Bijections of trees arising from Voiculescu’s free probability theory
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Quotient trees
Let a polygonal graph \( G \) with arbitrary orientations of the edges be fixed. Let \( x \) be a pairing between the edges of \( G \). We will always assume that \( x \) is non-crossing and compatible with the orientations of edges (in each pair of connected edges one is oriented clockwise and the other counterclockwise).

Let us glue together each pair of edges connected by \( x \). The resulting graph \( T_x \) is a tree (called quotient tree) and each edge inherits the orientation from the orientations of the edges in the original graph \( G \).

Orders on quotient trees
The orientations of edges define a partial order \( \prec \) on the vertices of tree \( T_x \) (convention: if \( A \leftrightarrow B \) we write \( A < B \)).

In the following we shall consider some total (linear) orders on the vertices of \( T_x \). From such an order \( \prec \) we can define a compatible linear order on the edges of \( T_x \) corresponding to the partial order \( \prec \).

Quotient trees considered above naturally have a structure of planar rooted trees with a root \( R \). By \( \prec \) we denote the order on the vertices given by pre-order. For example, in the above case we have \( v_1 \prec v_2 < v_3 \prec v_5 \prec v_6 \).

Regular polygonal graphs
For integers \( l, m \geq 1 \) we consider \((l, m)\)-regular graph. It is the polygonal graph with \( 2lm \) edges of the form below. It consists of \( 2m \) groups of edges, each group consists of \( l \) edges with the same orientation, consecutive groups have opposite orientations.

Generalized parking functions
Let integers \( l, m \geq 1 \) be fixed. We say that \( (a_1, \ldots, a_{lm}) \) is an \((l, m)\)-parking function if:

- \( a_1, \ldots, a_{lm} \in \{1, \ldots, m\} \);
- for each \( 1 \leq n < m \) in the sequence \( (a_1, \ldots, a_{lm}) \) there are at most \( n \) elements which belong to \( \{1, \ldots, n\} \).

Random lemma implies that the number of \((l, m)\)-parking functions is equal to \( m^m \).

Main result: Bijection between ordered trees and parking functions

**Theorem.** Let \( l, m \geq 1 \) be fixed. The algorithm MainBijection provides a bijection between:

- the set of pairs \((T_x, \prec)\), where \( T_x \) is a quotient tree corresponding to the \((l, m)\)-regular graph and \( \prec \) is a total order on vertices of \( T_x \) compatible with the orientations of edges;
- the set of \((l, m)\)-parking functions.

**Corollary: general Cauchy identities**

\[
x^p = \sum_{\mu} \binom{n}{\mu} y^\mu \\
x^q = \sum_{\mu} \binom{n}{\mu} y^\mu \sum_{\nu} \binom{\mu}{\nu} \binom{n-\mu}{n-\mu-\nu} \\
\]

Auxiliary bijection between ordered trees

**Theorem.** Let integers \( l, m \geq 1 \) be given. The algorithm SmallBijection provides a bijection between:

- the set of quotient trees \((T_x, \prec)\) corresponding to \((l, m)\)-regular graph equipped with a total order \( \prec \) compatible with the orientation of the edges;
- the set of quotient trees \((T_x, \prec)\) corresponding to \((l, m)\)-regular graph equipped with a total order \( \prec \) on the vertices with the following two properties:
  - on the set \( \{x \in T_x : x \geq R\} \) the orders \( \prec \) and \( \alpha \) coincide, where \( R \) denotes the root;
  - for all pairs of vertices \( v, w \in T_x \) such that \( R \neq v \neq w \neq R \) we have \( v < w \iff \alpha(v) \not\sim \alpha(w) \).

Applications
In the limit \( l \to \infty \) orders on trees can be interpreted as stochastic processes in \( \mathcal{X}^1 \) (Brownian motions, Brownian bridges). Above bijections give rise to measure-preserving maps related to Pitman transform.

Where is Voiculescu’s free probability theory?
Please, ask me about it!

References