On the Number of Factorizations of Full Cycles

John Irving

Department of Mathematics and Computing Science
Saint Mary's University, Halifax

FPSAC 2006

The Symmetric Group

- $\mathfrak{S}_n$ is the group of permutations on $\{1, 2, \ldots, n\}$
- $\pi \in \mathfrak{S}_n$ is of cycle type $[1^{m_1} 2^{m_2} \cdots] \vdash n$ if it consists of m_i disjoint i-cycles
- $\mathscr{C}_\alpha$ is the conjugacy class consisting of all permutations of cycle type $\alpha \vdash n$

Example

$$(1\,3\,7\,2)(5\,4)(8\,6)(9) \in \mathscr{C}_{[1\,2^2\,4]} \subset \mathfrak{S}_9$$

The Symmetric Group

- $\mathfrak{S}_n$ is the group of permutations on $\{1, 2, \ldots, n\}$
- $\pi \in \mathfrak{S}_n$ is of cycle type $[1^{m_1} 2^{m_2} \cdots] \vdash n$ if it consists of m_i disjoint i-cycles
- $\mathscr{C}_\alpha$ is the conjugacy class consisting of all permutations of cycle type $\alpha \vdash n$

Example

$$(1\,3\,7\,2)(5\,4)(8\,6)(9) \in \mathscr{C}_{[1\,2^2\,4]} \subset \mathfrak{S}_9$$

- $\mathfrak{S}_n$ is the group of permutations on $\{1, 2, \ldots, n\}$
- $\pi \in \mathfrak{S}_n$ is of cycle type $[1^{m_1} 2^{m_2} \cdots] \vdash n$ if it consists of m_i disjoint i-cycles
- $\mathscr{C}_\alpha$ is the conjugacy class consisting of all permutations of cycle type $\alpha \vdash n$

Example

$$(1\,3\,7\,2)(5\,4)(8\,6)(9) \in \mathscr{C}_{[1\,2^2\,4]} \subset \mathfrak{S}_9$$

The Symmetric Group

- $\mathfrak{S}_n$ is the group of permutations on $\{1, 2, \ldots, n\}$
- $\pi \in \mathfrak{S}_n$ is of cycle type $[1^{m_1} 2^{m_2} \cdots] \vdash n$ if it consists of m_i disjoint i-cycles
- $\mathscr{C}_\alpha$ is the conjugacy class consisting of all permutations of cycle type $\alpha \vdash n$

Example

$$(1\,3\,7\,2)(5\,4)(8\,6)(9) \in \mathscr{C}_{[1\,2^2\,4]} \subset \mathfrak{S}_9$$

Definition

Let $c^{(n)}_{\alpha_1,\dots,\alpha_m}$ be the number of ways of writing $(1\,2\,\cdots\,n) \in \mathfrak{S}_n$ as an ordered product $\sigma_1 \cdots \sigma_m$, where $\sigma_i \in \mathscr{C}_{\alpha_i}$.

Example

In $\mathfrak{S}_6$ the following factorizations are counted by $c^{(6)}_{[2^3],[2\,4]}$:

$$(1\,2\,3\,4\,5\,6) = \overbrace{(1\,3)(2\,5)(4\,6)}^{\sigma_1} \cdot \overbrace{(1\,5\,4\,2)(3\,6)}^{\sigma_2}$$
$$= (1\,5)(2\,4)(3\,6) \cdot (1\,4)(2\,6\,5\,3)$$

Definition

Let $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$ be the number of ways of writing $(1\,2\,\cdots\,n) \in \mathfrak{S}_n$ as an ordered product $\sigma_1 \cdots \sigma_m$, where $\sigma_i \in \mathscr{C}_{\alpha_i}$.

Example

In $\mathfrak{S}_6$ the following factorizations are counted by $c^{(6)}_{[2^3],[2\,4]}$:

$$\begin{aligned}(1\,2\,3\,4\,5\,6) &= \overbrace{(1\,3)(2\,5)(4\,6)}^{\sigma_1} \cdot \overbrace{(1\,5\,4\,2)(3\,6)}^{\sigma_2} \\ &= (1\,5)(2\,4)(3\,6) \cdot (1\,4)(2\,6\,5\,3)\end{aligned}$$

An Explicit Formula

For a partition $\lambda = [1^{m_1} 2^{m_2} 3^{m_3} \cdots]$, let

- $\ell(\lambda) := m_1 + m_2 + \cdots$
- $z_\lambda := \prod_i i^{m_i} m_i!$
- $\mathrm{Aut}(\lambda) := \prod_i m_i!$

Theorem (Goulden & Jackson, 1995)

Let $\alpha_1, \ldots, \alpha_m \vdash n$ with $\ell(\alpha_1) + \cdots + \ell(\alpha_m) = n(m-1)+1$. Then

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = n^{m-1} \prod_{i=1}^{m} \frac{(\ell(\alpha_i)-1)!}{\mathrm{Aut}(\alpha_i)}.$$

An Explicit Formula

For a partition $\lambda = [1^{m_1} 2^{m_2} 3^{m_3} \cdots]$, let

- $\ell(\lambda) := m_1 + m_2 + \cdots$
- $z_\lambda := \prod_i i^{m_i} m_i!$
- $\mathrm{Aut}(\lambda) := \prod_i m_i!$

Theorem (Goulden & Jackson, 1995)

Let $\alpha_1, \ldots, \alpha_m \vdash n$ *with* $\ell(\alpha_1) + \cdots + \ell(\alpha_m) = n(m-1) + 1$. *Then*

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = n^{m-1} \prod_{i=1}^{m} \frac{(\ell(\alpha_i) - 1)!}{\mathrm{Aut}(\alpha_i)}.$$

What Else is Known?

- 1998 (Goupil & Schaeffer): $c^{(n)}_{\alpha,\beta}$
- 2002 (Poulalhon & Schaeffer): $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$
- 2005 (Biane): $c^{(n)}_{\alpha,\beta}$

What Else is Known?

- 1998 (Goupil & Schaeffer): $c_{\alpha,\beta}^{(n)}$
- 2002 (Poulalhon & Schaeffer): $c_{\alpha_1,\ldots,\alpha_m}^{(n)}$
- 2005 (Biane): $c_{\alpha,\beta}^{(n)}$

What Else is Known?

- 1998 (Goupil & Schaeffer): $c^{(n)}_{\alpha,\beta}$
- 2002 (Poulalhon & Schaeffer): $c^{(n)}_{\alpha_1,\ldots,\alpha_m}$
- 2005 (Biane): $c^{(n)}_{\alpha,\beta}$

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

- $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ be the usual elementary symmetric function
- $2\lambda - 1 := (2\lambda_1 - 1, 2\lambda_2 - 1, 2\lambda_3 - 1, \ldots)$
- $R_\lambda(x, y) := \frac{1}{2y} \prod_{i \geq 1} ((x+y)^{\lambda_i} - (x-y)^{\lambda_i}) = \sum_{j+k=n-1} R_\lambda^j \frac{x^j}{j!} \frac{y^k}{k!}$.

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R_{\alpha_i}^{j_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ such that $0 \leq j_i \leq n$ for all i.

Main Result

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

- $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ be the usual elementary symmetric function
- $2\lambda - 1 := (2\lambda_1 - 1, 2\lambda_2 - 1, 2\lambda_3 - 1, \ldots)$
- $R_\lambda(x, y) := \frac{1}{2y} \prod_{i \geq 1} ((x+y)^{\lambda_i} - (x-y)^{\lambda_i}) = \sum_{j+k=n-1} R_\lambda^j \frac{x^j}{j!} \frac{y^k}{k!}$.

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then

$$c^{(n)}_{\alpha_1, \ldots, \alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R^{j_i}_{\alpha_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ such that $0 \leq j_i \leq n$ for all i.

Main Result

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

- $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ be the usual elementary symmetric function
- $2\lambda - 1 := (2\lambda_1 - 1, 2\lambda_2 - 1, 2\lambda_3 - 1, \ldots)$
- $R_\lambda(x, y) := \dfrac{1}{2y} \prod_{i \geq 1} \left((x+y)^{\lambda_i} - (x-y)^{\lambda_i}\right) = \sum_{j+k=n-1} R_\lambda^j \dfrac{x^j}{j!} \dfrac{y^k}{k!}$.

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{0 \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R^{j_i}_{\alpha_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ such that $0 \leq j_i \leq n$ for all i.

Main Result

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

- $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ be the usual elementary symmetric function
- $2\lambda - 1 := (2\lambda_1 - 1, 2\lambda_2 - 1, 2\lambda_3 - 1, \ldots)$
- $R_\lambda(x, y) := \dfrac{1}{2y} \displaystyle\prod_{i \geq 1} ((x+y)^{\lambda_i} - (x-y)^{\lambda_i}) = \sum_{j+k=n-1} R^j_\lambda \frac{x^j}{j!} \frac{y^k}{k!}.$

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{0 \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R^{j_i}_{\alpha_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ such that $0 \leq j_i \leq n$ for all i.

Main Result

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

- $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ be the usual elementary symmetric function
- $2\lambda - 1 := (2\lambda_1 - 1, 2\lambda_2 - 1, 2\lambda_3 - 1, \ldots)$
- $R_\lambda(x, y) := \frac{1}{2y} \prod_{i \geq 1} \left((x+y)^{\lambda_i} - (x-y)^{\lambda_i}\right) = \sum_{j+k=n-1} R_\lambda^j \frac{x^j}{j!} \frac{y^k}{k!}$.

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R^{j_i}_{\alpha_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ such that $0 \leq j_i \leq n$ for all i.

Main Result

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

- $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ be the usual elementary symmetric function
- $2\lambda - 1 := (2\lambda_1 - 1, 2\lambda_2 - 1, 2\lambda_3 - 1, \ldots)$
- $R_\lambda(x,y) := \frac{1}{2y} \prod_{i \geq 1} ((x+y)^{\lambda_i} - (x-y)^{\lambda_i}) = \sum_{j+k=n-1} R_\lambda^j \frac{x^j}{j!} \frac{y^k}{k!}$.

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{\mathbf{0 \leq j \leq n}} \prod_{i=1}^{m} R^{j_i}_{\alpha_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ such that $0 \leq j_i \leq n$ for all i.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

- $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ be the usual elementary symmetric function
- $2\lambda - 1 := (2\lambda_1 - 1, 2\lambda_2 - 1, 2\lambda_3 - 1, \ldots)$
- $R_\lambda(x, y) := \frac{1}{2y} \prod_{i \geq 1} ((x+y)^{\lambda_i} - (x-y)^{\lambda_i}) = \sum_{j+k=n-1} R_\lambda^j \frac{x^j}{j!} \frac{y^k}{k!}$.

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ and let $\mathbf{x} = (x_1, \ldots, x_m)$. Then

$$c^{(n)}_{\alpha_1, \ldots, \alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R_{\alpha_i}^{j_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ such that $0 \leq j_i \leq n$ for all i.

For a partition $\lambda = (\lambda_1, \lambda_2, \ldots)$, let

- $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots$ be the usual elementary symmetric function
- $2\lambda - 1 := (2\lambda_1 - 1, 2\lambda_2 - 1, 2\lambda_3 - 1, \ldots)$
- $R_\lambda(x, y) := \frac{1}{2y} \prod_{i \geq 1} ((x+y)^{\lambda_i} - (x-y)^{\lambda_i}) = \sum_{j+k=n-1} R_\lambda^j \frac{x^j}{j!} \frac{y^k}{k!}$.

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ *and let* $\mathbf{x} = (x_1, \ldots, x_m)$. *Then*

$$c^{(n)}_{\alpha_1, \ldots, \alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R^{j_i}_{\alpha_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda) = n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ *such that* $0 \leq j_i \leq n$ *for all* i.

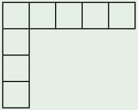
Hook Shapes

Definition

Let $(i\,|\,j)$ *denote the* ***hook*** *partition* $[1^j\,(i+1)]$.

Example

$$(4\,|\,3) = [1^3\,5] \vdash 8 \quad \longleftrightarrow$$



Lemma

- $\chi^{\lambda}_{[n]} = \begin{cases} (-1)^j & \textit{if } \lambda = (i \mid j) \textit{ with } i + j = n - 1 \\ 0 & \textit{otherwise} \end{cases}$
- $\chi^{(i|j)}_{[1^n]} = \binom{n-1}{j}$ *if* $i + j = n - 1$.
- $H_\lambda(x, y) := \sum_{i+j=n-1} \chi^{(i|j)}_{\lambda} x^i y^j = \frac{1}{x+y} \prod_{k \geq 1} (x^{\lambda_k} - (-y)^{\lambda_k})$.

Proof.

Murnaghan-Nakayama rule. □

Lemma

- $\chi^{\lambda}_{[n]} = \begin{cases} (-1)^j & \textit{if } \lambda = (i \mid j) \textit{ with } i+j = n-1 \\ 0 & \textit{otherwise} \end{cases}$
- $\chi^{(i|j)}_{[1^n]} = \binom{n-1}{j}$ *if* $i+j=n-1$.
- $H_{\lambda}(x,y) := \sum_{i+j=n-1} \chi^{(i|j)}_{\lambda} x^i y^j = \frac{1}{x+y} \prod_{k\geq 1} (x^{\lambda_k} - (-y)^{\lambda_k})$.

Proof.

Murnaghan-Nakayama rule. □

Hook Characters

Lemma

- $\chi^{\lambda}_{[n]} = \begin{cases} (-1)^j & \textit{if } \lambda = (i \mid j) \textit{ with } i + j = n - 1 \\ 0 & \textit{otherwise} \end{cases}$
- $\chi^{(i|j)}_{[1^n]} = \binom{n-1}{j}$ *if* $i + j = n - 1$.
- $H_\lambda(x, y) := \sum_{i+j=n-1} \chi^{(i|j)}_\lambda x^i y^j = \frac{1}{x+y} \prod_{k \geq 1} (x^{\lambda_k} - (-y)^{\lambda_k})$.

Proof.

Murnaghan-Nakayama rule. □

Hook Characters

Lemma

- $\chi^{\lambda}_{[n]} = \begin{cases} (-1)^j & \textit{if } \lambda = (i \,|\, j) \textit{ with } i + j = n - 1 \\ 0 & \textit{otherwise} \end{cases}$
- $\chi^{(i|j)}_{[1^n]} = \binom{n-1}{j}$ *if* $i + j = n - 1.$
- $H_\lambda(x, y) := \sum_{i+j=n-1} \chi^{(i|j)}_{\lambda} x^i y^j = \frac{1}{x+y} \prod_{k \geq 1} (x^{\lambda_k} - (-y)^{\lambda_k}).$

Proof.

Murnaghan-Nakayama rule. □

Hook Characters

Lemma

- $\chi^{\lambda}_{[n]} = \begin{cases} (-1)^j & \textit{if } \lambda = (i\,|\,j) \textit{ with } i+j=n-1 \\ 0 & \textit{otherwise} \end{cases}$
- $\chi^{(i|j)}_{[1^n]} = \binom{n-1}{j}$ *if* $i+j=n-1$.
- $H_\lambda(x,y) := \sum_{i+j=n-1} \chi^{(i|j)}_\lambda x^i y^j = \frac{1}{x+y} \prod_{k\geq 1} (x^{\lambda_k} - (-y)^{\lambda_k})$.

Proof.

Murnaghan-Nakayama rule. □

A Character Sum Formulation

Fact

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n!^{m-1}}{z_{\alpha_1}\cdots z_{\alpha_m}} \sum_{\lambda\vdash n} \frac{\chi^\lambda_{\alpha_1}\cdots\chi^\lambda_{\alpha_m}}{(\chi^\lambda_{[1^n]})^{m-1}} \chi^\lambda_{[n]}$$

From the Lemma there follows:

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{z_{\alpha_1}\cdots z_{\alpha_m}} \sum_{a+b=n-1} (a!\,b!)^{m-1} \chi^{(a|b)}_{\alpha_1}\cdots\chi^{(a|b)}_{\alpha_m}(-1)^b.$$

A Character Sum Formulation

Fact

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n!^{m-1}}{z_{\alpha_1}\cdots z_{\alpha_m}} \sum_{\lambda\vdash n} \frac{\chi^\lambda_{\alpha_1}\cdots\chi^\lambda_{\alpha_m}}{(\chi^\lambda_{[1^n]})^{m-1}} \chi^\lambda_{[n]}$$

From the Lemma there follows:

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{z_{\alpha_1}\cdots z_{\alpha_m}} \sum_{a+b=n-1} (a!\,b!)^{m-1} \chi^{(a|b)}_{\alpha_1}\cdots\chi^{(a|b)}_{\alpha_m}(-1)^b.$$

A Character Sum Formulation

Fact

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n!^{m-1}}{z_{\alpha_1}\cdots z_{\alpha_m}} \sum_{\lambda\vdash n} \frac{\chi^\lambda_{\alpha_1}\cdots\chi^\lambda_{\alpha_m}}{(\chi^\lambda_{[1^n]})^{m-1}} \chi^\lambda_{[n]}$$

From the Lemma there follows:

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{z_{\alpha_1}\cdots z_{\alpha_m}} \sum_{a+b=n-1} (a!\,b!)^{m-1}\chi^{(a|b)}_{\alpha_1}\cdots\chi^{(a|b)}_{\alpha_m}(-1)^b.$$

A Gaussian Integral

Definition

Let $d\mu(z)$ be the normalized Gaussian density on $\mathbb{C}$

$$d\mu(z) := \frac{1}{\pi} e^{-|z|^2} dz,$$

where $dz = ds\,dt$ for $z = s + t\sqrt{-1}$.

Lemma

$$\int_{\mathbb{C}} z^j \bar{z}^k \, d\mu(z) = j! \, \delta_{jk}$$

A Gaussian Integral

Definition

Let $d\mu(z)$ be the normalized Gaussian density on $\mathbb{C}$

$$d\mu(z) := \frac{1}{\pi} e^{-|z|^2} dz,$$

where $dz = ds\,dt$ for $z = s + t\sqrt{-1}$.

Lemma

$$\int_{\mathbb{C}} z^j \,\bar{z}^k \, d\mu(z) = j! \, \delta_{jk}$$

Proposition

Let $\alpha_1, \ldots, \alpha_m \vdash n+1$, and set $d\mu(\mathbf{u}, \mathbf{v}) := \prod_{i=1}^{m} d\mu(u_i) d\mu(v_i)$. Then

$$\sum_{a+b=n} (a!\, b!)^{m-1} \chi_{\alpha_1}^{(a|b)} \cdots \chi_{\alpha_m}^{(a|b)} (-1)^b$$

$$= \frac{1}{n!} \int_{\mathbb{C}^{2m}} (u_1 \cdots u_m - v_1 \cdots v_m)^n \prod_{i=1}^{m} H_{\alpha_i}(\bar{u}_i, \bar{v}_i)\, d\mu(\mathbf{u}, \mathbf{v}).$$

An Integral Representation

Proof.

$$\frac{1}{n!}(u_1\cdots u_m - v_1\cdots v_m)^n \prod_{i=1}^{m} H_{\alpha_i}(\bar{u}_i,\bar{v}_i)$$

$$= \sum_{a+b=n} \frac{u_1^a\cdots u_m^a\cdot v_1^b\cdots v_m^b}{a!\,b!}(-1)^b \prod_{i=1}^{m}\sum_{a_i+b_i=n} \chi_{\alpha_i}^{(a_i|b_i)}\bar{u}_i^{a_i}\bar{v}_i^{b_i}.$$

- Integrating with respect to $d\mu(\mathbf{u},\mathbf{v})$ forces $a_i = a$ and $b_i = b$.
- The RHS becomes

$$\sum_{a+b=n} (a!\,b!)^{m-1}\chi_{\alpha_1}^{(a|b)}\cdots\chi_{\alpha_m}^{(a|b)}(-1)^b.$$

An Integral Representation

Proof.

$$\frac{1}{n!}(u_1 \cdots u_m - v_1 \cdots v_m)^n \prod_{i=1}^{m} H_{\alpha_i}(\bar{u}_i, \bar{v}_i)$$

$$= \sum_{a+b=n} \frac{u_1^a \cdots u_m^a \cdot v_1^b \cdots v_m^b}{a!\, b!} (-1)^b \prod_{i=1}^{m} \sum_{a_i+b_i=n} \chi_{\alpha_i}^{(a_i|b_i)} \bar{u}_i^{a_i} \bar{v}_i^{b_i}.$$

- Integrating with respect to $d\mu(\mathbf{u}, \mathbf{v})$ forces $a_i = a$ and $b_i = b$.
- The RHS becomes

$$\sum_{a+b=n} (a!\, b!)^{m-1} \chi_{\alpha_1}^{(a|b)} \cdots \chi_{\alpha_m}^{(a|b)} (-1)^b.$$

An Integral Representation

Proof.

$$\frac{1}{n!}(u_1 \cdots u_m - v_1 \cdots v_m)^n \prod_{i=1}^{m} H_{\alpha_i}(\bar{u}_i, \bar{v}_i)$$

$$= \sum_{a+b=n} \frac{u_1^a \cdots u_m^a \cdot v_1^b \cdots v_m^b}{a!\, b!} (-1)^b \prod_{i=1}^{m} \sum_{a_i+b_i=n} \chi_{\alpha_i}^{(a_i|b_i)} \bar{u}_i^{a_i} \bar{v}_i^{b_i}.$$

- Integrating with respect to $d\mu(\mathbf{u}, \mathbf{v})$ forces $a_i = a$ and $b_i = b$.
- The RHS becomes

$$\sum_{a+b=n} (a!\, b!)^{m-1} \chi_{\alpha_1}^{(a|b)} \cdots \chi_{\alpha_m}^{(a|b)} (-1)^b.$$

□

Change of Variables

Recall

- $R_\lambda(x,y) = \frac{1}{2y}\prod_{k\geq 1}((x+y)^{\lambda_k} - (x-y)^{\lambda_k}),$
- $H_\lambda(x,y) = \frac{1}{x+y}\prod_{k\geq 1}(x^{\lambda_k} - (-y)^{\lambda_k}).$

Key Observations

Upon setting $u_i = \frac{1}{\sqrt{2}}(y_i + x_i)$ and $v_i = \frac{1}{\sqrt{2}}(y_i - x_i)$, get

- $H_\lambda(\bar{u}_i, \bar{v}_i) = 2^{-n/2} R_\lambda(\bar{x}_i, \bar{y}_i)$
- $d\mu(\mathbf{u}, \mathbf{v}) = d\mu(\mathbf{x}, \mathbf{y})$
- $u_1 \cdots u_m - v_1 \cdots v_m = 2^{1-m/2} y_1 \cdots y_m \sum_{s\geq 1} e_{2s-1}(\frac{x_1}{y_1}, \ldots, \frac{x_m}{y_m})$

Change of Variables

Recall

- $R_\lambda(x,y) = \frac{1}{2y} \prod_{k \geq 1} ((x+y)^{\lambda_k} - (x-y)^{\lambda_k}),$
- $H_\lambda(x,y) = \frac{1}{x+y} \prod_{k \geq 1} (x^{\lambda_k} - (-y)^{\lambda_k}).$

Key Observations

Upon setting $u_i = \frac{1}{\sqrt{2}}(y_i + x_i)$ and $v_i = \frac{1}{\sqrt{2}}(y_i - x_i)$, get

- $H_\lambda(\bar{u}_i, \bar{v}_i) = 2^{-n/2} R_\lambda(\bar{x}_i, \bar{y}_i)$
- $d\mu(\mathbf{u}, \mathbf{v}) = d\mu(\mathbf{x}, \mathbf{y})$
- $u_1 \cdots u_m - v_1 \cdots v_m = 2^{1-m/2} y_1 \cdots y_m \sum_{s \geq 1} e_{2s-1}(\frac{x_1}{y_1}, \ldots, \frac{x_m}{y_m})$

Change of Variables

Recall

- $R_\lambda(x,y) = \frac{1}{2y}\prod_{k\geq 1}((x+y)^{\lambda_k} - (x-y)^{\lambda_k}),$
- $H_\lambda(x,y) = \frac{1}{x+y}\prod_{k\geq 1}(x^{\lambda_k} - (-y)^{\lambda_k}).$

Key Observations

Upon setting $u_i = \frac{1}{\sqrt{2}}(y_i + x_i)$ and $v_i = \frac{1}{\sqrt{2}}(y_i - x_i)$, get

- $H_\lambda(\bar{u}_i, \bar{v}_i) = 2^{-n/2} R_\lambda(\bar{x}_i, \bar{y}_i)$
- $d\mu(\mathbf{u},\mathbf{v}) = d\mu(\mathbf{x},\mathbf{y})$
- $u_1 \cdots u_m - v_1 \cdots v_m = 2^{1-m/2} y_1 \cdots y_m \sum_{s\geq 1} e_{2s-1}(\frac{x_1}{y_1}, \ldots, \frac{x_m}{y_m})$

Change of Variables

Recall

- $R_\lambda(x,y) = \frac{1}{2y}\prod_{k\geq 1}((x+y)^{\lambda_k} - (x-y)^{\lambda_k}),$
- $H_\lambda(x,y) = \frac{1}{x+y}\prod_{k\geq 1}(x^{\lambda_k} - (-y)^{\lambda_k}).$

Key Observations

Upon setting $u_i = \frac{1}{\sqrt{2}}(y_i + x_i)$ and $v_i = \frac{1}{\sqrt{2}}(y_i - x_i)$, get

- $H_\lambda(\bar{u}_i, \bar{v}_i) = 2^{-n/2} R_\lambda(\bar{x}_i, \bar{y}_i)$
- $d\mu(\mathbf{u},\mathbf{v}) = d\mu(\mathbf{x},\mathbf{y})$
- $u_1 \cdots u_m - v_1 \cdots v_m = 2^{1-m/2} y_1 \cdots y_m \sum_{s\geq 1} e_{2s-1}(\frac{x_1}{y_1}, \ldots, \frac{x_m}{y_m})$

Recall

- $R_\lambda(x,y) = \frac{1}{2y}\prod_{k\geq 1}((x+y)^{\lambda_k} - (x-y)^{\lambda_k}),$
- $H_\lambda(x,y) = \frac{1}{x+y}\prod_{k\geq 1}(x^{\lambda_k} - (-y)^{\lambda_k}).$

Key Observations

Upon setting $u_i = \frac{1}{\sqrt{2}}(y_i + x_i)$ and $v_i = \frac{1}{\sqrt{2}}(y_i - x_i)$, get

- $H_\lambda(\bar{u}_i, \bar{v}_i) = 2^{-n/2} R_\lambda(\bar{x}_i, \bar{y}_i)$
- $d\mu(\mathbf{u},\mathbf{v}) = d\mu(\mathbf{x},\mathbf{y})$
- $u_1\cdots u_m - v_1\cdots v_m = 2^{1-m/2} y_1\cdots y_m \sum_{s\geq 1} e_{2s-1}(\frac{x_1}{y_1},\ldots,\frac{x_m}{y_m})$

Change of Variables

Thus

$$\int_{\mathbb{C}^{2m}} (u_1 \cdots u_m - v_1 \cdots v_m)^n \prod_{i=1}^{m} H_{\alpha_i}(\bar{u}_i, \bar{v}_i)\, d\mu(\mathbf{u}, \mathbf{v})$$

becomes

$$\frac{1}{2^{n(m-1)}} \int_{\mathbb{C}^{2m}} \left(y_1 \cdots y_m \sum_{s \geq 1} e_{2s-1}\left(\frac{\mathbf{x}}{\mathbf{y}}\right)\right)^n \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)\, d\mu(\mathbf{x}, \mathbf{y}).$$

Reversing the Integral Formulation

Now

$$\int_{\mathbb{C}^{2m}} \left(y_1 \cdots y_m \sum_{s \geq 1} e_{2s-1}\left(\tfrac{\mathbf{x}}{\mathbf{y}}\right) \right)^n \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)\, d\mu(\mathbf{x}, \mathbf{y})$$

$$= \sum_{\mathbf{j}+\mathbf{k}=\mathbf{n}} \mathbf{j}!\,\mathbf{k}!\,[\mathbf{x}^{\mathbf{j}}\mathbf{y}^{\mathbf{k}}] \left(y_1 \cdots y_m \sum_{s \geq 1} e_{2s-1}\left(\tfrac{\mathbf{x}}{\mathbf{y}}\right) \right)^n \cdot [\bar{\mathbf{x}}^{\mathbf{j}}\bar{\mathbf{y}}^{\mathbf{k}}] \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)$$

$$= \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} [\mathbf{x}^{\mathbf{j}}] \left(\sum_{s \geq 1} e_{2s-1}(\mathbf{x}) \right)^n \prod_{i=1}^{m} R_{\alpha_i}^{j_i}$$

$$= \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R_{\alpha_i}^{j_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)} \qquad \textbf{DONE!}$$

Now

$$\int_{\mathbb{C}^{2m}} \left(y_1 \cdots y_m \sum_{s \geq 1} e_{2s-1}\left(\tfrac{\mathbf{x}}{\mathbf{y}}\right) \right)^n \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)\, d\mu(\mathbf{x}, \mathbf{y})$$

$$= \sum_{\mathbf{j}+\mathbf{k}=\mathbf{n}} \mathbf{j}!\, \mathbf{k}!\, [\mathbf{x}^{\mathbf{j}} \mathbf{y}^{\mathbf{k}}] \left(y_1 \cdots y_m \sum_{s \geq 1} e_{2s-1}\left(\tfrac{\mathbf{x}}{\mathbf{y}}\right) \right)^n \cdot [\bar{\mathbf{x}}^{\mathbf{j}} \bar{\mathbf{y}}^{\mathbf{k}}] \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)$$

$$= \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} [\mathbf{x}^{\mathbf{j}}] \left(\sum_{s \geq 1} e_{2s-1}(\mathbf{x}) \right)^n \prod_{i=1}^{m} R_{\alpha_i}^{j_i}$$

$$= \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R_{\alpha_i}^{j_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

DONE!

Reversing the Integral Formulation

Now

$$\int_{\mathbb{C}^{2m}} \left(y_1 \cdots y_m \sum_{s\geq 1} e_{2s-1}\left(\tfrac{\mathbf{x}}{\mathbf{y}}\right)\right)^n \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)\, d\mu(\mathbf{x}, \mathbf{y})$$

$$= \sum_{\mathbf{j}+\mathbf{k}=\mathbf{n}} \mathbf{j}!\,\mathbf{k}!\,[\mathbf{x}^{\mathbf{j}}\mathbf{y}^{\mathbf{k}}]\left(y_1 \cdots y_m \sum_{s\geq 1} e_{2s-1}\left(\tfrac{\mathbf{x}}{\mathbf{y}}\right)\right)^n \cdot [\bar{\mathbf{x}}^{\mathbf{j}}\bar{\mathbf{y}}^{\mathbf{k}}] \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)$$

$$= \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{n}} [\mathbf{x}^{\mathbf{j}}] \left(\sum_{s\geq 1} e_{2s-1}(\mathbf{x})\right)^n \prod_{i=1}^{m} R_{\alpha_i}^{j_i}$$

$$= \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{n}} \prod_{i=1}^{m} R_{\alpha_i}^{j_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

DONE!

Reversing the Integral Formulation

Now

$$\int_{\mathbb{C}^{2m}} \Big(y_1 \cdots y_m \sum_{s \geq 1} e_{2s-1}\big(\tfrac{\mathbf{x}}{\mathbf{y}}\big) \Big)^n \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)\, d\mu(\mathbf{x}, \mathbf{y})$$

$$= \sum_{\mathbf{j}+\mathbf{k}=\mathbf{n}} \mathbf{j}!\,\mathbf{k}!\,[\mathbf{x}^{\mathbf{j}}\mathbf{y}^{\mathbf{k}}] \Big(y_1 \cdots y_m \sum_{s \geq 1} e_{2s-1}\big(\tfrac{\mathbf{x}}{\mathbf{y}}\big) \Big)^n \cdot [\bar{\mathbf{x}}^{\mathbf{j}}\bar{\mathbf{y}}^{\mathbf{k}}] \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)$$

$$= \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} [\mathbf{x}^{\mathbf{j}}] \Big(\sum_{s \geq 1} e_{2s-1}(\mathbf{x}) \Big)^n \prod_{i=1}^{m} R_{\alpha_i}^{j_i}$$

$$= \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R_{\alpha_i}^{j_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

DONE!

Reversing the Integral Formulation

Now

$$\int_{\mathbb{C}^{2m}} \left(y_1 \cdots y_m \sum_{s\geq 1} e_{2s-1}(\tfrac{\mathbf{x}}{\mathbf{y}}) \right)^n \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)\, d\mu(\mathbf{x}, \mathbf{y})$$

$$= \sum_{\mathbf{j}+\mathbf{k}=\mathbf{n}} \mathbf{j}!\,\mathbf{k}!\,[\mathbf{x}^{\mathbf{j}}\mathbf{y}^{\mathbf{k}}] \left(y_1 \cdots y_m \sum_{s\geq 1} e_{2s-1}(\tfrac{\mathbf{x}}{\mathbf{y}}) \right)^n \cdot [\bar{\mathbf{x}}^{\mathbf{j}}\bar{\mathbf{y}}^{\mathbf{k}}] \prod_{i=1}^{m} R_{\alpha_i}(\bar{x}_i, \bar{y}_i)$$

$$= \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{n}} [\mathbf{x}^{\mathbf{j}}] \left(\sum_{s\geq 1} e_{2s-1}(\mathbf{x}) \right)^n \prod_{i=1}^{m} R_{\alpha_i}^{j_i}$$

$$= \sum_{\mathbf{0}\leq\mathbf{j}\leq\mathbf{n}} \prod_{i=1}^{m} R_{\alpha_i}^{j_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

DONE!

Main Result

Theorem

Fix $\alpha_1, \ldots, \alpha_m \vdash n$ *and let* $\mathbf{x} = (x_1, \ldots, x_m)$. *Then*

$$c^{(n)}_{\alpha_1,\ldots,\alpha_m} = \frac{n^{m-1}}{2^{(n-1)(m-1)} \prod_i z_{\alpha_i}} \sum_{\mathbf{0} \leq \mathbf{j} \leq \mathbf{n}} \prod_{i=1}^{m} R^{j_i}_{\alpha_i} \cdot [\mathbf{x}^{\mathbf{j}}] \sum_{\ell(\lambda)=n-1} \frac{e_{2\lambda-1}(\mathbf{x})}{\mathrm{Aut}(\lambda)}$$

where the outer sum extends over all $\mathbf{j} = (j_1, \ldots, j_m)$ *such that* $0 \leq j_i \leq n$ *for all* i.

A Binomial Identity

The "integral trick" is equivalent to the identity

$$\sum_{i,s,t} \frac{\binom{k}{s}\binom{\ell}{t}\binom{n-k}{i-s}\binom{n-\ell}{i-t}(-1)^{s+t}}{\binom{n}{i}} = \begin{cases} \dfrac{2^n}{\binom{n}{k}} & \text{if } k = \ell \\ 0 & \text{otherwise.} \end{cases}$$