Kreweras Walks and Loopless Triangulations

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Kreweras walks

Walks made of *West*, *South* and *North – East* steps, starting and ending at the origin and confined in the first quadrant.
Preliminary remarks

Kreweras walks are words $w$ on \{a, b, c\} such that

- $|w|_a = |w|_b = |w|_c$,
- for any prefix $w'$, $|w'|_a \leq |w'|_c$ and $|w'|_b \leq |w'|_c$.

$$w = caccbbcbcbbaaaa$$
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Kreweras walks

Theorem (Kreweras 65): The number of Kreweras walks of size $n$ (3n steps) is

$$k_n = \frac{4^n}{(n + 1)(2n + 1)} \binom{3n}{n}.$$
Kreweras walks

**Theorem (Kreweras 65):** The number of Kreweras walks of size $n$ ($3n$ steps) is

$$k_n = \frac{4^n}{(n+1)(2n+1)} \binom{3n}{n}.$$ 

[Kreweras 65, Niederhausen 82, 83, Gessel 86, Bousquet-Mélou 05]
Kreweras walks and cubic maps

- Cubic maps and depth-trees.

- Bijection:
  \[ \text{Kreweras walk } \iff \text{Cubic map } + \text{ Depth-tree.} \]

- Counting Kreweras walks and cubic maps.

- Open problems.
Cubic maps and depth-trees
Maps

A map is a connected planar graph properly embedded in the sphere. The map is considered up to deformation.

\[
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{map1}
\quad = \quad \includegraphics[width=0.3\textwidth]{map2}
\end{array}
\neq
\begin{array}{c}
\includegraphics[width=0.3\textwidth]{map3}
\end{array}
\]
Maps

A **map** is a connected planar graph properly embedded in the sphere. The map is considered up to deformation.

A map is **rooted** if a half-edge is distinguished as the root.
Cubic maps

A map is cubic if every vertex has degree 3.
Cubic maps

A map is **cubic** if every vertex has degree 3.

We focus on cubic maps *without isthmus*.
Cubic maps and triangulations

Cubic maps without isthmus are the dual of loopless triangulations.
Cubic maps and triangulations

*Cubic maps* without isthmus are the *dual* of loopless triangulations.
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*Cubic maps* without isthmus are the *dual* of loopless triangulations.
Cubic maps - counting result

**Remark:** The number of edges of a cubic map is always a multiple of 3.

A cubic map of size $n$ has $3n$ edges, $2n$ vertices and $n + 2$ faces.
Cubic maps - counting result

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A cubic map of size $n$ has $3n$ edges, $2n$ vertices and $n + 2$ faces.

**Theorem [Mullin 65, Poulalhon & Schaeffer 03]:**
The number of cubic maps without isthmus of size $n$ is

$$c_n = \frac{2^n}{(n + 1)(2n + 1)} \binom{3n}{n} = \frac{k_n}{2^n}.$$
Depth-trees

We consider *spanning trees* of rooted maps.
Depth-trees

A spanning tree of a rooted map is a depth-tree if every external edge links a vertex to one of its ancestors.
Counting depth-trees

**Theorem:** For any cubic map of size $n$ ($3n$ edges), there are $2^n$ depth-trees not containing the root.
Counting depth-trees

**Theorem:** For any cubic map of size $n$ ($3n$ edges), there are $2^n$ depth-trees not containing the root.

**Example:** $n=3$
Counting depth-trees

(Idea of the) proof:

- The depth-trees are the trees that can be obtained by a *depth-first search algorithm* (DFS).

- During a DFS, there are $n$ *real* binary choices. (One for each external edge.)
Kreweras walk

Cubic map + Depth-tree
Kreweras walk

\[ k_n = c_n \times 2^n \]

Cubic map + Depth-tree
Bijection

Example:

\[ w = cacccbbcbcbbaaaa \]
Bijection

Example:

\[ w = caccbbcbcbbaaa.a \]
Bijection

Example:

\[ w = caccbbcbbcbbaaaa \]
Bijection

Example:

\[ w = caccbbcbcbbaaaa \]
Bijection

Example:

\[ w = caccbbcbcbbaaaa \]
Bijection

Example:

\[ w = \text{caccbbcbcbbaaaa} \]
Bijection

Example:

\[ w = cacccbbcbcbbaaaa \]
Bijection

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Bijection

Example:

\[ w = caccbbcbcbbaaaaa \]
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\[ w = caccbbcbcbaaaaa \]
Bijection

Example:

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Bijection

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Bijection

Example:

\[ w = caccbbcbcbbaaaa \]
Bijection

Example:

\[ w = caccbbcbbbaaaa \]
**Theorem:** This construction is a bijection between Kreweras walks of size $n$ and cubic maps of size $n + \text{depth-tree}$.

**Corollary:** $k_n = c_n \times 2^n$. 
Proof: The reverse bijection

\[ w = caccbbcbcbbaaaa \]
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\[ w = caccbbcbcbbaaa \]
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\[ w = caccbbcbcbbaaa \]
Proof: The reverse bijection

\[ w = caaccbcbbcbbaaa \]
Proof: The reverse bijection

\[ w = cacbbcbbbaaaa \]
Proof: The reverse bijection

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Proof: The reverse bijection

\[ w = caccbbcbbbaaa \]
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\[ w = caccbbbbbcbbaaaa \]
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\[ w = cacccbcbcbbaaaa \]
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Proof: The reverse bijection

\[ w = caccbbcbcbbaaaa \]
Counting Kreweras walks and cubic maps
Relaxing some constraints

Kreweras walks are the words $w$ on $\{a, b, c\}$ such that

1. $|w|_a = |w|_b = |w|_c$,
2. for any prefix $w'$, $|w'|_a \leq |w'|_c$ and $|w'|_b \leq |w'|_c$. 
Relaxing some constraints

Kreweras walks are the words $w$ on $\{a, b, c\}$ such that
- $|w|_a = |w|_b = |w|_c$,
- for any prefix $w'$, $|w'|_a \leq |w'|_c$ and $|w'|_b \leq |w'|_c$.

What about words $w$ on $\{a, b, c\}$ such that
- $|w|_a + |w|_b = 2|w|_c$,
- for any prefix $w'$, $|w'|_a + |w'|_b \leq 2|w'|_c$?

We call them excursions.
Kreweras

\[ w = caccaacbcbbbaaaa \]

\[ |w'|_a \leq |w'|_c \text{ and } |w'|_b \leq |w'|_c \]

Excursion

\[ w = caccbbcbcbbaaaa \]

\[ |w'|_a + |w'|_b \leq 2|w'|_c \]
Proposition: There are \( e_n = \frac{4^n}{2n + 1} \binom{3n}{n} \) excursions of size \( n \).
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Proof: The excursions $w$ are such that:

$$|w|_a + |w|_b = 2|w|_c,$$

for all prefix $w'$, $|w'|_a + |w'|_b \leq 2|w'|_c$.

- Position of the c’s: $\frac{1}{2n+1} \binom{3n}{n}$.

Cycle lemma: There are $\frac{1}{2n+1} \binom{3n}{n}$ (one-dimensional) walks with $3n$ steps +2 and -1.

- Position of the a’s and b’s: $2^{2n}$.
Extending the bijection

Example:

\[ w = caccaacbebbaaaa \]
Extending the bijection

Example:

\[ w = caccaacbcbbaaa \]
Extending the bijection

Example:

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Extending the bijection

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**Theorem:** This construction is a bijection between ecursions of size $n$ and cubic maps of size $n +$ depth-tree $+$ marked external edge.

**Corollary:** $e_n = c_n \times 2^n \times (n + 1)$. 
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**Corollary:** $e_n = c_n \times 2^n \times (n + 1)$.

Thus,

$$c_n = \frac{2^n}{(n + 1)(2n + 1)} \binom{3n}{n} \quad \text{and} \quad k_n = \frac{4^n}{(n + 1)(2n + 1)} \binom{3n}{n}.$$
Concluding remarks
Results

- We established a bijection between Kreweras walks and cubic maps with a depth-tree.

  \[ \Rightarrow \text{Coding of triangulations with } \log_2(27) \text{ bits per vertex.} \]
  \[ \text{(Optimal coding: } \log_2(27) - 1 \text{ bits per vertex.)} \]
Results

- We established a bijection between Kreweras walks and cubic maps with a depth-tree.

  ⇒ Coding of triangulations with $\log_2(27)$ bits per vertex. (Optimal coding: $\log_2(27) - 1$ bits per vertex.)

- We extended the bijection to a more general class of walks.

  ⇒ Counting results.

  ⇒ Random sampling of triangulations in linear time.

\[ k_n = \frac{4^n}{(n + 1)(2n + 1)} \binom{3n}{n}. \]
Open problems

Can we count Kreweras walks ending at \((i, 0)\) ? at \((i, j)\) ?
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Can we count Kreweras walks ending at \((i, 0)\) ? at \((i, j)\) ?

Theorem [Kreweras 65] : There are

\[
k_{n,i} = 4^n \binom{2i}{i} \frac{2i + 1}{(n + i + 1)(2n + 2i + 1)} \binom{3n + 2i}{n}
\]

Kreweras walks ending at \((i, 0)\).
Open problems

Can we count Kreweras walks ending at $(i, 0)$? at $(i, j)$?

**Theorem [Kreweras 65]:** There are

$$k_{n,i} = 4^n \binom{2i}{i} \frac{2i + 1}{(n + i + 1)(2n + 2i + 1)} \binom{3n + 2i}{n}$$

Kreweras walks ending at $(i, 0)$.

**Remark:** Kreweras walks ending at $(i, 0)$ and $(i + 2)$-near-cubic maps are related:

$$k_{n,i} = 2^n \times c_{n,i}.$$
Open problems

There are similar counting results:

- Non-separable maps [Tutte].
- Two-stack sortable permutations [West, Zeilberger].

\[\mathcal{NS}_n = \frac{2}{(n+1)(2n+1)} \binom{3n}{n}.\]

[Dulucq, Gire & Guibert 96, Goulden & West 96]
Thanks.