ (recall that $\mathrm{Rec}(\sigma) = \mathrm{Des}(\sigma^{-1})$). Note that σ is the permutation of minimal length appearing in v_σ. By triangularity, it follows that the family $(v_\sigma)_{\sigma\in\mathfrak{S}_n}$ forms a vector space basis of $\mathbb{C}\mathfrak{S}_n$. The usefulness of this basis comes from the fact that

Proposition 1. *For any composition $I := (i_1, \dots, i_k)$ of sum n, the families*

$$\{v_I \cdot \sigma \mid \sigma \in \mathfrak{S}_n,\ \mathrm{Rec}(\sigma) \cap \mathrm{Des}(I) = \emptyset\} \qquad \text{and} \qquad \{v_\sigma \mid \sigma \in \mathfrak{S}_n,\ \mathrm{Rec}(\sigma) \cap \mathrm{Des}(I) = \emptyset\} \tag{28}$$

are both vector space bases of P_I; in particular, P_I is of dimension $\frac{n!}{i_1!i_2!\dots i_k!}$.

Since $\mathfrak{S}_n$ and $\mathrm{H}_n(0)$ are both sub-algebras of $\mathrm{H}\mathfrak{S}_n$, the space P_I is naturally a module over them. The following proposition elucidates its structure.

Proposition 2. *Let (-1) denote the sign representation of the symmetric group as well as the corresponding representation of the Hecke algebra $\mathrm{H}_n(0)$ (sending T_i to -1, or equivalently π_i to 0).*

(a) *As a $\mathfrak{S}_n$ module, $P_I \approx (-1) \uparrow_{\mathfrak{S}_I}^{\mathfrak{S}_n}$; its character is the symmetric function $e_I := e_{i_1} \cdots e_{i_k}$.*

(b) *As a $\mathrm{H}_n(0)$ module, $P_I \approx (-1) \uparrow_{\mathrm{H}_I(0)}^{\mathrm{H}_n(0)}$; it is a projective module whose character is the noncommutative symmetric function $\Lambda^I := \Lambda_{i_1} \cdots \Lambda_{i_k}$.*

(c) *In particular the P_I's are non isomorphic as $\mathrm{H}_n(0)$-modules and thus as $\mathrm{H}\mathfrak{S}_n$-modules.*

We are now in position to state the main theorem of this section.

Theorem 3.3. *For $\sigma \in \mathfrak{S}_n$, let $p_\sigma \in \mathrm{End}(\mathbb{C}\mathfrak{S}_n)$ denote the projector on $\mathbb{C}v_\sigma$ parallel to $\oplus_{\tau \neq \sigma} \mathbb{C}v_\tau$. Then,*

(a) *The ideal $p_\sigma \mathrm{H}\mathfrak{S}_n$ is isomorphic to $P_{\mathrm{Rec}(\sigma)} = P_{\mathrm{Des}(\sigma^{-1})}$ as an $\mathrm{H}\mathfrak{S}_n$ module;*

(b) *The idempotents p_σ all belong to $\mathrm{H}\mathfrak{S}_n$; they give a maximal decomposition of the identity into orthogonal idempotents in $\mathrm{H}\mathfrak{S}_n$;*

(c) *The family of modules $(P_I)_{I \vDash n}$ forms a complete set of representatives of the indecomposable projective modules of $\mathrm{H}\mathfrak{S}_n$.*

Proof. Item (a) is an easy consequence of Proposition 1. To prove (b) one needs to check that p_σ belongs to $\mathrm{H}\mathfrak{S}_n$. This is done by showing that it preserves left antisymmetries. Then, since the p_σ's give a maximal decomposition of the identity in $\mathrm{End}(\mathbb{C}\mathfrak{S}_n)$, they are as well a maximal decomposition of the identity in $\mathrm{H}\mathfrak{S}_n$. Finally, Item (c) follows from (a) and (b) and Item (c) of Proposition 2. $\square$

3.3.2. *Simple modules.* The simple modules are obtained as quotients of the projective modules by their radical:

Theorem 3.4. *The modules $S_I := P_I / \sum_{J \subsetneq I} P_J$ form a complete set of representatives of the simple modules of $\mathrm{H}\mathfrak{S}_n$. Moreover, the projection of the family $\{v_\sigma, \mathrm{Rec}(\sigma) = I\}$ in S_I forms a vector space basis of S_I.*

The modules S_I are closely related to the projective modules of the 0-Hecke algebra:

Proposition 3. *The restriction of the simple module S_I to $\mathrm{H}_n(0)$ is an indecomposable projective module whose characteristic is the noncommutative symmetric function R_{I^c}.*

3.3.3. *Cartan's invariants matrix and the boolean lattice.* We now turn to the description of the Cartan matrix. Let $p_I := p_\alpha$ where α is the shortest permutation such that $\mathrm{Rec}(\alpha) = I$ (this choice is in fact irrelevant).

Proposition 4. *Let I and J be two subsets of $\{1, \dots, n\}$. Then,*

$$\dim Hom(P_I, P_J) = \dim p_I \mathrm{H}\mathfrak{S}_n p_J = \begin{cases} 1 & \text{if } I \subset J, \\ 0 & \text{otherwise.} \end{cases} \tag{29}$$

In other words, the Cartan matrix of $\mathrm{H}\mathfrak{S}_n$ is the incidence matrix of the boolean lattice. This suggests that there is a close relation between $\mathrm{H}\mathfrak{S}_n$ and the incidence algebra of the boolean lattice. Recall that the *incidence algebra* $\mathbb{C}[P]$ of a partially ordered set $(P, \leq_P)$ is the algebra whose basis elements are indexed by the couples $(u, v) \in P^2$ such that $u \leq_P v$ with the multiplication rule

$$(u, v) \cdot (u', v') = \begin{cases} (u, v') & \text{if } v = u', \\ 0 & \text{otherwise.} \end{cases} \tag{30}$$

An algebra is called *elementary* (or sometimes reduced) if its simple modules are all one dimensional. Starting from an algebra A, it is possible to get a canonical elementary algebra by the following process. Start with a maximal decomposition of the identity $1 = \sum_i e_i$ into orthogonal idempotents. Two idempotents e_i and e_j are *conjugate* if e_i can be written as ae_jb where a and b belongs to A, or equivalently, if the projective modules e_iA and e_jA are isomorphic. Select an idempotent e_c in each conjugacy classes c and put $e := \sum e_c$. Then, it is well known [**4**] that the algebra eAe is elementary and that the functor $M \mapsto Me$ which sends a right A module to a eAe module is an equivalence of category. Recall finally that two algebra A and B such that the category of A-modules and B-modules are equivalent are said *Morita equivalent*. Thus A and eAe are Morita-equivalent.

Applying this to $\mathrm{H}\mathfrak{S}_n$, one gets

THEOREM 3.5. *Let e be the idempotent defined by $e := \sum_{I \vDash n} p_I$. Then the algebra $e\mathrm{H}\mathfrak{S}_n e$ is isomorphic to the incidence algebra $\mathbb{C}[B_{n-1}]$ of the boolean lattice B_{n-1} of subsets of $\{1, \ldots, n-1\}$. Consequently, $\mathrm{H}\mathfrak{S}_n$ and $\mathbb{C}[B_{n-1}]$ are Morita equivalent.*

3.3.4. *Induction, restriction, and Grothendieck rings.* Let $\mathcal{G} := \mathcal{G}\left((\mathrm{H}\mathfrak{S}_n)_n\right)$ and $\mathcal{K} := \mathcal{K}\left((\mathrm{H}\mathfrak{S}_n)_n\right)$ be respectively the Grothendieck rings of the characters of the simple and projective modules of the tower of algebras $(\mathrm{H}\mathfrak{S}_n)_n$. Let furthermore C be the cartan map from $\mathcal{K}$ to $\mathcal{G}$. It is the algebra and coalgebra morphism which gives the projection of a module onto the direct sum of its composition factors. It is given by

$$C(P_I) = \sum_{I \succeq J} S_J \,. \tag{31}$$

Since the indecomposable projective modules are indexed by compositions, it comes out as no surprise that the structure of algebras and coalgebras of $\mathcal{G}$ and $\mathcal{K}$ are each isomorphic to QSym and NCSF. However, we do not get Hopf algebras, because the structures of algebras and coalgebras are not compatible.

PROPOSITION 5. *The following diagram gives a complete description of the structures of algebras and of coalgebras on $\mathcal{G}$ and $\mathcal{K}$.*

$$\begin{tikzcd}
(\mathrm{QSym}, .) & (\mathcal{G}, .) \arrow[l, hook, two heads, "\chi(S_I)\mapsto M_{I^c}"'] & (\mathcal{K}, .) \arrow[l, hook, two heads, "C"'] \arrow[r, hook, two heads, "\chi(P_I)\mapsto F_{I^c}"] & (\mathrm{QSym}, .) \\
(\mathrm{NCSF}, \Delta) & (\mathcal{G}, \Delta) \arrow[l, hook, two heads, "\chi(S_I)\mapsto R_{I^c}"] & (\mathcal{K}, \Delta) \arrow[l, hook, two heads, "C"] \arrow[r, hook, two heads, "\chi(P_I)\mapsto \Lambda^I"'] & (\mathrm{NCSF}, \Delta)
\end{tikzcd} \tag{32}$$

PROOF. The bottom line is already known from Proposition 2 and the fact that, for all m and n, the following diagram commutes

$$\begin{tikzcd}
\mathrm{H}_m(0) \otimes \mathrm{H}_n(0) \arrow[r, hook] \arrow[d, hook] & \mathrm{H}_{m+n}(0) \arrow[d, hook] \\
\mathrm{H}\mathfrak{S}_m \otimes \mathrm{H}\mathfrak{S}_n \arrow[r, hook] & \mathrm{H}\mathfrak{S}_{m+n}
\end{tikzcd} \tag{33}$$

Thus the map which sends a module to the characteristic of its restriction to $\mathrm{H}_n(0)$ is a coalgebra morphism. The isomorphism from $(\mathcal{K}, .)$ to $QSym$ is then obtained by Frobenius duality between induction of projective modules and restriction of simple modules. And the last case is obtained by applying the Cartan map C. $\square$

It is important to note that the algebra $(\mathcal{G}, .)$ is not the dual of the coalgebra $(\mathcal{K}, \Delta)$ because the dual of the restriction of projective modules is the so called *co-induction* of simple modules which is, in general, not the same as the induction for non self-injective algebras.

Finally the same process applied to the adjoint algebra which preserve symmetries would have given the following diagram

$$\begin{tikzcd}
(\mathrm{QSym}, .) & (\mathcal{G}, .) \arrow[l, hook, two heads, "\chi(S_I)\mapsto X_{I^c}"'] & (\mathcal{K}, .) \arrow[l, hook, two heads, "C"'] \arrow[r, hook, two heads, "\chi(P_I)\mapsto F_I"] & (\mathrm{QSym}, .) \\
(\mathrm{NCSF}, \Delta) & (\mathcal{G}, \Delta) \arrow[l, hook, two heads, "\chi(S_I)\mapsto R_I"] & (\mathcal{K}, \Delta) \arrow[l, hook, two heads, "C"] \arrow[r, hook, two heads, "\chi(P_I)\mapsto S^I"'] & (\mathrm{NCSF}, \Delta)
\end{tikzcd} \tag{34}$$

where $(X_I)_I$ is the dual basis of the elementary basis $(\Lambda_I)_I$ of NCSF. Thus we have a representation theoretical interpretation of many bases of NCSF and QSym.

3.4. Links with the affine Hecke algebra. Recall that, for any complex number q, the extended affine Hecke algebra $\hat{\mathrm{H}}_n(q)$ of type A_{n-1} is the $\mathbb{C}$-algebra generated by $(T_i)_{i=1\cdots n-1}$ together with an extra generator Ω verifying the defining relations of the Hecke algebra and the relation:

$$\Omega T_i = T_{i-1}\Omega \qquad \text{for } 1 \le i \le n. \tag{35}$$

The center of the affine Hecke algebra is isomorphic to the ring of symmetric polynomials in some variables $\xi_1, \dots, \xi_n$ and it can thus be specialized. Let us denote $\mathcal{H}_n(q)$ the specialization of the center $\hat{\mathrm{H}}_n(q)$ to the alphabet $1, q, \dots q^{n-1}$. That is

$$\mathcal{H}_n(q) := \hat{\mathrm{H}}_n(q) / \langle e_i(\xi_1, \dots, \xi_n) - e_i(1, q, \dots q^{n-1}) \mid i = 1 \dots n\rangle \,. \tag{36}$$

It is well known that the simple modules S_I of $\mathcal{H}_n(q)$ are indexed by compositions I and that their bases are indexed by descent classes of permutations. Thus one expects a strong link between $\mathrm{H}\mathfrak{S}_n$ and $\mathcal{H}_n(q)$. It comes out as follows. Let q be a generic complex number (i.e.: not 0 nor a root of the unity). Sending Ω to $\sigma_1\sigma_2\cdots\sigma_{n-1}$ and T_i to itself yields a surjective morphism from $\mathcal{H}_n(q)$ to $\mathrm{H}\mathfrak{S}_n$. Thus, the simple modules of $\mathcal{H}_n(q)$ are the simple modules of $\mathrm{H}\mathfrak{S}_n$ lifted back through this morphism. This also explains the link between the projective modules of $\mathrm{H}_n(0)$ and the simple modules of $\mathcal{H}_n(q)$, thanks to Proposition 2.

4. The algebra of non-decreasing functions

Definition 2. Let NDF_n be the set of *non-decreasing functions* from $\{1, \dots, n\}$ to itself. The composition and the neutral element id_n make NDF_n into a monoid. Its cardinal is $\binom{2n-1}{n-1}$, and we denote by $\mathbb{C}[\mathrm{NDF}_n]$ its monoid algebra.

The monoid $\mathrm{NDF}_n \times \mathrm{NDF}_m$ can be identified as the submonoid of NDF_{n+m} whose elements stabilize both $\{1, \dots, n\}$ and $\{n+1, \dots, n+m\}$. This makes $(\mathbb{C}[\mathrm{NDF}_n])_n$ into a tower of algebras.

One can take as generators for NDF_n and A_n the functions π_i et $\overline{\pi}_i$, such that $\pi_i(i+1) = i$, $\pi_i(j) = j$ for $j \neq i+1$, $\overline{\pi}_i(i) = i+1$, and $\pi_i(j) = j$ for $j \neq i$. The functions π_i are idempotents, and satisfy the braid relations, together with a new relation:

$$\pi_i^2 = \pi_i \qquad \text{and} \qquad \pi_{i+1}\pi_i\pi_{i+1} = \pi_i\pi_{i+1}\pi_i = \pi_{i+1}\pi_i \,. \tag{37}$$

This readily defines a morphism $\phi : \pi_{\mathrm{H}_n(0)} \mapsto \pi_{\mathbb{C}[\mathrm{NDF}_n]}$ of $\mathrm{H}_n(0)$ into $\mathbb{C}[\mathrm{NDF}_n]$. Its image is the monoid algebra of *non-decreasing parking functions* which will be discussed in Section 5. The same properties hold for the operators $\overline{\pi}_i$'s. Although this is not a priori obvious, it will turn out that the two morphisms $\phi : \pi_{\mathrm{H}_n(0)} \mapsto \pi_{\mathbb{C}[\mathrm{NDF}_n]}$ and $\overline{\phi} : \overline{\pi}_{\mathrm{H}_n(0)} \mapsto \overline{\pi}_{\mathbb{C}[\mathrm{NDF}_n]}$ are compatible, making $\mathbb{C}[\mathrm{NDF}_n]$ into a quotient of $\mathrm{H}\mathfrak{S}_n$.

4.1. Representation on exterior powers. We now want to construct a suitable representation of $\mathbb{C}[\mathrm{NDF}_n]$ where the existence of the epimorphism from $\mathrm{H}\mathfrak{S}_n$ onto $\mathbb{C}[\mathrm{NDF}_n]$, and the representation theory of $\mathbb{C}[\mathrm{NDF}_n]$ become clear.

The *natural representation* of $\mathbb{C}[\mathrm{NDF}_n]$ is obtained by taking the vector space $\mathbb{C}^n$ with canonical basis $e_1, \dots, e_n$, and letting a function f act on it by $e_i.f = e_{f(i)}$. For $n > 2$, this representation is a faithful representation of the monoid NDF_n but not of the algebra, as $\dim \mathbb{C}[\mathrm{NDF}_n] = \binom{2n-1}{n-1} \gg n^2$. However, since NDF_n is a monoid, the diagonal action on *exterior powers*

$$(x_1 \wedge \cdots \wedge x_k) \cdot f := (x_1 \cdot f) \wedge \cdots \wedge (x_k \cdot f) \tag{38}$$

still define an action. Taking the *exterior powers* $\bigwedge^k \mathbb{C}^n$ of the natural representation gives a new representation, whose basis $\{e_S := e_{s_1} \wedge \cdots \wedge e_{s_k}\}$ is indexed by subsets $S := \{s_1, \dots, s_k\}$ of $\{1, \dots, n\}$. The action of a function f in NDF_n is simply given by (note the absence of sign!):

$$e_S.f := \begin{cases} e_{f(S)} & \text{if } |f(S)| = |S|, \\ 0 & \text{otherwise.} \end{cases} \tag{39}$$

We call *representation of* $\mathbb{C}[\mathrm{NDF}_n]$ *on exterior powers* the representation of $\mathbb{C}[\mathrm{NDF}_n]$ on $\bigoplus_{k=1}^n \bigwedge^k \mathbb{C}^n$, which is of dimension $2^n - 1$ (it turns out that we do not need to include the component $\bigwedge^0 \mathbb{C}^n$ for our purposes).

Lemma 4.1. *The representation of* $\mathbb{C}[\mathrm{NDF}_n]$ *on* $\bigoplus_{k=1}^n \bigwedge^k \mathbb{C}^n \bigwedge \mathbb{C}^n$ *is faithful.*

We now want to realize the representation of $\mathbb{C}[\mathrm{NDF}_n]$ on the k-th exterior power as a representation of $\mathrm{H}\mathfrak{S}_n$. To this end, we use a variation on the standard construction of the Specht module $V_{k,1,\dots,1}$ of $\mathfrak{S}_n$ to make it a $\mathrm{H}\mathfrak{S}_n$-module. The trick is to use an appropriate quotient of $\mathbb{C}\mathfrak{S}_n$ to simulate the symmetries that we usually get by working with polynomials, while preserving the $\mathrm{H}\mathfrak{S}_n$-module structure. Namely, consider the following $\mathrm{H}\mathfrak{S}_n$-module:

$$P_n^k := P_{k,1,\dots,1}/\bigcup P_{k,1,\dots,1,2,1,\dots,1}. \tag{40}$$

An element in P_n^k is left-antisymmetric on the values $1,\dots,k-1$ and symmetric on the values $k+1,\dots,n-1$, the effect of the quotient being to identify two permutations which differ by a permutation of the values $\{k+1,\dots,n\}$. A basis of P_n^k indexed by subsets of size k of $\{1,\dots,n\}$ is obtained by taking for each such subset S the image in the quotient P_n^k of

$$e_S := \sum_{\sigma,\sigma(S)=\{1,\dots,k\},\sigma(i)<\sigma(j) \text{ for } i<j\notin S} (-1)^{\mathrm{sign}\sigma}\sigma\,. \tag{41}$$

It is straightforward to check that the actions of π_i and $\overline{\pi}_i$ of $\mathrm{H}\mathfrak{S}_n$ on e_S of P_k coincide with the actions of π_i and $\overline{\pi}_i$ of $\mathbb{C}[\mathrm{NDF}_n]$ on e_S of $\bigwedge^k\mathbb{C}^n$ (justifying a posteriori the identical notations). In the sequel, we identify the modules P_n^k and $\bigwedge^k\mathbb{C}^n$ of $\mathrm{H}\mathfrak{S}_n$ and $\mathbb{C}[\mathrm{NDF}_n]$, and we call *representation on exterior powers of* $\mathrm{H}\mathfrak{S}_n$ its representation on $\bigoplus_{k=1}^n\bigwedge^k\mathbb{C}^n$. Using Lemma 4.1 we are in position to state the following

PROPOSITION 6. *$\mathbb{C}[\mathrm{NDF}_n]$ is the quotient of $\mathrm{H}\mathfrak{S}_n$ obtained by considering its representation on exterior powers. The restriction of this representation of $\mathrm{H}\mathfrak{S}_n$ to $\mathbb{C}[\mathfrak{S}_n]$, $\mathrm{H}_n(0)$, and $\mathrm{H}_n(-1)$ yield respectively the usual representation of $\mathfrak{S}_n$ on exterior powers, the algebra of non-decreasing parking functions (see Section 5), and the Temperley-Lieb algebra.*

4.2. Representation theory.

4.2.1. *Projective modules, simple modules, and Cartan's invariant matrix.* Let δ be the usual homology border map:

$$\delta : \begin{cases} P_n^k & \to P_n^{k-1} \\ S := \{s_1,\dots,s_k\} & \mapsto \sum_{i\in\{1,\dots,k\}}(-1)^{k-i}S\backslash\{s_i\} \end{cases}. \tag{42}$$

This map is naturally a morphism of $\mathbb{C}[\mathrm{NDF}_n]$-module. For each k in $1,\dots,n$, let $S_k := P_k/\ker\delta$. It turns out that together with the identity, δ is essentially the only $\mathbb{C}[\mathrm{NDF}_n]$-morphism. We are now in position to describe the projective and simple modules, as well as the Cartan matrix of $\mathbb{C}[\mathrm{NDF}_n]$.

PROPOSITION 7. *The modules $(P_n^k)_{k=1,\dots,n}$ form a complete set of representatives of the indecomposable projective modules of $\mathbb{C}[\mathrm{NDF}_n]$.*

The modules $(S_n^k)_{k=1,\dots,n}$ form a complete set of representatives of the simple modules of $\mathbb{C}[\mathrm{NDF}_n]$.

Let k and l be two integers in $\{1,\dots,n\}$. Then,

$$\dim Hom(P_n^k,P_n^l) = \begin{cases} 1 & \textit{if } l\in\{k,k-1\}, \\ 0 & \textit{otherwise.}\end{cases} \tag{43}$$

The proof relies essentially on the following lemma:

LEMMA 4.2. *There exists a minimal decomposition of the identity of $\mathbb{C}[\mathrm{NDF}_n]$ into 2^n-1 orthogonal idempotents. In particular, the representation on exterior powers is the smallest faithful representation of $\mathbb{C}[\mathrm{NDF}_n]$.*

4.2.2. *Induction, restriction, and Grothendieck groups.*

PROPOSITION 8. *The restriction and induction of indecomposable projective modules and simple modules are described by:*

$$P^k_{n_1+n_2}\downarrow^{\mathbb{C}[\mathrm{NDF}_{n_1+n_2}]}_{\mathbb{C}[\mathrm{NDF}_{n_1}]\otimes\mathbb{C}[\mathrm{NDF}_{n_2}]}\approx \bigoplus_{\substack{n_1+n_2=n\\ k_1+k_2=k\\ 1\le k_i\le n_i \textit{ or } k_i=n_i=0}} P^{k_1}_{n_1}\otimes P^{k_2}_{n_2} \tag{44}$$

$$P^{k_1}_{n_1}\otimes P^{k_2}_{n_2}\uparrow^{\mathbb{C}[\mathrm{NDF}_{n_1+n_2}]}_{\mathbb{C}[\mathrm{NDF}_{n_1}]\otimes\mathbb{C}[\mathrm{NDF}_{n_2}]}\approx P^{k_1+k_2}_{n_1+n_2}\oplus P^{k_1+k_2-1}_{n_1+n_2} \tag{45}$$

$$S^k_{n_1+n_2}\downarrow^{\mathbb{C}[\mathrm{NDF}_{n_1+n_2}]}_{\mathbb{C}[\mathrm{NDF}_{n_1}]\otimes\mathbb{C}[\mathrm{NDF}_{n_2}]} = \bigoplus_{\substack{n_1+n_2=n\\ k_1+k_2\in\{k,k+1\}\\ 1\le k_i\le n_i \text{ or } k_i=n_i=0}} S^{k_1}_{n_1}\otimes S^{k_2}_{n_2} \tag{46}$$

$$S^{k_1}_{n_1}\otimes S^{k_2}_{n_2}\uparrow^{\mathbb{C}[\mathrm{NDF}_{n_1+n_2}]}_{\mathbb{C}[\mathrm{NDF}_{n_1}]\otimes\mathbb{C}[\mathrm{NDF}_{n_2}]} \approx S^{k_1+k_2}_{n_1+n_2} \tag{47}$$

Those rules yield structures of commutative algebras and cocommutative coalgebras on $\mathcal{G}$ and $\mathcal{K}$ which can be realized as quotients or sub(co)algebras of Sym, QSym, and NCSF. However, we do not get Hopf algebras, because the structures of algebras and coalgebras are not compatible (compute for example $\Delta(\chi(P^1_1)\chi(P^1_1))$ in the two ways, and check that the coefficients of $\chi(P^1_1)\otimes\chi(P^1_1)$ differ).

5. The algebra of non-decreasing parking functions

Definition 3. A *nondecreasing parking function* of size n is a nondecreasing function f from $\{1,2,\dots n\}$ to $\{1,2,\dots n\}$ such that $f(i)\le i$, for all $i\le n$.

The composition of maps and the neutral element id_n make the set of nondecreasing parking function of size n into a monoid denoted NDPF_n.

It is well known that the nondecreasing parking functions are counted by the Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$. It is also clear that NDPF_n is the sub-monoid of NDF_n generated by the π_i's.

5.1. Simple modules. The goal of the sequel is to study the representation theory of NDPF_n, or equivalently of its algebra $\mathbb{C}[\mathrm{NDPF}_n]$. The following remark allows us to deduce the representations of $\mathbb{C}[\mathrm{NDPF}_n]$ from the representations of $\mathrm{H}_n(0)$.

Proposition 9. *The kernel of the algebra epi-morphism $\phi:\mathrm{H}_n(0)\to\mathbb{C}[\mathrm{NDPF}_n]$ defined by $\phi(\pi_i)=\pi_i$ is a sub-ideal of the radical of* $\mathrm{H}_n(0)$.

Proof. It is well known (see [**12**]) that the quotient of $\mathrm{H}_n(0)$ by its radical is a commutative algebra. Consequently, $\pi_i\pi_{i+1}\pi_i-\pi_i\pi_{i+1}=[\pi_i\pi_{i+1},\pi_i]$ belongs to the radical of $\mathrm{H}_n(0)$. □

As a consequence, taking the quotient by their respective radical shows that the projection ϕ is an isomorphism from $\mathbb{C}[\mathrm{NDPF}_n]/\mathrm{rad}(\mathbb{C}[\mathrm{NDPF}_n])$ to $\mathrm{H}_n(0)/\mathrm{rad}(\mathrm{H}_n(0))$. Moreover $\mathbb{C}[\mathrm{NDPF}_n]/\mathrm{rad}(\mathbb{C}[\mathrm{NDPF}_n])$ is isomorphic to the commutative algebra generated by the π_i such that $\pi_i^2=\pi_i$. As a consequence, $\mathrm{H}_n(0)$ and $\mathrm{H}\mathfrak{S}_n$ share, roughly speaking, the same simple modules:

Corollary 1. *There are 2^{n-1} simple $\mathbb{C}[\mathrm{NDPF}_n]$-modules S_I, and they are all one dimensional. The structure of the module S_I, generated by η_I, is given by*

$$\begin{cases}\eta_I\cdot\pi_i = 0 & \textit{if } i\in\mathrm{Des}(I),\\ \eta_I\cdot\pi_i=\eta_I & \textit{otherwise.}\end{cases} \tag{48}$$

5.2. Projective modules. The projective modules of NDPF_n can be deduced from the ones of NDF_n.

Theorem 5.1. *Let I be a composition of n, and $S:=\mathrm{Des}(I)=\{s_1,\dots,s_k\}$ be its associated set. Then, the principal sub-module*

$$P_I := (e_1\wedge e_{s_1+1}\wedge\cdots\wedge e_{s_1+1})\cdot\mathbb{C}[\mathrm{NDPF}_n] \quad\subset\quad \bigwedge^{k+1}\mathbb{C}^n \tag{49}$$

is an indecomposable projective module. Moreover, the set $(P_I)_{I\models n}$ is a complete set of representatives of indecomposable projective modules of $\mathbb{C}[\mathrm{NDPF}_n]$.

This suggests an alternative description of the algebra $\mathbb{C}[\mathrm{NDPF}_n]$. Let $G_{n,k}$ be the lattice of subsets of $\{1,\dots,n\}$ of size k for the *product order* defined as follows. Let $S:=\{s_1<s_2<\cdots<s_k\}$ and $T:=\{t_1<t_2<\cdots<t_k\}$ be two subsets. Then,

$$S\le_G T \qquad\text{if and only if}\qquad s_i\le t_i,\text{ for } i=1,\dots,k. \tag{50}$$

One easily sees that $S\le_G T$ if and only if there exists a nondecreasing parking function f such that $e_S=e_T\cdot f$. This lattice appears as the Bruhat order associated to the Grassman manifold G^n_k of k-dimensional subspaces in $\mathbb{C}^n$.

THEOREM 5.2. *There is a natural algebra isomorphism*

$$\mathbb{C}[\mathrm{NDPF}_n] \;\approx\; \bigoplus_{k=0}^{n-1} \mathbb{C}[G_{n-1,k}]\;. \tag{51}$$

In particular the Cartan map $C : \mathcal{K} \to \mathcal{G}$ is given by the lattice $\leq_G$:

$$C(P_I) \;=\; \sum_{J,\ \mathrm{Des}(J)\leq_G \mathrm{Des}(I)} S_J \tag{52}$$

On the other hand, due to the commutative diagram

$$\begin{tikzcd} \mathrm{H}_m(0)\otimes \mathrm{H}_n(0) \arrow[r, hook] \arrow[d] & \mathrm{H}_{m+n}(0) \arrow[d] \\ \mathrm{NDPF}_m \otimes \mathrm{NDPF}_n \arrow[r, hook] & \mathrm{NDPF}_{m+n} \end{tikzcd} \tag{53}$$

it is clear that the restriction of simple modules and the induction of indecomposable projective modules follow the same rule as for $\mathrm{H}_n(0)$. The induction of simple modules can be deduced via the Cartan map, giving rise to a new basis G_I of NCSF. It remains finally to compute the restrictions of indecomposable projective modules. It can be obtained by a not yet completely explicit algorithm. All of this is summarized by the following diagram:

$$\begin{tikzcd} (\mathrm{NCSF},.) & (\mathcal{G},.) \arrow[l, two heads, "\chi(S_I)\mapsto G_I"'] & (\mathcal{K},.) \arrow[l, two heads, "C"'] \arrow[r, hook, two heads, "\chi(P_I)\mapsto R_I"] & (\mathrm{NCSF},.) \\ (\mathrm{QSym},\Delta) & (\mathcal{G},\Delta) \arrow[l, two heads, "\chi(S_I)\mapsto F_I"] & (\mathcal{K},\Delta) \arrow[l, two heads, "C"] \arrow[r, hook, two heads, "\chi(P_I)\mapsto ???"'] & ??? \end{tikzcd} \tag{54}$$

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