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Pattern avoiding doubly alternating permutations

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Abstract. We study pattern avoiding doubly alternating (DA) permutations, i.e., alternating (or zigzag) permutations whose inverse is also alternating. We exhibit a bijection between the 1234-avoiding permutations and the 1234-avoiding DA permutations of twice the size using the Robinson-Schensted correspondence. Further, we present a bijection between the 1234- and 2134-avoiding DA permutations and we prove that the 2413-avoiding DA permutations are counted by the Catalan numbers.

Résumé. Nous étudions les permutations qui évitent les motifs double-alternant (DA), c'est à dire, les permutations alternantes dont l'inverse est alternante. Nous montrons, en utilisant le correspondance de Robinson-Schensted, une bijection entre les permutations de longueur n évitant 1234 et les permutations DA de longueur $2n$ évitant 1234. Nous montrons aussi, une bijection entre les permutations DA évitant 1234 et celles évitant 2134, et que les permutations DA évitant 2413 sont dénombrées par les nombres de Catalan.

1. Introduction

A permutation $\sigma \in \mathcal{S}_n$ is said to *contain* the pattern $\tau \in \mathcal{S}_m$ if there is a subsequence of (the word representation of) σ which is order equivalent to (the word representation of) τ. To distinguish between patterns and other permutations, we will use slightly different notation. For example, the permutation $(1,3,2,4)$ will be written as 1324 if it is used as a pattern. We will often use the matrix representation of σ, which is the $n \times n$ 0-1-matrix having ones in the positions with matrix coordinates $(i, \sigma(i))$. It can also be written as $([\![\sigma(i) = j]\!])_{i,j=1}^n$, using Iverson's bracket notation [**9**] for the characteristic function, $[\![S]\!] = 1$ if S is true and 0 otherwise. In the figures we will use dots instead of ones and leave the zeroes empty, as in Figure 1, to make the picture clearer. In this notation σ contains the pattern τ if some submatrix of (the matrix representation of) σ is equal to (the matrix representation of) τ. The permutations not containing τ are called τ*-avoiding*, and we write

$$\mathcal{S}_n(\tau_1, \tau_2, \ldots, \tau_t) = \{\sigma \in \mathcal{S}_n : \sigma \text{ is } \tau_i\text{-avoiding for all } i = 1, \ldots, t\}.$$

A word, e.g., a permutation, $w = (w_i)_{i=1}^n$, is *(up-down)-alternating* if $w_{2i-1} < w_{2i}$ and $w_{2i} \geqslant w_{2i+1}$ for all applicable i. This means that the word alternates between *rises* and *descents*, beginning with a rise. If it instead starts with a descent, it is called *down-up-alternating*.

A permutation σ is *doubly alternating* (DA) if both σ and σ^{-1} are alternating. The set of pattern avoiding doubly alternating permutations is denoted by

$$\mathrm{DA}_n(\tau_1, \ldots, \tau_t) = \{\sigma \in \mathcal{S}_n(\tau_1, \ldots, \tau_t) : \sigma \text{ is doubly alternating }\}.$$

Pattern avoiding permutations have been subject to much attention since the pioneering work by Knuth's [**10**], where he used them for studying stack sortable permutations. For a thorough summary of the current status of research, see Bóna's book [**4**]. Alternating permutations have a long history, they were studied already in the 19th century by André [**1**], and it is well know that they are counted by the tangent and secant numbers, also known as Euler numbers, E_k, and thus, their exponential generating function

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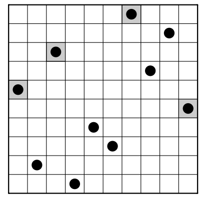

Figure 1. The permutation $(7, 9, 3, 8, 1, 10, 5, 6, 2, 4) \in \mathrm{DA}(1234)$ contains the pattern 3214, but avoids 1234.

is $\tan(x)+\sec(x)$. Alternating permutation avoiding patterns have been studied by Mansour [**11**], but there are still many open questions remaining.

The doubly alternating permutations were first counted by Foulkes [**6**], up to $n = 10$, using a result which we state as Theorem 5.1. The only formula known is due to Stanley [**13**]:

$$\sum_{n \text{ odd}} \mathrm{DA}_n\, x^n = \sum_{k \text{ odd}} E_k^2((\log(1+x)/(1-x))/2)^k/k!,$$
$$\sum_{n \text{ even}} \mathrm{DA}_n\, x^n = (1-x^2)^{-1/2} \sum_{k \text{ even}} E_k^2((\log(1+x)/(1-x))/2)^k/k!,$$

from which we get that the first few numbers for $\mathrm{DA}_n, n \geqslant 1$, are

$$1, 1, 1, 2, 3, 8, 19, 64, 880, 3717, 18288, 92935, \ldots$$

The motivation for studying doubly alternating permutations came from work by Guibert and Linusson [**8**] who showed that doubly alternating Baxter permutations are counted by the Catalan numbers. It was a natural step to study other restrictions to see whether interesting results could be found.

Using computer enumerations Guibert came up with several conjectures that indicated there are surprising connections between doubly alternating permutations and ordinary permutations. Some of these are proved in this paper, see proposition 4.1 and Theorems 5.2 and 6.2, whereas others still remain unproved and are listed in conjecture 7.1.

In this paper we study doubly alternating permutations avoiding patterns of lengths three and four. The patterns of length three are covered in Section 3. In Section 4, we show that doubly alternating permutations avoiding 2413 are counted by Catalan numbers, and are closely related to the doubly alternating Baxter permutations. Section 5 contains a bijection between $\mathrm{DA}_{2n}(1234)$ and $\mathcal{S}_n(1234)$ and in Section 6 we use a result by Babson and West [**2**] to construct a bijection between $\mathrm{DA}_n(12\tau)$ and $\mathrm{DA}_n(21\tau)$, where τ is any permutation of $\{3, 4, \ldots, m\}$, $m \geqslant 3$. In Section 7 other patterns giving the same sequence are investigated and in the final section some remarks on a few DA permutations avoiding two patterns of length four are given.

I like to thank Olivier Guibert for introducing me to the problem and for interesting discussions. Thanks also to Svante Linusson, Bruce Sagan and Mark Dukes for numerous comments and suggestions.

2. Notation and basic facts

First we define the *reverse*, the *complement* and the *rotation* of a permutation σ,

$$\sigma^r = (\sigma(n+1-i))_{i=1}^n$$
$$\sigma^c = (n+1-\sigma(i))_{i=1}^n$$
$$\sigma^{\#} = (\sigma^c)^r = (\sigma^r)^c = (n+1-\sigma(n+1-i))_{i=1}^n$$

In terms of matrices, the first two correspond to flipping the matrix vertically and horizontally, respectively, whereas the last operation rotates the matrix 180 degrees. However, these bijections do not in general preserve the doubly alternating property, which means that we lose some symmetry compare with ordinary permutations, so that more genuinely different patterns need to be examined. However, it is obvious from the definition that inverting and, if n is even, rotating a permutation does preserve the property of being doubly alternating.

Lemma 2.1.

(a) $\sigma \in \mathrm{DA}_n \iff \sigma^{-1} \in \mathrm{DA}_n$
(b) $\sigma \in \mathrm{DA}_{2n} \iff \sigma^{\#} \in \mathrm{DA}_{2n}$

Another simple, but very useful, property that follows from the DA condition is that some areas on the border of the matrix can never have a dot, see Figure 2.

LEMMA 2.2.

(a) *Let* $\sigma \in \mathrm{DA}_{2n}$*, then*
  (i) $\sigma(1)$ *is odd,*
  (ii) $\sigma(2) \in \{3, 5, 7, \ldots, 2n-1, 2n\}$,
  (iii) $\sigma(2) = 2n$ *iff* $\sigma(1) = 2n-1$,
  (iv) $\sigma(2n)$ *is even,*
  (v) $\sigma(2n-1) \in \{1, 2, 4, 6, \ldots, 2n-2\}$,
  (vi) $\sigma(2n-1) = 1$ *iff* $\sigma(2n) = 2$.

(b) *Let* $\sigma \in \mathrm{DA}_{2n+1}$*, then*
  (i) $\sigma(1)$ *is odd and less than* $2n+1$,
  (ii) $\sigma(2)$ *is odd and greater than* 1,
  (iii) $\sigma(2n+1)$ *is even,*
  (iv) $\sigma(2n) \in \{4, 6, 8, \ldots, 2n, 2n+1\}$,
  (v) $\sigma(2n) = 2n+1$ *iff* $\sigma(2n+1) = 2n$.

PROOF. First for the case a(i), if $\sigma(1) = k > 1$, then $\sigma^{-1}(k) = 1$, so $\sigma^{-1}(k-1) > 1$, which implies that k is odd, since $\sigma \in \mathrm{DA}$. For a(ii), assume $\sigma(2) = m < 2n$, $m > \sigma(1) \geqslant 1$. Then $\sigma^{-1}(m-1) > 2$ or $\sigma^{-1}(m+1) > 2$, so m is odd. The equivalence a(iii) is a direct consequence of the definition of doubly alternating, since $\sigma(2) > \sigma(1)$ and $\sigma^{-1}(2n-1) < \sigma^{-1}(2n)$. The other cases are similar. □

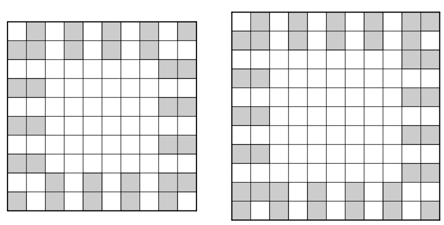

FIGURE 2. Illustration of Lemma 2.2. Shaded areas are forbidden.

Note that this lemma could be applied to σ^{-1} as well, because of Lemma 2.1. From the two lemmas it is clear that there is a difference between odd and even sizes, so they require separate treatment in many of the proofs. This disparity is also reflected in the fact that $\mathrm{DA}_n(\tau_1, \tau_2, \ldots, \tau_t)$ is not an increasing function of n for all patterns. Some counterexamples are, $\mathrm{DA}_4(321) = 2 > 1 = \mathrm{DA}_5(321)$, $\mathrm{DA}_6(2431) = 6 > 5 = \mathrm{DA}_7(2431)$ and, if Conjecture 8.1 is true, $\mathrm{DA}_{27}(1234, 2134) = 2681223 > 2674440 = \mathrm{DA}_{28}(1234, 2134)$.

3. Patterns of length three

For normal permutations, patterns of length three are the first non-trivial cases; they are all counted by the Catalan numbers. However, for the doubly alternating permutations, it turns out that all the patterns of length three are (more or less) trivial.

PROPOSITION 3.1.

(i) $|\,\mathrm{DA}_n(123)| = |\,\mathrm{DA}_n(213)| = |\,\mathrm{DA}_n(231)| = |\,\mathrm{DA}_n(312)| = 1$
(ii) $|\,\mathrm{DA}_n(132)| = [\![n \text{ even or } n = 1]\!]$
(iii) $|\,\mathrm{DA}_n(321)| = 1 + [\![n \text{ even and } n \geqslant 4]\!]$

PROOF. Omitted in the extended abstract. □

4. 2413-avoiding doubly alternating permutations

The doubly alternating 2413-avoiding permutations were conjectured by Guibert to be counted by the Catalan numbers. We prove this by showing them to possess a fairly simple block structure. First we need a technical lemma.

Lemma 4.1. *Let* $\sigma \in \mathrm{DA}_n(2413)$*, then*

(i) n *odd* $\implies$ $\sigma(1) = 1$

(ii) n *even* $\implies$ $\sigma(1) = 1$ *and* $\sigma(n) = n$ *or there is a* k*,* $2 < k \leqslant n-1$*, such that* $\sigma(i) > \sigma(j)$ *for all* $i < k \leqslant j$*.*

Proof. We can assume that $n \geqslant 3$, the smaller cases are trivial. Let $a = \sigma(1)$, and assume $\sigma(1) \neq 1$. By Lemma 2.2, a must be odd. Now let $b = \sigma(\beta)$, where β is the smallest number such that $\sigma(\beta) < a$. Note that $\beta \geqslant 3$, since $\sigma(2) > a$. Also, β is odd since $\sigma(\beta - 1) > \sigma(\beta)$.

Let c be the largest number such that $\gamma = \sigma^{-1}(c) < \beta$, see Figure 3. Thus $c > a$ and c must be odd or $c = n$, since $\sigma^{-1}(c+1) > \beta > \sigma^{-1}(c) = \gamma$ if $c < n$.

If $\kappa > \beta$, then $\sigma(\kappa) < a$ or $\sigma(\kappa) > c$, otherwise we get the 2413 pattern. Therefore the rectangle, with NW corner $(2, a+1)$ and SE corner $(\beta - 1, c)$ contains exactly one dot in each row and column, so it is square, and hence $c = a + \beta - 2$ is even and thus $c = n$ is the only possibility. This proves the first assertion.

If n is even, $\sigma(1) = 1$ implies $\sigma(n)$, by rotational symmetry, so the second assertion follows from $i < \beta \leqslant j \Rightarrow \sigma(j) < a \leqslant \sigma(i)$. □

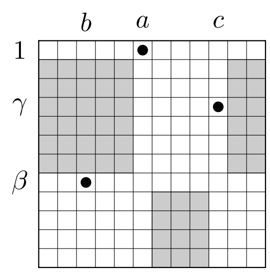

Figure 3. Illustration of Lemma 4.1, the shaded areas are empty.

As a direct consequence, we get the following corollary, which gives a very explicit description of what the DA 2413-avoiding permutations look like.

Corollary 4.2.

(i) $\sigma \in \mathrm{DA}_{2n+1}(2413)$ *iff* $\sigma = (1, \tilde{\sigma})$*, where* $(\tilde{\sigma}^r)^{-1} \in \mathrm{DA}_{2n}(2413)$*.*

(ii) $\sigma \in \mathrm{DA}_{2n}(2413)$ *iff the permutation matrix of* σ *is a block matrix, where all but the anti-diagonal blocks are empty. Any non-empty block* ν *has even size,* $2k$*, and can be written* $\nu = (1, \tilde{\nu}, 2k)$*, where* $(\tilde{\nu}^r)^{-1} \in \mathrm{DA}_{2k-2}(2413)$*.*

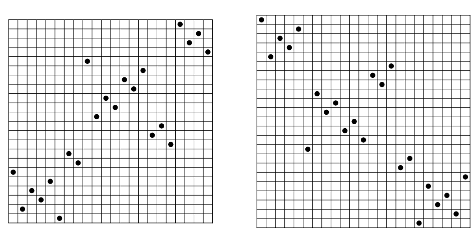

Figure 4. Example of the block structure of 2413-avoiding DA permutations.

The block structure condition in the corollary is in fact invariant under taking inverses, even though the pattern 2413 is not, therefore we get the following, slightly surprising result: $\mathrm{DA}_n(2413)$ and $\mathrm{DA}_n(3142)$ are not only the same size, but are actually the same sets. Therefore also $\mathrm{DA}_n(2413, 3142)$ is the same set. They are all counted by the Catalan numbers.

Proposition 4.1. $|\operatorname{DA}_n(2413)| = C_{\lfloor n/2 \rfloor}$.

Proof. First if n is odd, we have as a direct consequence of Corollary 4.2(i) that $|\operatorname{DA}_n(2413)| = |\operatorname{DA}_{n-1}(2413)|$. If n is even, then Corollary 4.2(ii) tells us that $\sigma \in \operatorname{DA}_n(2413)$ can be factored into blocks $\sigma_1, \sigma_2, \ldots, \sigma_m$, where $\sigma_i = (1, \tilde{\sigma}_i, |\sigma_i|)$ and $\tilde{\sigma}_i^r \in \operatorname{DA}(2413)$.

Let $D(x)$ be the generating function $D(x) = \sum_k |\operatorname{DA}_{2k}(2413)| x^k$. Then

$$D(x) = \sum_{i=0}^{\infty} (xD(x))^i = \frac{1}{1 - xD(x)}$$

which implies $xD(x)^2 - D(x) + 1 = 0$, i.e., the well know equation for the generating function of the Catalan numbers. Since $D(0) = 1$, we get $|\operatorname{DA}_{2k}(2413)| = C_k$. □

Another way to prove this is to construct a bijection with Dyck paths. We define $\Theta : \operatorname{DA}_{2n}(2413) \leftrightarrow \{\text{Dyck paths of length } 2n\}$ recursively, by using Corollary 4.2.

(i) $\Theta(\emptyset) = \emptyset$
(ii) If σ consists of a single block, so that $\sigma = (1, \tilde{\sigma}, 2n)$, then $\Theta(\sigma)$ is the Dyck path starting with a rise, ending with a descent and having the Dyck path $\Theta(\tilde{\sigma}^r)$ as the middle part.
(iii) If σ can be factored into k blocks, $\sigma_1, \ldots, \sigma_k$ (starting with the leftmost block), then $\Theta(\sigma)$ is the concatenation of the Dyck paths $\Theta(\sigma_1), \ldots, \Theta(\sigma_k)$.

The inverse is similarly defined, using recursion.

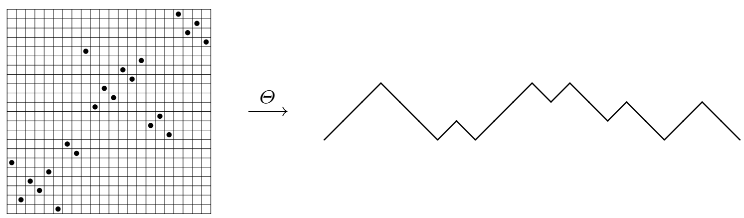

Figure 5. Example of the bijection between $\operatorname{DA}_{2n}(2413)$ and Dyck paths.

4.1. Doubly alternating Baxter permutations.

A *Baxter permutation* is defined to be a permutation, $\sigma = (\sigma_i)_{i=1}^n$, such that for all $1 \leqslant i < j < k < l \leqslant n$,

$$\begin{aligned} \sigma_i + 1 = \sigma_l \text{ and } \sigma_j > \sigma_l &\implies \sigma_k > \sigma_l \quad \text{and} \\ \sigma_l + 1 = \sigma_i \text{ and } \sigma_k > \sigma_i &\implies \sigma_j > \sigma_i. \end{aligned}$$

It is clear from this definition that if σ avoids both 2413 and 3142 then it is a Baxter permutation, so we have

$$\operatorname{DA}_n(2413, 3142) \subset \{\sigma \in \operatorname{DA}_n : \sigma \text{ is Baxter}\}.$$

However, in [**8**], Guibert and Linusson showed that the doubly alternating Baxter permutations are counted by the Catalan numbers, so the sets must in fact be the same:

Corollary 4.3.

$$\{\sigma \in \operatorname{DA}_n : \sigma \textit{ is Baxter}\} = \operatorname{DA}_n(2413, 3142) = \operatorname{DA}_n(2413) = \operatorname{DA}_n(3142).$$

It is also possible to prove this directly, without referring to the result by Guibert and Linusson.

Lemma 4.4. $\{\sigma \in \operatorname{DA}_n : \sigma \textit{ is Baxter}\} \subset \operatorname{DA}_n(2413)$.

Proof. Assume σ is Baxter, but not 2413-avoiding, and d_1, d_2, d_3, d_4, with $d_k = (i_k, j_k)$, constitute a 2413 pattern, such that $j_4 - j_1$ is as small as possible and given j_1 and j_4, $i_3 - i_2$ is as small as possible , as in Figure 6. The four areas shaded in the figure are empty, otherwise we would use one of those dots for the 2413-pattern. Now, let ν be the permutation having as permutation matrix the submatrix of σ, consisting of the rows $i_2 + 1, i_2 + 2, \ldots, i_3 - 1$ and columns $j_1 + 1, j_1 + 2, \ldots, j_4 - 1$. Since σ is Baxter, ν cannot be empty. The DA condition implies that ν is up-down-alternating, ν^{-1} is down-up-alternating and $|\nu|$ is even.

Also ν is 2413-avoiding, since otherwise this occurrence of 2413 would have been used instead of $d_1, \dots, d_4$, and $\nu(1) \neq 1$, since ν^{-1} is down-up-alternating.

Rehashing the argument for Lemma 4.1, we can see that in fact no such permutation ν can exist. Defining a, b, c, β and γ in the same way as in the proof of Lemma 4.1 we get once again Figure 3. The difference is that now a and c must be even, whereas β is still odd. But then $c = a + \beta - 2$ is odd, a contradiction. $\square$

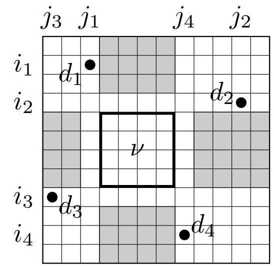

FIGURE 6. Illustration of Lemma 4.4. The shaded areas do not contain any dots.

5. 1234-avoiding doubly alternating permutations

In this section we construct a bijection between the doubly alternating 1234-avoiding permutations of size $2n$ and the ordinary 1234-avoiding permutations of size n by using the Robinson-Schensted correspondence. Let λ be a *Young diagram*, and denote by $\mathrm{SYT}(\lambda)$ the set of standard Young tableaux of shape λ. For a standard Young tableaux, T, let $\mathsf{row}_k(T)$ ($\mathsf{col}_k(T)$) denote the number of the row (column) for the entry k, counting from the top row (leftmost column), which is given the number one. The vector $\mathsf{row}(T)$ ($\mathsf{col}(T)$) is called the *row (column) reading* of T.

We define the set of *alternating standard tableaux* as

$$\begin{aligned}\mathrm{Alt}(\lambda) =& \{T \in \mathrm{SYT}(\lambda) : \mathsf{col}(T) \text{ is up-down-alternating}\}\\ =& \{T \in \mathrm{SYT}(\lambda) : \mathsf{row}(T) \text{ is down-up-alternating}\},\end{aligned}$$

where the second equality is a consequence of the relative positions of two consecutive entries in a standard tableau, as shown in Figure 7.

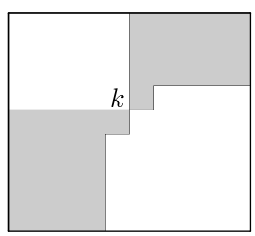

FIGURE 7. The shaded area denotes the possible positions for the entry $k+1$, relative to the entry k, in a standard Young tableau.

The Robinson-Schensted correspondence, RSK, is a well known bijection between a permutation and a pair of standard Young tableaux of the same shape, see for example the book by Fulton [**7**]. An interesting fact is that doubly alternating permutations can be recognised by their RSK tableaux: They are both alternating if and only if the permutation is DA. In fact, Foulkes [**5**] proved a more general theorem, in which he counts the number of permutations with any given sequences of ups and downs for the permutation and its inverse. The following lemma is the key for proving Foulkes theorem.

LEMMA 5.1. *Let $\sigma \in \mathcal{S}_n$, $\mathrm{RSK}(\sigma) = (P, Q)$ and $1 \leqslant k < n$, then*

$$k \text{ comes before } k+1 \text{ in } \sigma \iff \mathsf{row}_k(P) \geqslant \mathsf{row}_{k+1}(P).$$

PROOF. First assume k is inserted before $k+1$ in the RSK bumping process. This means that $k+1$ can never end up below k, since whenever they are in the same row only k can be bumped down.

For the converse, assume $k+1$ is inserted before k. Then k will always be strictly above $k+1$, since if k is in the row exactly above $k+1$ and is being bumped down (or $k+1$ is in the first row and k is about to be inserted), k has to bump down $k+1$ in the next step and thus stay above. □

Let the *signature* of a word, $w = w_1 w_2 \dots w_n$, be a sequence of $+$'s and $-$'s which has a $+$ in position i iff $w_i < w_{i+1}$. For example, signature$(4,1,5,5,6,2,2) = (-,+,-,+,-,-)$. We now get Foulkes theorem as a consequence of Lemma 5.1 and the fact that $\mathrm{RSK}(\sigma) = (P,Q)$ iff $\mathrm{RSK}(\sigma^{-1}) = (Q,P)$:

THEOREM 5.1 (Foulkes). *Let $\sigma \in \mathcal{S}_n$ and $\mathrm{RSK}(\sigma) = (P,Q)$. Then*

$$\begin{aligned}\mathrm{signature}(\sigma^{-1}) &= \mathrm{signature}(\mathsf{col}(P)) = -\,\mathrm{signature}(\mathsf{row}(P)),\\ \mathrm{signature}(\sigma) &= \mathrm{signature}(\mathsf{col}(Q)) = -\,\mathrm{signature}(\mathsf{row}(Q)).\end{aligned}$$

The doubly alternating permutations are a special case:

COROLLARY 5.2. *Let $\sigma \in \mathcal{S}_n$ and $\mathrm{RSK}(\sigma) = (P,Q)$. Then*

$$\sigma \in \mathrm{DA}_n \quad \Longleftrightarrow \quad P, Q \in \mathrm{Alt}(\lambda).$$

Let T be an alternating standard tableau with $2n$ entries and at most three columns. We define the *pair column reading* $\mathsf{colpair}(T) = (w_i)_{i=1}^n$, where $w_i = \mathsf{col}_{2i-1}(T) + \mathsf{col}_{2i}(T) - 2$, i.e, $(1,2) \mapsto 1, (1,3) \mapsto 2$ and $(2,3) \mapsto 3$, since Lemma 5.1 tells us that the only possibilities for the pairs are $(1,2), (1,3)$ and $(2,3)$.

Let $w = (w_i)_{i=1}^l$ be a word, and $\mathsf{weight}(w) \stackrel{\mathrm{def}}{=} (|\{i : w_i = k\}|)_{k\geqslant 1}$ be the *weight* vector of w. We call w *Yamanouchi* (or a *ballot sequence*) if the weight of each prefix of w is a partition, i.e., it is weakly decreasing.

LEMMA 5.3. $\mathsf{colpair}$ *is a bijection between alternating standard tableaux with $2n$ elements and at most three columns and Yamanouchi words of length n on three letters.*

PROOF. Let T be an alternating standard tableaux, with at most three columns. Then $\mathsf{colpair}(T) = (w_i)_{i=1}^n$ is, as noted above, a word on the letters 1,2 and 3, so we need to show that $\mathsf{colpair}(T)$ is Yamanouchi iff $\mathsf{col}(T)$ is an alternating Yamanouchi word.

First assume $\mathsf{colpair}(T)$ is Yamanouchi and let $v = (w_i)_{i=1}^k$ be an arbitrary prefix of $\mathsf{colpair}(T)$. Then $\mathsf{weight}(v) = (a,b,c)$, is a partition, i.e., $a \geqslant b \geqslant c$. Hence $\mathsf{weight}((\mathsf{col}_j(T)_{j=1}^{2k}) = (a+b, a+c, b+c)$ is also a partition. Since $\mathsf{col}(T)$ is alternating and $\mathsf{weight}((\mathsf{col}_i(T))_{i=1}^{2k+1})$ is a partition if $\mathsf{weight}((\mathsf{col}_i(T))_{i=1}^{2k+2})$ is, it follows that $\mathsf{col}(T)$ is an alternating Yamanouchi word.

For the converse, assume $\mathsf{col}(T)$ is an alternating Yamanouchi word and let $u = (\mathsf{col}_i(T))_{i=1}^{2k}$ be a prefix of $\mathsf{col}(T)$. Then $\mathsf{weight}(u) = (d,e,f)$ is a partition, so $\mathsf{weight}((\mathsf{colpair}_i(T))_{i=1}^k) = \frac{1}{2}(d+e-f, d+f-e, e+f-d)$ is a partition, which proves that $\mathsf{colpair}(T)$ is Yamanouchi. □

Now we are ready to combine the bijections to get the bijection $\Phi : \mathcal{S}_n(1234) \to \mathrm{DA}_{2n}(1234)$, defined by

$$\Phi(\sigma) = \mathrm{RSK}^{-1}(\mathsf{colpair}^{-1}(\mathsf{col}(P)), \mathsf{colpair}^{-1}(\mathsf{col}(Q)),$$

where $\mathrm{RSK}(\sigma) = (P,Q)$. See Figure 8 for an illustrative example.

THEOREM 5.2. *Φ is a bijection, hence*

$$|\,\mathrm{DA}_{2n}(1234)| = |\,\mathcal{S}_n(1234)|.$$

PROOF. Let $\sigma \in \mathcal{S}_n(1234)$ and $\mathrm{RSK}(\sigma) = (P,Q)$. RSK is a bijection between permutations and pairs of standard tableaux of the same shape such that if the permutation is 1234-avoiding iff the shape does not have more than three columns. From the definitions we know that $\mathsf{col}(P)$ and $\mathsf{col}(Q)$ are Yamanouchi words, so, by Lemma 5.3, $\mathsf{colpair}^{-1}(\mathsf{col}(P))$ and $\mathsf{colpair}^{-1}(\mathsf{col}(Q))$ are alternating standard tableaux with at most three columns. Their shapes are the same since the weights of $\mathsf{col}(P)$ and $\mathsf{col}(Q)$ are the same, which is a consequence of P and Q having the same shape. Applying the inverse of RSK and using Corollary 5.2, we get that $\Phi(\sigma) \in \mathrm{DA}_{2n}(1234)$.

The converse is similar. □

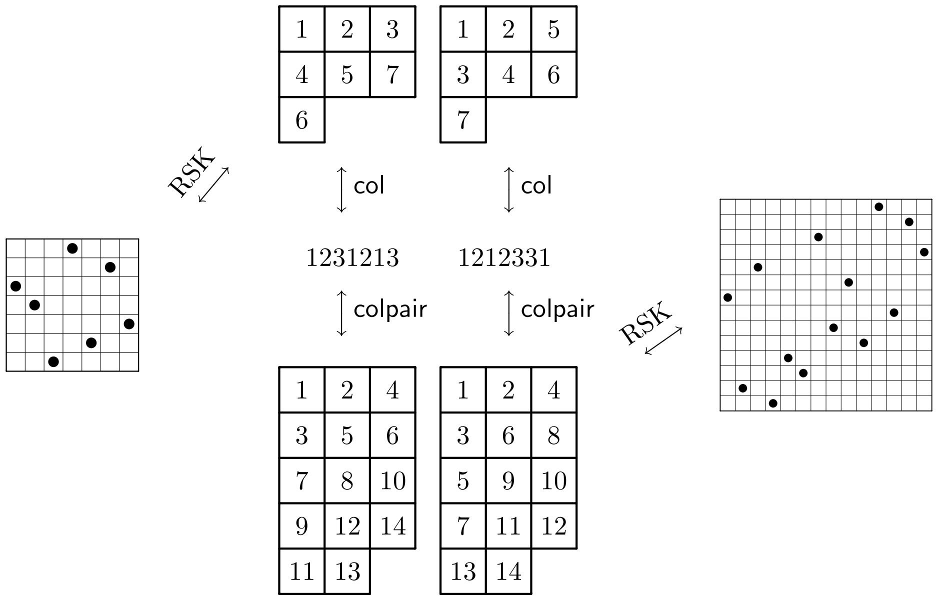

FIGURE 8. Example of the bijection $\Phi : \mathcal{S}_n(1234) \to \mathrm{DA}_{2n}(1234)$.

6. 21τ-avoiding doubly alternating permutations

The goal of this section is to find a bijection between $\mathrm{DA}_n(12\tau)$ and $\mathrm{DA}_n(21\tau)$, where τ is any permutation of $\{3, 4, \ldots, m\}$, $m \geqslant 3$, by using a well known bijection due to Babson and West [**2**]. The problem is that it is far from obvious that it will preserve the property of being doubly alternating. To show this we need a few definitions. During this section we assume τ to be fixed.

A dot, d, is called *active* if d is the 1 or 2 in any 12τ or 21τ pattern in σ and other dots are called *inactive*. Also the pair of dots, (d_1, d_2), is called an *active pair* if $d_1 d_2$ is the 12 in a 12τ-pattern or the 21 in a 21τ-pattern.

LEMMA 6.1. *Assume* $\sigma \in \mathrm{DA}_n(12\tau) \cup \mathrm{DA}_n(21\tau)$ *and* $d = (i, j)$ *is any active dot. Then* i *and* j *are odd.*

PROOF. First assume $\sigma \in \mathrm{DA}_n(12\tau)$ and that σ has a 21τ-pattern, otherwise there are no active dots. By inversion symmetry, we can assume that d is the 1 in a 21τ pattern. If $j = 1$, i.e. $\sigma^{-1}(1) = j$, then j is odd by Lemma 2.2, and if $j > 1$, then $\sigma^{-1}(j-1) > \sigma^{-1}(j)$, since a dots to the north-west of d would give a 12τ pattern. Hence j is odd. Also, to avoid the 12τ, $\sigma(i-1) > \sigma(i)$, so i is odd as well.

Now assume instead $\sigma \in \mathrm{DA}_n(21\tau)$. Let $d_1, d_2, \ldots, d_m$, be the dots in a 12τ pattern, with $d_k = (i_k, j_k)$. If i_1 is even then there is a descent from i_1 to $i_1 + 1$ and so the corresponding points along with tau will make the forbidden pattern. So i_1 is odd and the same argument applies to i_2, j_1, and j_2. □

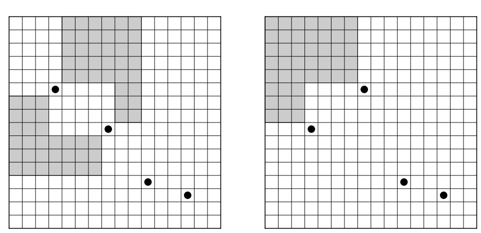

FIGURE 9. Illustration of the proof for Lemma 6.1, with $\tau = (3, 4)$. Shaded areas are forbidden.

We now define a Young diagram, λ_σ, consisting of the part of the board which contains the active dots. For a pair of dots, d_1, d_2, let R_{d_1,d_2} to be the smallest rectangle with top left coordinates $(1, 1)$, such that

$d_1, d_2 \in R_{d_1,d_2}$. Define

$$\lambda_\sigma \stackrel{\text{def}}{=} \bigcup R_{d_1,d_2},$$

where the union is over all active pairs (d_1, d_2). It is clear from the definition that λ_σ is indeed a Young diagram (see Figure 10).

A *rook placement* (also known as *traversal* or *transversal*) of a Young diagram, λ, is a placement of dots, such that all rows and columns contain exactly one dot. If some of the rows or columns are empty we call it a *partial rook placement*. Furthermore, we say that a rook placement on λ avoids the pattern τ if no rectangle, $R \subset \lambda$, contain τ.

The definition of λ_σ implies the following useful fact:

LEMMA 6.2. *Let $\sigma \in \mathrm{DA}_n$ and $\mathrm{rp}(\lambda_\sigma)$ be the partial rook placement on λ_σ induced by σ. Then*

$$\begin{aligned} \sigma \in \mathrm{DA}_n(12\tau) \quad &\Longleftrightarrow \quad \mathrm{rp}(\lambda_\sigma) \text{ is 12-avoiding,} \\ \sigma \in \mathrm{DA}_n(21\tau) \quad &\Longleftrightarrow \quad \mathrm{rp}(\lambda_\sigma) \text{ is 21-avoiding.} \end{aligned}$$

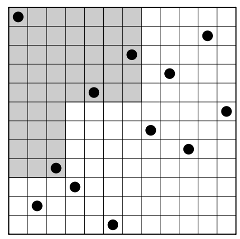

FIGURE 10. Example of λ_σ for $\sigma = (1, 11, 7, 9, 5, 12, 8, 10, 3, 4, 2, 6)$ and $\tau = (3, 4)$.

The bijection we will use is due to Babson and West [**2**], which built on work by Simion and Schmidt [**12**] and West [**14**]. But we give here the more general result by Backelin, West, and Xin [**3**].

THEOREM 6.1 (Backelin, West, Xin). *Let τ be any permutation of $\{t+1, t+2, \ldots, m\}$. Then for every Young diagram λ, the number of $(t, t-1, \ldots, 1, \tau)$-avoiding rook placements on λ equals the number of $(1, 2, \ldots, t, \tau)$-avoiding rook placements on λ.*

We call two permutations of the same size *a-equivalent* if all the inactive dots are the same, and write $\sigma_1 \sim_a \sigma_2$. We shall see in Lemma 6.4 that this implies $\lambda_{\sigma_1} = \lambda_{\sigma_2}$.

LEMMA 6.3. *If $\sigma \in \mathrm{DA}_n(12\tau) \cup \mathrm{DA}_n(21\tau)$ and $\nu \sim_a \sigma$, then ν is doubly alternating.*

PROOF. Let $d = (i, j) \in \nu$ be a dot in an odd row, so that, by Lemma 6.1, both the dots in row $i-1$ and row $i+1$ are inactive (if they exists). If d is inactive then all three dots also belong to σ so $\nu(i-1) > \nu(i) < \nu(i+1)$. If d is active there is a τ-pattern to the SE of d, so if either of the dots in row $i-1$ or row $i+1$ are to the left of d, then this dot is active, since it creates either a 12τ or 21τ pattern together with d and the τ pattern, giving a contradiction, so again $\nu(i-1) > \nu(i) < \nu(i+1)$. The same applies, by symmetry, to dots in odd columns. □

LEMMA 6.4. *If $\sigma, \nu \in \mathrm{DA}_n$, then*

$$\sigma \sim_a \nu \quad \Longrightarrow \quad \lambda_\sigma = \lambda_\nu.$$

PROOF. Let $\sigma \sim_a \nu$ be two arbitrary a-equivalent DA permutations and assume $s = (i, j)$ is a SE corner of λ_σ. We need to show that $s \in \lambda_\nu$, so that $\lambda_\sigma \subseteq \lambda_\nu$ and thus, since $\sim_a$ is reflexive, $\lambda_\nu = \lambda_\sigma$.

Let $R_{d_1,d_2} \subset \lambda_\sigma$ be a rectangle, such that $s \in R_{d_1,d_2}$. Such a rectangle must exist, otherwise could not s belong to λ_σ. Hence there is a τ-pattern to the SE of s and one of the d_k is in row i and one (possibly the same one) is in column j. But, as $\nu \sim_a \sigma$, they have the same inactive dots, so there must also exist a dot $d_1' \in \lambda_\nu$ in row i and a dot $d_2' \in \lambda_\nu$ in column j. If d_1' is east of s or if d_2' is south of s then $s \in \lambda_\nu$. Hence we can assume d_1' and d_2' to be weakly NW of s. If $d_1' \neq d_2'$, then $s \in R_{d_1',d_2'} \subset \lambda_\nu$, since the τ-pattern is still SE of s, and if $d_1' = d_2' = s$ then clearly $s \in \lambda_\nu$. □

Now we are ready to construct a bijection $\Psi : \mathrm{DA}_n(12\tau) \to \mathrm{DA}_n(21\tau)$. Let $\sigma \in \mathrm{DA}_n(12\tau)$, so that the restriction of σ to λ_σ is a partial 12-avoiding rook placement. By Theorem 6.1 (ignoring the empty rows and columns) and Lemma 6.4, there exists a unique 21-avoiding (partial) rook placement on λ_σ, with the same rows and columns empty, which we combine with the inactive dots of σ to get $\Psi(\sigma)$. By Lemma 6.3, $\Psi(\sigma)$ is DA, and Lemma 6.2 says that it avoids 21τ. It is also clear from Theorem 6.1 that it is indeed a bijection. We have thus bijectively shown:

THEOREM 6.2. *Let τ be any permutation of $\{3, 4, \ldots, m\}$, $m \geqslant 3$. Then*

$$|\,\mathrm{DA}_n(21\tau)| = |\,\mathrm{DA}_n(12\tau)|.$$

As a special case we have

COROLLARY 6.5. $\quad |\,\mathrm{DA}_n(2134)| = |\,\mathrm{DA}_n(1234)|.$

7. Other patterns with the same number sequence as $\mathcal{S}_n(1234)$

By examining all the patterns of length four with computer, Guibert found 15 different cases that all seem to give rise to the same sequence, $|\,\mathcal{S}_n(1234)|$. Using Theorems 5.2 and 6.2, inversion, rotation and Proposition 7.1 below, we get altogether ten bijections, see Figure 11. However, to prove that all of them are indeed the same we would need five more bijections. In fact, we conjecture that the number of permutations are the same in all the cases given below.

CONJECTURE 7.1 (Guibert).

$$\begin{aligned} |\,\mathrm{DA}_{2n}(1234)| &= |\,\mathrm{DA}_{2n+1}(1243)| \\ &= |\,\mathrm{DA}_{2n}(1432)| \\ &= |\,\mathrm{DA}_{2n+1}(1432)| \\ &= |\,\mathrm{DA}_{2n}(2341)| \\ &= |\,\mathrm{DA}_{2n}(3421)| \end{aligned}$$

One can note that many of the patterns in the conjecture are of the same type as treated in Theorem 6.1, but the proof does not work here, except for 2134, since the bijections destroy the DA property.

PROPOSITION 7.1. $\quad |\,\mathrm{DA}_{2n}(2143)| = |\,\mathrm{DA}_{2n+1}(3412)| = |\,\mathrm{DA}_{2n+2}(3412)|.$

PROOF. Let $\sigma \in \mathrm{DA}_n(3412)$, with $n \geqslant 4$. If $\sigma(1) > 1$, we get the forbidden pattern on the rows 1, 2, $\sigma^{-1}(1)$, $\sigma^{-1}(2)$, so $\sigma(1) = 1$. Let $\tilde{\sigma}$ be the permutation with the first row and column of σ removed. It is clear that if n is odd then $\tilde{\sigma}^c \in \mathrm{DA}_{n-1}(2143)$ iff $\sigma \in \mathrm{DA}_n(3412)$, and if n is even then $\tilde{\sigma}^\# \in \mathrm{DA}_{n-1}(3412)$ iff $\sigma \in \mathrm{DA}_n(3412)$. □

$\mathcal{S}_n(1234)$	$\xleftrightarrow{\text{Th. 5.2}}$	$\mathrm{DA}_{2n}(1234)$	$\xleftrightarrow{\text{Th. 6.2}}$	$\mathrm{DA}_{2n}(2134)$	$\xleftrightarrow{\#}$	$\mathrm{DA}_{2n}(1243)$
	$\xleftrightarrow{\text{Th. 6.2}}$	$\mathrm{DA}_{2n}(2143)$	$\xleftrightarrow{\text{Pr. 7.1}}$	$\mathrm{DA}_{2n+1}(3412)$	$\xleftrightarrow{\text{Pr. 7.1}}$	$\mathrm{DA}_{2n+2}(3412)$
$\mathrm{DA}_{2n+1}(1243)$	$\xleftrightarrow{\text{Th. 6.2}}$	$\mathrm{DA}_{2n+1}(2143)$				
$\mathrm{DA}_{2n}(1432)$	$\xleftrightarrow{\#}$	$\mathrm{DA}_{2n}(3214)$				
$\mathrm{DA}_{2n}(2341)$	$\xleftrightarrow{-1}$	$\mathrm{DA}_{2n}(4123)$				
$\mathrm{DA}_{2n}(3421)$	$\xleftrightarrow{\#}$	$\mathrm{DA}_{2n}(4312)$				
$\mathrm{DA}_{2n+1}(1432)$						

FIGURE 11. Known bijections between the sequences conjectured to be $|\,\mathcal{S}_n(1234)|$.

8. Avoiding pairs of patterns of length four

When we have two patterns of length four, there are a huge number of cases. We have not yet studied many of these, but would like to give a flavour of what can happen by presenting one result and two conjectures. Combining the results in Sections 4 and 5 we get

PROPOSITION 8.1.

$$|\,\mathrm{DA}_n(1234, 2413)| = \begin{cases} F_{n/2}, & \textit{if } n \textit{ is even,} \\ 2, & \textit{if } n = 5, \\ 1, & \textit{otherwise,} \end{cases}$$

where the F_n are the Fibonacci numbers.

PROOF. Let $\sigma \in \mathrm{DA}_{2n}(1234, 2413)$. By Corollary 4.2, σ can be factored into blocks, $\sigma_1, \sigma_2, \ldots \sigma_k$. As σ avoids 1234 must each block be either 12 or 1324, since each of them have a dot in the NW corner and the SE corner. Hence

$$|\,\mathrm{DA}_{2n}(1234, 2413)| = |\,\mathrm{DA}_{2n-2}(1234, 2413)| + |\,\mathrm{DA}_{2n-4}(1234, 2413)|,$$

and since $|\,\mathrm{DA}_0(1234, 2413)| = |\,\mathrm{DA}_2(1234, 2413)| = 1$, we get the Fibonacci numbers.

If $\sigma \in \mathrm{DA}_{2n+1}(1234, 2413)$, then $\sigma(1) = 1$. Let $\tilde{\sigma} = (2n + 1 - \sigma(i))_{i=2}^{2n+1}$ be the permutation constructed from σ by removing the first row and column and then flipping horizontally. Then $\tilde{\sigma} \in \mathrm{DA}_{2n}(321, 2413)$, which by Proposition 3.1(iii) gives two possibilities if $2n \geqslant 4$, namely $(1, 3, 2, 5, 4, \ldots, 2n-1, 2n-2, 2n)$ and $(3, 5, 1, 7, 2, 9, 4, \ldots, n, n-4, n-2)$. However, only the former avoids 2413 if $n \geqslant 6$, so we get the desired result. □

The following two conjectures have been verified by computer calculations up to $n = 23$.

CONJECTURE 8.1.

$$|\,\mathrm{DA}_n(1234, 3214)| = \begin{cases} F_{n-1}, & \textit{if } n \textit{ is even,} \\ 1, & \textit{if } n = 1 \textit{ or } n = 3, \\ F_{n-1} - F_{n-7}, & \textit{otherwise.} \end{cases}$$

CONJECTURE 8.2.

$$|\,\mathrm{DA}_n(1234, 2134)| = \begin{cases} C_{n/2}, & \textit{if } n \textit{ is even,} \\ 1, & \textit{if } n = 1 \textit{ or } n = 3, \\ C^{(4)}_{(n-5)/2}, & \textit{otherwise.} \end{cases}$$

Here $C^{(4)}_n$ is the fourth difference of the Catalan numbers, defined recursively by $C^{(0)}_n = C_n$ and $C^{(i+1)}_n = C^{(i)}_{n+1} - C^{(i)}_n$. By collecting the terms and simplifying we get

$$\begin{aligned} C^{(4)}_n &= C_{n+4} - 4C_{n+3} + 6C_{n+2} - 4C_{n+1} + C_n \\ &= 9C_n \frac{9n^4 + 54n^3 + 135n^2 + 122n + 40}{(n+2)(n+3)(n+4)(n+5)}. \end{aligned}$$

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