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A Proof of the q,t-Square Conjecture

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Abstract. We prove a combinatorial formula conjectured by Loehr and Warrington for the coefficient of the sign character in $\nabla(p_n)$. Here ∇ denotes the Bergeron-Garsia nabla operator, and p_n is a power-sum symmetric function. The combinatorial formula enumerates lattice paths in an $n \times n$ square according to two suitable statistics.

Résumé. Nous démontrons une formule combinatoire conjecturée par Loehr et Warrington concernant le coefficient du caratère signe dans $\nabla(p_n)$. Nous dénotons par ∇ l'operateur nabla de Bergeron-Garsia, et par p_n une fonction symmétrique en les puissances n-èmes. La formule combinatoire énumère les chemins, sur un rèseau carré de dimension $n \times n$, vérifiant deux statistiques.

1. Introduction

We begin with a quick overview of the remarkable q,t-*Catalan theorem* and the q,t-*square conjecture*. The q,t-Catalan theorem is the culmination of a series of papers by Garsia, Haglund, and Haiman [**2, 3, 4, 8, 9**]. The theorem states that, for every n, the following seemingly unrelated quantities are in fact equal:

1. the weighted sum of all *Dyck paths* of order n, weighted by *area* and *bounce score*;
2. the *Hilbert series* of the module of *diagonal harmonic alternants* of order n;
3. the n'th Garsia-Haiman q,t-*Catalan number*, which is a certain sum of complicated rational functions constructed from partitions;
4. the coefficient of the sign character in $\nabla(e_n)$, where ∇ is the *Bergeron-Garsia nabla operator* [**1, 10**], and e_n is an elementary symmetric function.

Precise definitions of the terms mentioned here will be given later (§2).

Loehr and Warrington [**12**] recently found an analogue of this theorem that involves q,t-analogues of lattice paths inside squares. Their result, which we call the q,t-*square conjecture*, states that the following five quantities are equal for every n:

1. the weighted sum of all $n \times n$ *square paths* ending in a north step, weighted by *area* and *bounce score*;
2. the weighted sum of all $n \times n$ *square paths* ending in an east step, weighted by *area* and *bounce score*;
3. a certain sum of rational functions analogous to the Garsia-Haiman q,t-Catalan number;
4. the coefficient of the sign character in $(-1)^{n-1}\nabla(p_n)$, where p_n is a power-sum symmetric function.

Again, we defer precise definitions of these quantities to §2. Loehr and Warrington proved that items 1 and 2 were equal, and also proved that items 3 and 4 were equal. Based on extensive computer calculations, they conjectured that all four items were equal.

The main theorem of our paper is a proof of this q,t-square conjecture. Here is a rough outline of the proof strategy. In light of previous results, it suffices to prove that item 1 equals item 4 for all n. We will

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establish a refinement of this equality based on an expansion of p_n in terms of certain symmetric functions $E_{n,k}$ that appeared in the proof of the q,t-Catalan theorem. Explicitly, we will prove that

$$(-1)^{n-1}p_n = \sum_{k=1}^{n} \frac{1-q^n}{1-q^k} E_{n,k}. \tag{1.1}$$

(In contrast, $e_n = \sum_{k=1}^{n} E_{n,k}$.) Applying nabla and taking the coefficient of s_{1^n} gives

$$\langle (-1)^{n-1}\nabla(p_n), s_{1^n}\rangle = \sum_{k=1}^{n} \frac{1-q^n}{1-q^k} \langle \nabla(E_{n,k}), s_{1^n}\rangle. \tag{1.2}$$

Garsia and Haglund previously found combinatorial formulas and recursions for $\langle \nabla(E_{n,k}), s_{1^n}\rangle$. Comparing these results to recursions involving q,t-analogues of square lattice paths, we will show that each summand on the right side of (1.2) enumerates a suitable subcollection of the q,t-square paths mentioned in item 1. The equality of item 1 and item 4 will readily follow.

The rest of this paper is organized as follows. Section 2 reviews the minimal framework of definitions needed to give precise statements of the q,t-Catalan theorem and the q,t-square conjecture. Section 3 discusses some (previously known) technical results needed in our proof of the q,t-square conjecture. Section 4 uses a plethystic calculation to prove the fundamental expansion (1.1). Section 5 analyzes a combinatorial recursion that lets us identify the square q,t-lattice paths enumerated by each summand in (1.2). Section 6 concludes by discussing some natural open problems pertaining to the q,t-Catalan theorem and the q,t-square conjecture.

2. Definitions

This section reviews the definitions of the concepts appearing in the q,t-Catalan theorem and the q,t-square conjecture. Precise statements of these two results are given at the end of this section. We assume familiarity with standard background material on partitions, symmetric functions, representation theory, Macdonald polynomials, and lattice paths [**6, 13, 14, 15**]. Readers who find this section too terse may wish to consult the more leisurely treatment contained in the introduction of [**12**].

2.1. Partition Definitions. We write $\mu \vdash n$ to indicate that μ is a partition of n. The *diagram* of μ is

$$\mathrm{dg}(\mu) = \{(i,j) \in \mathbb{N}\times\mathbb{N} : 1 \le i \le \mu_j\}.$$

Let $c = (i_0, j_0)$ be a cell in $\mathrm{dg}(\mu)$. The *arm* of c is $a(c) = |\{(i,j_0) \in \mathrm{dg}(\mu) : i > i_0\}|$. The *coarm* of c is $a'(c) = |\{(i,j_0) \in \mathrm{dg}(\mu) : i < i_0\}|$. The *leg* of c is $l(c) = |\{(i_0,j) \in \mathrm{dg}(\mu) : j > j_0\}|$. The *coleg* of c is $l'(c) = |\{(i_0,j) \in \mathrm{dg}(\mu) : j < j_0\}|$. Let $n(\mu) = \sum_{c\in\mathrm{dg}(\mu)} l(c)$, and let μ' be the transpose of μ. In the ring $\mathbb{Z}[q,t] \subseteq \mathbb{Q}(q,t)$, define $M = (1-q)(1-t)$, $B_\mu = \sum_{c\in\mathrm{dg}(\mu)} q^{a'(c)}t^{l'(c)}$, $\Pi_\mu = \prod_{(1,1)\neq c\in\mathrm{dg}(\mu)}(1-q^{a'(c)}t^{l'(c)})$, $T_\mu = q^{n(\mu')}t^{n(\mu)}$, and $w_\mu = \prod_{c\in\mathrm{dg}(\mu)}[(q^{a(c)} - t^{l(c)+1})(t^{l(c)} - q^{a(c)+1})]$.

The n'th Garsia-Haiman q,t-Catalan number is defined by the formula

$$\sum_{\mu\vdash n} \frac{T_\mu^2 M B_\mu \Pi_\mu}{w_\mu} \in \mathbb{Q}(q,t). \tag{2.1}$$

This sum of rational functions evaluates to a polynomial in $\mathbb{N}[q,t]$, although this is quite hard to prove. The analogous expression appearing in the q,t-square conjecture is

$$\sum_{\mu\vdash n} \frac{T_\mu^2 M B_{(n^n)} \Pi_\mu}{w_\mu} \in \mathbb{Q}(q,t). \tag{2.2}$$

The only difference is that B_μ has been replaced by the constant $B_{(n^n)}$, where $(n^n) \vdash n^2$ consists of n parts equal to n. It is easy to see that this expression can also be written

$$(1-t^n)(1-q^n)\sum_{\mu\vdash n} \frac{T_\mu^2 \Pi_\mu}{w_\mu}. \tag{2.3}$$

2.2. Symmetric Function Definitions. We write Λ to denote the ring of symmetric functions with coefficients in the field $F = \mathbb{Q}(q,t)$. As usual, e_n will denote the n'th elementary symmetric function, p_n will denote the n'th power-sum symmetric function, s_μ will denote the Schur function indexed by a partition μ, and $\tilde{H}_\mu$ will denote the modified Macdonald polynomial indexed by μ. The Bergeron-Garsia *nabla operator* is the unique linear operator on Λ such that $\nabla(\tilde{H}_\mu) = T_\mu \tilde{H}_\mu$. The *Hall scalar product* $\langle \cdot,\cdot \rangle$ on Λ is defined by requiring that the Schur functions be an orthonormal basis. If $f \in \Lambda$ is the Frobenius character of some S_n-module M, then $\langle f, s_{1^n}\rangle$ gives the multiplicity of the sign representation in M. Accordingly, for any $f \in \Lambda$, we often call $\langle f, s_{1^n}\rangle$ the "coefficient of the sign character in f." We remark in passing that the well-known identities

$$e_n = \sum_{\mu \vdash n} (MB_\mu \Pi_\mu / w_\mu)\tilde{H}_\mu, \qquad (-1)^{n-1} p_n = \sum_{\mu \vdash n} (MB_{(n^n)} \Pi_\mu / w_\mu)\tilde{H}_\mu,$$

and $\langle \tilde{H}_\mu, s_{1^n}\rangle = T_\mu$ easily imply that $\langle \nabla(e_n), s_{1^n}\rangle$ and $\langle (-1)^{n-1}\nabla(p_n), s_{1^n}\rangle$ are given by formulas (2.1) and (2.2), respectively.

2.3. Path Definitions. A *Dyck path* of order n is a lattice path in the x,y-plane that starts at the origin, consists of n unit north steps (N) and n unit east steps (E), and always stays weakly above the line $y = x$. We let $\mathcal{DP}_n$ denote the set of Dyck paths of order n. For $P \in \mathcal{DP}_n$, define area(P) to be the number of complete lattice squares bounded by P and the line $y = x$. Next, we define Haglund's *bounce path* for P, which consists of certain quantities $v_i(P)$ for $i \geq 0$. We imagine a ball bouncing from (n,n) to $(0,0)$ and being deflected by P and the diagonal $y = x$. At stage $i \geq 0$, the ball is on the line $y = x$ and goes $v_i(P)$ units west until it is blocked by the upper end of a north step of P. The ball then "bounces" south $v_i(P)$ units back to the diagonal. This bouncing continues until the ball reaches $(0,0)$. The *bounce score* of P is bounce$(P) = \sum_{i\geq 0} iv_i(P)$. For example, the path P encoded by NNNENNEENENENEEE lies in $\mathcal{DP}_8$ and has area$(P) = 14$, $v_0(P) = 3$, $v_1(P) = 4$, $v_2(P) = 1$, and bounce$(P) = 6$.

A *square path* of order n is a lattice path in the x,y-plane that starts at the origin and consists of n unit north steps and n unit east steps. We write $\mathcal{SQP}_n$, $\mathcal{SQP}_n^N$, and $\mathcal{SQP}_n^E$ to denote (respectively) the set of all such square paths, the set of all such paths ending with a north step, and the set of all such paths ending in an east step. To define analogues of bounce and area as in [**12**], we need some auxiliary concepts. Fix a square path P. Consider the diagonal lines $y = x - c$, for $c = 0,1,2,\ldots$. The lowest such diagonal that meets the path P is called the *base diagonal*. The lowest point on P touching the base diagonal is called the *breakpoint*. We now let area(P) be the number of complete lattice squares in the region bounded on the left by P, on the right by the base diagonal, on the top by $y = n$ and on the bottom by $y = 0$. For example, the path P encoded by NEENEENNENEEENNENENENEENNNENNNEE lies in $\mathcal{SQP}_{15}^E$ and has base diagonal $y = x - 3$, breakpoint $(8,5)$, and area 25.

Now we define square bounce paths. Given P, let $y = x - c$ be the base diagonal for P. This time there are two bouncing balls. The first ball starts at (n,n) and moves vertically c units south to the base diagonal. Thereafter, the ball bounces west and south as in the Dyck path case, with all southward moves terminating on the base diagonal, until it reaches the breakpoint. The second ball starts at $(0,0)$ and moves horizontally c units east to the base diagonal. This ball proceeds to bounce northeast to the breakpoint as follows. Starting at the base diagonal, the ball moves north until it is blocked by the end of an east step of P. It then moves the same distance east to reach the base diagonal. (This is *not* simply a reflected version of the bouncing policy followed by the first ball!) Let the vertical moves made by the first ball (in order) have lengths $v_0'(P) = c, v_1'(P), \ldots, v_s'(P)$, and let the vertical moves made by the second ball have lengths $v_0''(P), \ldots, v_t''(P)$. We then define $(v_0(P), v_1(P), \ldots) = (v_t''(P), \ldots, v_0''(P), v_0'(P), \ldots, v_s'(P))$ and set bounce$(P) = \sum_{i\geq 0} iv_i(P)$ as before. For the specific example considered above, the first ball moves south 3, west 3, south 3, west 2, south 2, west 2, south 2, while the second ball moves east 3, north 2, east 2, north 2, east 2, north 1, east 1. Accordingly, $(v_0(P), v_1(P), \ldots) = (1,2,2,3,3,2,2)$ and bounce$(P) = 49$. See Figure 1.

2.4. q-Definitions. For $n \geq 1$, define $[n]_q = 1 + q + q^2 + \cdots + q^{n-1} = (1-q^n)/(1-q)$, $(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1})$, and $[n]!_q = (q;q)_n/(1-q)^n = \prod_{i=1}^n [i]_q$. We also set $[0]_q = 1 = (a;q)_0$.

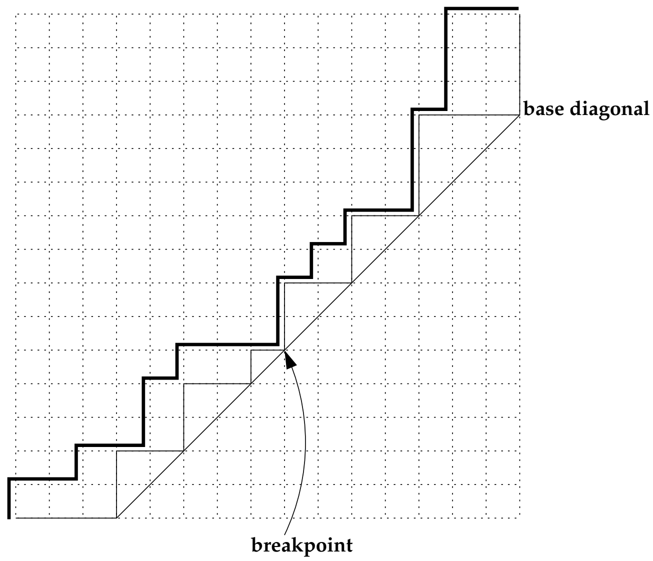

FIGURE 1. Square bouncing.

We define the q-binomial coefficient by setting

$$\begin{bmatrix} m+n \\ m,n \end{bmatrix}_q = \frac{(q;q)_{m+n}}{(q;q)_m(q;q)_n} = \frac{[m+n]!_q}{[m]!_q[n]!_q}.$$

It is well-known that $\left[\begin{smallmatrix} m+n \\ m,n \end{smallmatrix}\right]_q = \sum_{\mu\subseteq(m^n)} q^{|\mu|}$. Thus the q-binomial coefficient enumerates partitions (or lattice paths) contained in an $m \times n$ rectangle, weighted by area.

2.5. Precise Statements of Theorems. We summarize the preceding definitions by giving precise versions of the q,t-Catalan theorem and the q,t-square conjecture.

THEOREM 2.1 (Garsia-Haglund-Haiman q,t-Catalan Theorem). *For all $n \geq 1$, we have*

$$\sum_{P\in\mathcal{DP}_n} q^{\mathrm{area}(P)}t^{\mathrm{bounce}(P)} = \mathrm{Hilb}(DHA_n) = \sum_{\mu\vdash n} \frac{T_\mu^2 MB_\mu \Pi_\mu}{w_\mu} = \langle \nabla(e_n), s_{1^n}\rangle.$$

In particular, the last two expressions are elements of $\mathbb{N}[q,t]$.

THEOREM 2.2 (Loehr-Warrington). *For all $n \geq 1$, we have*

$$\sum_{P\in\mathcal{SQP}_n^N} q^{\mathrm{area}(P)}t^{\mathrm{bounce}(P)} = \sum_{P\in\mathcal{SQP}_n^E} q^{\mathrm{area}(P)}t^{\mathrm{bounce}(P)}$$

$$\textit{and} \qquad \sum_{\mu\vdash n} \frac{T_\mu^2 MB_{(n^n)}\Pi_\mu}{w_\mu} = \langle (-1)^{n-1}\nabla(p_n), s_{1^n}\rangle.$$

CONJECTURE 2.3 (Loehr-Warrington). For all $n \geq 1$, all four quantities in the previous theorem are equal. In particular, the last two expressions are elements of $\mathbb{N}[q,t]$.

The rest of this paper is devoted to a proof of this conjecture. More specifically, we will prove a refinement of the identity

$$\langle (-1)^{n-1}\nabla(p_n), s_{1^n}\rangle = \sum_{P\in\mathcal{SQP}_n^N} q^{\mathrm{area}(P)}t^{\mathrm{bounce}(P)}.$$

3. Technical Results

This section states without proof some known results of a somewhat technical nature that will be needed to establish the q,t-square conjecture.

3.1. Plethysm. We begin with some fundamental facts about plethystic notation. This material is treated in much greater detail in [**6, 11**].

Recall that Λ can be viewed as a polynomial ring $\Lambda = F[p_1, p_2, \ldots, p_k, \ldots]$. Like any polynomial ring, Λ enjoys a universal mapping property (UMP) that says that any function g mapping the set $\{p_k : k \geq 1\}$ into an F-algebra S extends uniquely to an F-algebra homomorphism from Λ into S. This homomorphism is often called the *evaluation homomorphism* determined by g.

Now, if $f \in \Lambda$ and A is a "plethystic alphabet," the plethystic substitution $f[A]$ is defined to be the image of f under the evaluation homomorphism determined by a certain function g_A. This function g_A is itself determined by A according to the rules for interpreting plethystic alphabets. We shall only need three special cases of this definition:

(1) $f[X(1-z)/(1-q)]$ is the image of f under the F-algebra homomorphism from Λ to $\Lambda[z]$ such that $p_k \mapsto p_k(1-z^k)/(1-q^k)$.
(2) $f[X/(1-q)]$ is the image of f under the F-algebra homomorphism from Λ to Λ such that $p_k \mapsto p_k/(1-q^k)$.
(3) $f[1-z]$ is the image of f under the F-algebra homomorphism from Λ to $F[z]$ such that $p_k \mapsto 1-z^k$.

We also have the trivial substitutions $f[X] = f$ and $f[0] = 0$ for $f \in \Lambda$.

We now state three (standard) facts about plethysm needed in our proof. First, for any alphabets A and B, we have the dual Cauchy identity

$$e_n[AB] = \sum_{\mu \vdash n} s_\mu[A] s_{\mu'}[B]. \tag{3.1}$$

Second, for all partitions μ, we have

$$s_\mu[1-z] = \begin{cases} (-z)^a(1-z) & \text{if } \mu = (n-a, 1^a) \text{ for some } a \in \{0,1,2,\ldots,n-1\}, \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

Third, $e_n[X(1-z)/(1-q)]$ is an element of the polynomial ring $\Lambda[z]$ of degree at most n in z.

3.2. Definition of $E_{n,k}$ and $F_{n,k}$. We can now define the symmetric functions $E_{n,k}$ mentioned in the introduction. Let M be the subset of $\Lambda[z]$ consisting of polynomials of degree at most n in z. Clearly, M is a free Λ-module with basis $1, z, z^2, \ldots, z^n$. Easy degree considerations show that the set $\{(z;q)_k/(q;q)_k : 0 \leq k \leq n\}$ is also a basis for M. Combining this observation with the third fact from the last subsection, we see that there exist unique elements $E_{n,k} \in \Lambda$ such that

$$e_n\left[\frac{X(1-z)}{1-q}\right] = \sum_{k=0}^{n} \frac{(z;q)_k}{(q;q)_k} E_{n,k}. \tag{3.3}$$

Setting $z = 1$, we see that $E_{n,0} = 0$, while setting $z = q$ shows that $e_n = \sum_{k=1}^n E_{n,k}$.[1]

Define $F_{n,k} = \langle \nabla(E_{n,k}), s_{1^n} \rangle \in \mathbb{Q}(q,t)$. Garsia and Haglund showed [**3, 5**] that the $F_{n,k}$ satisfy the recurrence

$$F_{n,k} = q^{k(k-1)/2} t^{n-k} \sum_{r=0}^{n-k} \begin{bmatrix} r+k-1 \\ r, k-1 \end{bmatrix}_q F_{n-k,r} \tag{3.4}$$

with initial conditions $F_{n,0} = \delta_{n0}$. On the other hand, let $\mathcal{DP}_{n,k}$ be the set of all Dyck paths P of order n that end in exactly k east steps. Equivalently, $\mathcal{DP}_{n,k} = \{P \in \mathcal{DP}_n : v_0(P) = k\}$. Let $F'_{n,k} = \sum_{P \in \mathcal{DP}_{n,k}} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)}$. By "removing the first bounce" in the bounce path for P, one easily sees that the $F'_{n,k}$ satisfy the same recurrence and initial conditions as $F_{n,k}$. Therefore, $F_{n,k} = F'_{n,k}$ for all n and k, i.e.,

$$\langle \nabla(E_{n,k}), s_{1^n} \rangle = \sum_{P \in \mathcal{DP}_{n,k}} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)} \in \mathbb{N}[q,t]. \tag{3.5}$$

This formula provides the fundamental link between the nabla operator and the combinatorics of q,t-Dyck paths.

[1]These statements, while true, require a subtle additional justification. See [**11**] for a detailed discussion.

4. Expansion of p_n via $E_{n,k}$'s

THEOREM 4.1. *For all $n \geq 1$,*

$$(-1)^{n-1}p_n = \sum_{k=1}^{n} \frac{1-q^n}{1-q^k} E_{n,k}.$$

PROOF. Using (3.1) and (3.2), we compute

$$\begin{aligned} e_n\left[\frac{X(1-z)}{1-q}\right] &= \sum_{\mu \vdash n} s_\mu[X/(1-q)]s_{\mu'}[1-z] \\ &= \sum_{\substack{\mu\vdash n \\ \mu'=(n-a,1^a)}} s_\mu[X/(1-q)](-z)^a(1-z) \\ &= \sum_{a=1}^{n} s_{(a,1^{n-a})}[X/(1-q)](-z)^{a-1}(1-z). \end{aligned}$$

On the other hand, using (3.3) and $E_{n,0}=0$, we get

$$e_n\left[\frac{X(1-z)}{1-q}\right] = \sum_{k=1}^{n} \frac{(z;q)_k}{(q;q)_k}E_{n,k} = \sum_{k=1}^{n} \frac{(1-z)(zq;q)_{k-1}}{(q;q)_k}E_{n,k}.$$

Comparing the two expressions for $e_n[X(1-z)/(1-q)]$ and cancelling $1-z$ in the integral domain $\Lambda[z]$, we obtain

$$\sum_{a=1}^{n}(-z)^{a-1}s_{(a,1^{n-a})}[X/(1-q)] = \sum_{k=1}^{n} \frac{(zq;q)_{k-1}}{(q;q)_k}E_{n,k}.$$

Now apply the evaluation homomorphism $\Lambda[z] \to \Lambda$ sending z to 1:

$$\sum_{a=1}^{n}(-1)^{a-1}(s_{(a,1^{n-a})}[X/(1-q)]) = \sum_{k=1}^{n} \frac{(q;q)_{k-1}}{(q;q)_k}E_{n,k} = \sum_{k=1}^{n} \frac{E_{n,k}}{1-q^k}.$$

By the Pieri rule and linearity of plethysm, the left side here is

$$\begin{aligned} \sum_{a=1}^{n}(-1)^{a-1}(s_{(a,1^{n-a})}[X/(1-q)]) &= \left(\sum_{a=1}^{n}(-1)^{a-1}s_{(a,1^{n-a})}\right)[X/(1-q)] \\ &= ((-1)^{n-1}p_n)[X/(1-q)] = (-1)^{n-1}p_n/(1-q^n). \end{aligned}$$

Putting this into the previous formula and multiplying through by $1-q^n$, we obtain the theorem. $\square$

By linearity of ∇ and the Hall scalar product, we immediately deduce the following corollary.

COROLLARY 4.2.

$$\langle(-1)^{n-1}\nabla(p_n), s_{1^n}\rangle = \sum_{k=1}^{n} \frac{[n]_q}{[k]_q}F_{n,k}.$$

5. Combinatorial Recursion Analysis

In this section, we will identify each summand in the last corollary as the weighted sum of a suitable subcollection of square lattice paths. Specifically, define

$$S_{n,k} = \sum_{P \in \mathcal{SQP}_n^N : v_0(P)=k} q^{\mathrm{area}(P)}t^{\mathrm{bounce}(P)}.$$

We will prove that $S_{n,k} = \frac{[n]_q}{[k]_q}F_{n,k}$. The q,t-square conjecture will easily follow from this fact and the corollary. To obtain these results, we first derive a recursion characterizing $S_{n,k}$.

THEOREM 5.1. *We have $S_{n,n} = q^{n(n-1)/2} = F_{n,n}$ for all n. For all $n \geq 1$ and $1 \leq k < n$, we have*

$$S_{n,k} = F_{n,k} + q^{k(k-1)/2}t^{n-k}\sum_{r=1}^{n-k} q^k \begin{bmatrix} r-1+k \\ r-1,k \end{bmatrix}_q S_{n-k,r} \tag{5.1}$$

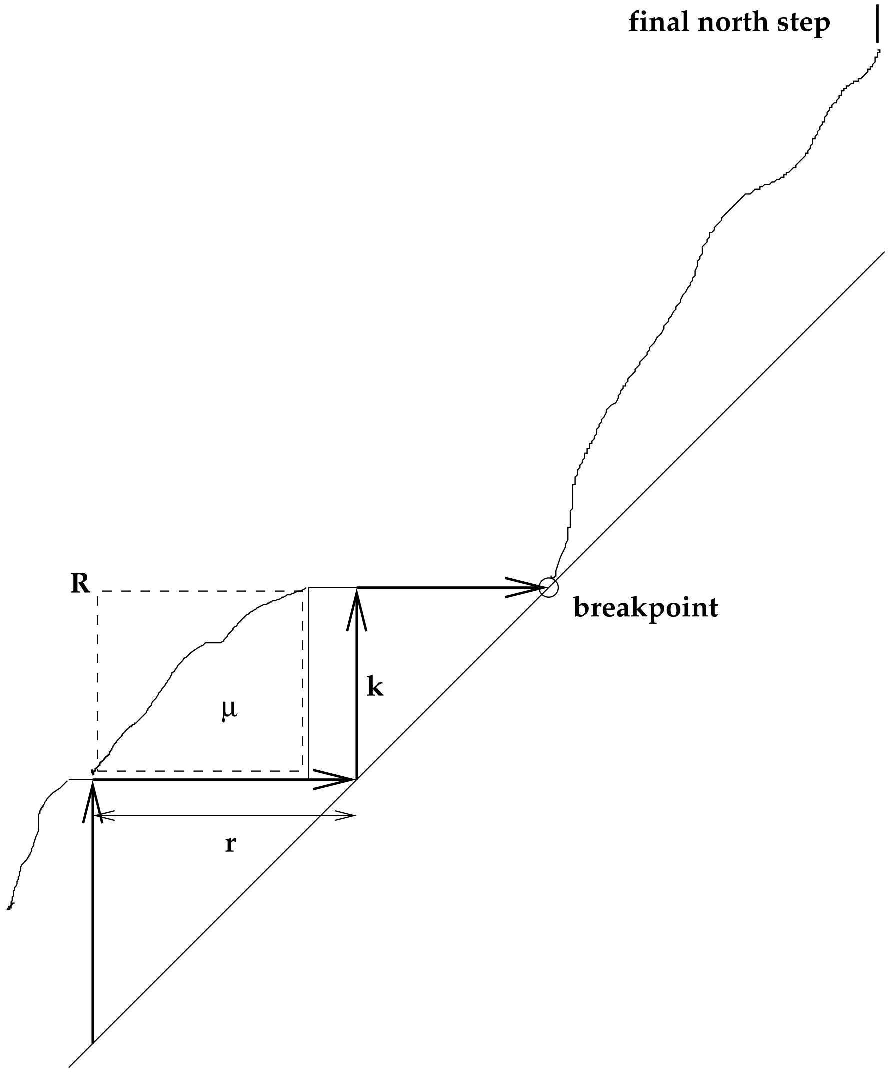

FIGURE 2. Removing the last negative bounce in case 2.

PROOF. Recall the combinatorial description of $F_{n,k}$ from §3.2:

$$F_{n,k} = \sum_{P \in \mathcal{DP}_n : v_0(P)=k} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)}.$$

The proof of the recurrence for $S_{n,k}$ is so similar to the proof of an analogous recurrence in [**12**] that we only sketch the details. (See Theorem 7 in [**12**] — the difference between the $R_{n,k}$ appearing there and the $S_{n,k}$ appearing here is that we demand that our paths end in a north step. This extra condition simplifies the recursion considerably.) Let P be a path counted by $S_{n,k}$. Observe that P is not a Dyck path, since it ends in a north step. Now consider two cases.

Case 1: The break point of P lies on the line $y = 0$. Then P must begin with k east steps. Moving these east steps to the end of the path and translating the break point to the origin, we obtain a typical path P' counted by $F_{n,k}$. (Note that $v_0(P') = k$ because P ends in a north step.) The map $P \mapsto P'$ defines a bijection between the paths P occurring in case 1 and the paths P' counted by $F_{n,k}$. Area and bounce are clearly preserved, so we have explained the first summand in (5.1).

Case 2: The break point of P lies above the line $y = 0$. We know $v_0(P) = k$; define $r = v_1(P)$, which is always the length of the horizontal move preceding the last vertical move made by the second bouncing ball.

A typical situation is pictured in Figure 2 — but note that the horizontal move of length r may also occur on the line $y = 0$. We can map P to a certain triple (r, μ, P'), where $r = v_1(P) \in \{1, 2, \ldots, n-k\}$, μ is the partition contained in the rectangle $R = ((r-1)^k)$ shown in the figure, and P' is a typical path counted by $S_{n-k,r}$. We obtain P' by merely erasing everything in the k rows immediately below the breakpoint, and then translating the part of P above the breakpoint k units down along the base diagonal. This has the effect of "removing the last bounce" made by the second bouncing ball. The rest of the bounce paths are unaffected by this shift, and it readily follows that $v_i(P') = v_{i+1}(P)$ and $\mathrm{bounce}(P) = \mathrm{bounce}(P') + n - k$. The breakpoint cannot be located at (n, n), so P' still ends in a north step. Since $P' \in \mathcal{SQP}_{n-k}^N$ and $v_0(P') = r$, P' is a path counted by $S_{n-k,r}$. It is not hard to see that $\mathrm{area}(P) = \mathrm{area}(P') + k(k-1)/2 + k + |\mu|$; here the $k(k-1)/2$ accounts for area cells to the right of the last vertical move made by the second ball, and the k accounts for the area cells in the column just left of this bounce, which is not part of μ. Finally, the passage from P (in case 2) to triples (r, μ, P') with $r \in \{1, 2, \ldots, n-k\}$, $\mu \subseteq (r-1)^k$, and $P' \in \mathcal{SQP}_{n-k}^N$ with $v_0(P') = r$ is clearly a bijection. Combining all these facts, we obtain the remaining terms in the recurrence (5.1). $\square$

THEOREM 5.2. *For all $n \geq 1$ and all $k \leq n$, $S_{n,k} = \frac{[n]_q}{[k]_q} F_{n,k}$.*

PROOF. The theorem holds for all n when $k = n$, since $S_{n,n} = F_{n,n}$ in this case. For the remaining cases, we use induction on n. Using the induction hypothesis to replace $S_{n-k,r}$ in the recursion (5.1), we first obtain

$$S_{n,k} = F_{n,k} + q^{k(k-1)/2} t^{n-k} \sum_{r=1}^{n-k} \frac{[r+k-1]!_q}{[r-1]!_q [k]!_q} \left(\frac{[n-k]_q}{[r]_q} F_{n-k,r} \right) q^k.$$

Rearranging the q-numbers here, the right side can be written

$$F_{n,k} + \frac{q^k [n-k]_q}{[k]_q} \left(q^{k(k-1)/2} t^{n-k} \sum_{r=1}^{n-k} \begin{bmatrix} r+k-1 \\ r, k-1 \end{bmatrix}_q F_{n-k,r} \right).$$

Comparing to (3.4), we see that the term in parentheses is just $F_{n,k}$ again! So the calculation continues:

$$S_{n,k} = F_{n,k} \left(1 + \frac{q^k [n-k]_q}{[k]_q} \right) = \frac{[n]_q}{[k]_q} F_{n,k}.$$

This completes the induction step and the proof. $\square$

Evidently $\mathcal{SQP}_n^N$ is the disjoint union of its subsets $\{P \in \mathcal{SQP}_n^N : v_0(P) = k\}$ as k ranges from 1 to n. Combining this fact with Theorem 5.2 and Corollary 4.2, we obtain our desired result:

COROLLARY 5.3.

$$\langle (-1)^{n-1} \nabla(p_n), s_{1^n} \rangle = \sum_{P \in \mathcal{SQP}_n^N} q^{\mathrm{area}(P)} t^{\mathrm{bounce}(P)}.$$

In particular, $\langle (-1)^{n-1} \nabla(p_n), s_{1^n} \rangle = \sum_{\mu \vdash n} \frac{T_\mu^2 M B_{(n^n)} \Pi_\mu}{w_\mu}$ is an element of $\mathbb{N}[q,t]$.

6. Conclusion

Comparing the q,t-Catalan theorem to the q,t-square theorem (as we shall now call it), one obvious difference is apparent: the latter theorem does not identify $\langle (-1)^{n-1} \nabla(p_n), s_{1^n} \rangle$ as the Hilbert series of some doubly graded module. Of course, one could define such a module by taking a direct sum of sign representations indexed by square paths, using the area and bounce statistics to determine the bigrading. We leave it as an open problem to find a less artificial solution, i.e., "naturally occurring" modules M_n carrying only the sign representation such that $\mathrm{Hilb}(M_n)$ is given by the quantities in the q,t-square theorem. More generally, one could seek modules whose Frobenius characters are given by $(-1)^{n-1} \nabla(p_n)$, just as the Frobenius characters of the diagonal harmonics modules DH_n are given by $\nabla(e_n)$ [**9**].

In closing, we recall that combinatorial interpretations have been proposed [**7, 12**] for the monomial expansions of the symmetric functions $\nabla(e_n)$, $\nabla(E_{n,k})$, and $\nabla(p_n)$. These conjectures involve labelled versions of Dyck paths and square lattice paths. At the time of this writing, all these conjectures are still open.

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