Crystal Bases for Quantum Generalized Kac-Moody Algebras

Seok-Jin Kang

Department of Mathematical Sciences
Seoul National University

20 June 2006

1.Monstrous Moonshine

M: Monster simple group

$=$ largest sporadic simple group

1.Monstrous Moonshine

M: Monster simple group

$=$ largest sporadic simple group

$$
\begin{aligned}
|M| &= 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 \\
&\simeq 8.08 \times 10^{53}
\end{aligned}
$$

1.Monstrous Moonshine

M: Monster simple group

$\quad =$ largest sporadic simple group

$$|M| = 2^{46} \cdot 3^{20} \cdot 5^{9} \cdot 7^{6} \cdot 11^{2} \cdot 13^{3} \cdot 17 \cdot 19 \cdot 23 \cdot 29 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71$$
$$\simeq 8.08 \times 10^{53}$$

恒河沙 $= 10^{52} \leq |M| \leq 10^{56} =$ 阿僧祇

Moonshine Conjecture *(Conway-Norton 1979)*

Moonshine Conjecture *(Conway-Norton 1979)*

1. $^{\exists}$ a $\mathbb{Z}$-graded representation $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ of M s.t.
$dim V_n^\natural = c(n)$, where
$j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n) q^n, q = e^{2\pi i\tau}, Im\tau > 0$

Moonshine Conjecture *(Conway-Norton 1979)*

1. $\exists$ a $\mathbb{Z}$-graded representation $V^\natural = \bigoplus_{n=-1}^{\infty} V_n^\natural$ of M s.t.
$dim V_n^\natural = c(n)$, where
$j(\tau) - 744 = \sum_{n=-1}^{\infty} c(n)q^n, q = e^{2\pi i \tau}, Im\tau > 0$
2. $T_g(\tau) = \sum_{n=-1}^{\infty} tr(g|V_n^\natural)q^n$: Mckay-Thompson series

$\Longrightarrow T_g(\tau)$ is the Hauptmodul for some discrete genus 0 subgroup Γ_g of $PSL_2(\mathbb{R})$

That is,
i) $\widehat{\mathcal{H}}/\Gamma_g \simeq S^2$
ii) Every modular function of weight 0 w.r.t Γ_g is a rational function in $T_g(\tau)$

Proof of Moonshine Conjecture

Proof of Moonshine Conjecture

1. Borcherds, Frenkel-Lepowsky-Meurman (1985) : introduced vertex(operator) algebras

Proof of Moonshine Conjecture

1. Borcherds, Frenkel-Lepowsky-Meurman (1985) : introduced vertex(operator) algebras
2. Frenkel-Lepowsky-Meurman(1988) : constructed the Moonshine module $V^\natural$

Proof of Moonshine Conjecture

1. Borcherds, Frenkel-Lepowsky-Meurman (1985) :
   introduced vertex(operator) algebras
2. Frenkel-Lepowsky-Meurman(1988) :
   constructed the Moonshine module $V^\natural$
3. Borcherds(1992) : completed the proof

   key ingredient : Monster Lie algebra
   = an example of generalized Kac-Moody algebra

2.Quantum GKM Algebra

Definition *Borcherds-Cartan matrix*

Let I be a finite or countably infinite index set.

A real matrix $A = (a_{ij})_{i,j \in I}$ is called a *Borcherds-Cartan matrix* if it satisfies the following conditions:

i) $a_{ii} = 2$ or $a_{ii} \leq 0$ for all $i \in I$,

ii) $a_{ij} \leq 0$ if $i \neq j$,

iii) $a_{ij} \in \mathbb{Z}$ if $a_{ii} = 2$,

iv) $a_{ij} = 0$ if and only if $a_{ji} = 0$.

Assume that A is even, integral and symmetrizable. That is,

i) a_{ii} is even , ${}^{\forall} i \in I$

ii) $a_{ij} \in \mathbf{Z}$, ${}^{\forall} i, j \in I$

iii) ${}^{\exists}$diagonal matrix $D = diag(s_i \in \mathbf{Z}_{>0} | i \in I)$ s.t. DA :symmetric

Borcherds-Cartan datum

$(A, P, P^\vee, \Pi, \Pi^\vee)$ consists of

i) $A = (a_{ij})_{i,j \in I}$, a Borcherds-Cartan matrix,
ii) a free abelian group P, the *weight lattice*,
iii) $P^\vee = Hom(P, \mathbb{Z})$, the *dual weight lattice*,
iv) $\Pi = \{\alpha_i \in P | i \in I\}$, the set of *simple roots*,
v) $\Pi^\vee = \{h_i | i \in I\} \subset P^\vee$, the set of *simple coroots*,

satisfying the properties:

a) $\langle h_i, \alpha_j \rangle = a_{ij}$ for all $i, j \in I$,
b) $^\forall i \in I$, there exists $\Lambda_i \in P$ s.t. $\langle h_j, \Lambda_i \rangle = \delta_{ij} \quad ^\forall j \in I$,
c) Π is linearly independent.

Definition *Quantum GKM Algebra*

Quantum GKM Algebra $U_q(\mathfrak{g})$ associated with $(A, P, P^\vee, \Pi, \Pi^\vee)$

$=$ the associative algebra over $\mathbf{Q}(\mathfrak{g})$ with 1 generated by the elements e_i, f_i $(i \in I)$, q^h $(h \in P^\vee)$ with

i) $q^0 = 1, \quad q^h q^{h'} = q^{h+h'}$ for $h, h' \in P^\vee$

ii) $q^h e_i q^{-h} = q^{\alpha_i(h)} e_i, \quad q^h f_i q^{-h} = q^{-\alpha_i(h)} f_i$, for $h \in P^\vee$ $i \in I$

iii) $e_i f_j - f_j e_i = \delta_{ij} \dfrac{K_i - K_i^{-1}}{q_i - q_i^{-1}}$ for $i, j \in I$, where $K_i = q^{s_i h_i}$

Definition *Quantum GKM Algebra*

iv) $\sum_{k=0}^{1-a_{ij}}(-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i e_i^{1-a_{ij}-k} e_j e_i^k = 0$ if $i \in I^{re}$ and $i \neq j$

v) $\sum_{k=0}^{1-a_{ij}}(-1)^k \begin{bmatrix} 1-a_{ij} \\ k \end{bmatrix}_i f_i^{1-a_{ij}-k} f_j f_i^k = 0$ if $i \in I^{re}$ and $i \neq j$

vi) $e_i e_j - e_j e_i = f_i f_j - f_j f_i = 0$ if $a_{ij} = 0$

3.Crystal Bases

Definition *Category O_{int}*

the abelian category O_{int} of $U_g(\mathfrak{g})$-module M satisfying the following properties:

i) $M = \oplus_{\lambda \in P} M_\lambda$, where

$$M_\lambda := \left\{ u \in M \,;\, q^h u = q^{\lambda(h)} u \quad \text{for any } h \in P^\vee \right\}$$

ii) $\dim U_q^+(\mathfrak{g})u < \infty$ for any $u \in M$

iii) $\mathrm{wt}(M) := \{\lambda \in P \,;\, M_\lambda \neq 0\} \subset \left\{\lambda \in P \,;\, \langle h_i, \lambda \rangle \geq 0, {}^\forall i \in I^{\mathrm{im}}\right\}$

iv) $f_i M_\lambda = 0$, ${}^\forall i \in I^{\mathrm{im}}$ and $\lambda \in P$ s.t. $\langle h_i, \lambda \rangle = 0$

v) $e_i M_\lambda = 0$, ${}^\forall i \in I^{\mathrm{im}}$ and $\lambda \in P$ s.t. $\langle h_i, \lambda \rangle \leq -a_{ii}$

Proposition *(Jeong-Kashiwara-K 2005)*

Proposition *(Jeong-Kashiwara-K 2005)*

1. The abelian category O_{int} is semisimple

Proposition *(Jeong-Kashiwara-K 2005)*

1. The abelian category O_{int} is semisimple
2. { irreducible objects in O_{int} } $= \{V(\lambda) | \lambda \in P^+\}$, where

$$P^+ = \{\lambda \in P | \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$$

Proposition *(Jeong-Kashiwara-K 2005)*

1. The abelian category O_{int} is semisimple
2. { irreducible objects in O_{int} } $= \{V(\lambda) | \lambda \in P^+\}$, where

$$P^+ = \{\lambda \in P | \langle h_i, \lambda \rangle \geq 0 \text{ for all } i \in I\}$$

3. $V(\lambda) = U_q(\mathfrak{g}) u_\lambda$, where

i) u_λ has weight λ

ii) $e_i u_\lambda = 0$ for all $i \in I$

iii) $f_i^{\langle h_i, \lambda \rangle + 1} u_\lambda = 0$ for any $i \in I^{\mathrm{re}}$

vi) $f_i u_\lambda = 0$ if $i \in I^{\mathrm{im}}$ and $\langle h_i, \lambda \rangle = 0$

Kashiwara operators

Let M be a $U_q(\mathfrak{g})$-module in $\mathcal{O}_{\text{int}}$.
The Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ ($i \in I$) are defined by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k \qquad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k$$

where

$$u = \sum_{k \geq 0} f_i^{(k)} u_k \quad \text{with } u_k \in M_{\mu + n\alpha_i} \text{ s.t. } e_i u_k = 0, \ ^\forall u \in M_\mu$$

and

$$f_i^{(k)} = \begin{cases} f_i^k / [k]_i! & \text{if } \ i \in I^{\text{re}} \\ f_i^k & \text{if } \ i \in I^{\text{im}} \end{cases}$$

Definition *Crystal Bases*

Let $\mathbf{A}_0 = \{ f/g \in \mathbf{Q}(q)\,;\, f, g \in \mathbf{Q}[q], g(0) \neq 0 \}$ and $M \in O_{int}$.

A Crystal Basis of M is a pair (L, B), where

i) L is a free $\mathbf{A}_0$-submodule L of M s.t. $M \simeq \mathbf{Q}(q) \bigotimes_{\mathbf{A}_0} L$

ii) B is a $\mathbf{Q}$-basis of L/gL

iii) $\tilde{f}_i L \subset L$ and $\tilde{e}_i L \subset L \quad {}^\forall i \in I$

iv) $\tilde{f}_i B \subset B \cup \{0\}$ and $\tilde{e}_i B \subset B \cup \{0\} \quad {}^\forall i \in I$

v) $\tilde{f}_i b = b' \iff b = \tilde{e}_i b', \quad$ for $b, b' \in B$ and $i \in I$

Theorem *(Jeong-Kashiwara-K 2005)*

For $\lambda \in P^+$, let

$$L(\lambda) = \mathbf{A}_0\text{-submodule of } V(\lambda) \text{ generated by} \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda \,;\, r \geq 0, i_k \in I \right\}$$

$$B(\lambda) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda + qL(\lambda) \,;\, r \geq 0, i_k \in I \right\} \setminus \{0\}$$

Theorem *(Jeong-Kashiwara-K 2005)*

For $\lambda \in P^{+}$, let

$$L(\lambda) = \mathbf{A}_0\text{-submodule of } V(\lambda) \text{ generated by } \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda \,;\, r \geq 0, i_k \in I \right\}$$

$$B(\lambda) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda + qL(\lambda) \,;\, r \geq 0, i_k \in I \right\} \setminus \{0\}$$

$\Longrightarrow (L(\lambda), B(\lambda))$ is a unique crystal basis of $V(\lambda)$

Crystal basis for $U_q^-(\mathfrak{g})$

Fix $i \in I$. For any $u \in U_q^-(\mathfrak{g})$, $^\exists$ unique $v, w \in U_q^-(\mathfrak{g})$ s.t.

$$e_i u - u e_i = \frac{K_i v - K_i^{-1} w}{q_i - q_i^{-1}}$$

We define the endomorphism $e_i' \colon U_q^-(\mathfrak{g}) \to U_q^-(\mathfrak{g})$ by $e_i'(u) = w$. Then every $u \in U_q^-(\mathfrak{g})$ has a unique i-string decomposition

$$u = \sum_{k \geq 0} f_i^{(k)} u_k, \quad \text{where} \quad e_i' u_k = 0 \quad \text{for all} \quad k \geq 0,$$

and the Kashiwara operators $\tilde{e}_i$, $\tilde{f}_i$ $(i \in I)$ are defined by

$$\tilde{e}_i u = \sum_{k \geq 1} f_i^{(k-1)} u_k, \qquad \tilde{f}_i u = \sum_{k \geq 0} f_i^{(k+1)} u_k.$$

Definition

A Crystal Basis of $U_q^-(\mathfrak{g})$ is a pair (L, B), where

i) L is a free $\mathbf{A}_0$-submodule L of $U_q^-(\mathfrak{g})$ s.t.
$U_q^-(\mathfrak{g}) \simeq \mathbf{Q}(q) \bigotimes_{\mathbf{A}_0} L$

ii) B is a $\mathbf{Q}$-basis of L/gL

iii) $\tilde{f}_i L \subset L$ and $\tilde{e}_i L \subset L \quad {}^{\forall} i \in I$

iv) $\tilde{f}_i B \subset B$ and $\tilde{e}_i B \subset B \cup \{0\} \quad {}^{\forall} i \in I$

v) $\tilde{f}_i b = b' \iff b = \tilde{e}_i b', \quad$ for $b, b' \in B$ and $i \in I$

Theorem *(Jeong-Kashiwara-K 2005)*

$$L(\infty) = \mathbf{A}_0\text{-submodule of } U_q^-(\mathfrak{g}) \text{ generated by } \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} \,;\, r \geq 0, i_k \in I \right\}$$

$$B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} + qL(\infty) \,;\, r \geq 0, i_k \in I \right\} \setminus \{0\}$$

Theorem *(Jeong-Kashiwara-K 2005)*

$$L(\infty) = \mathbf{A}_0\text{-submodule of } U_q^-(\mathfrak{g}) \text{ generated by}$$
$$\left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} \,;\, r \geq 0, i_k \in I \right\}$$

$$B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} + qL(\infty) \,;\, r \geq 0, i_k \in I \right\} \setminus \{0\}$$

$\Longrightarrow (L(\infty), B(\infty))$ is a unique crystal basis of $U_q^-(\mathfrak{g})$

Theorem *(Jeong-Kashiwara-K 2005)*

$$L(\infty) = \mathbf{A}_0\text{-submodule of } U_q^-(\mathfrak{g}) \text{ generated by } \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} \,;\, r \geq 0, i_k \in I \right\}$$

$$B(\infty) = \left\{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} + qL(\infty) \,;\, r \geq 0, i_k \in I \right\} \setminus \{0\}$$

$\Longrightarrow (L(\infty), B(\infty))$ is a unique crystal basis of $U_q^-(\mathfrak{g})$

Problem How to realize $B(\lambda)$ and $B(\infty)$?

4.Abstract Crystals

Definition *Abstract Crystals*

A set B with the maps $\mathrm{wt}\colon B \to P$, $\tilde{e}_i, \tilde{f}_i\colon B \to B \sqcup \{0\}$ and $\varepsilon_i, \varphi_i\colon B \to \mathbf{Z} \sqcup \{-\infty\}$ $(i \in I)$ where

i) $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}\, b + \alpha_i$ if $i \in I$ and $\tilde{e}_i b \neq 0$,

ii) $\mathrm{wt}(\tilde{f}_i b) = \mathrm{wt}\, b - \alpha_i$ if $i \in I$ and $\tilde{f}_i b \neq 0$,

iii) ${}^{\forall} i \in I$ and $b \in B$, $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}\, b\rangle$,

iv) ${}^{\forall} i \in I$ and $b, b' \in B$, $\tilde{f}_i b = b' \iff b = \tilde{e}_i b'$,

Notation $\mathrm{wt}_i(b) = \langle h_i, \mathrm{wt}\, b\rangle$ for $i \in I$ and $b \in B$

Definition *Abstract Crystals*

v) $^\forall i \in I$ and $b \in B$ s.t. $\tilde{e}_i b \neq 0$, we have

$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ if $i \in I^{\mathrm{re}}$,

$\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + a_{ii}$ if $i \in I^{\mathrm{im}}$,

vi) $^\forall i \in I$ and $b \in B$ s.t. $\tilde{f}_i b \neq 0$, we have

$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ if $i \in I^{\mathrm{re}}$,

$\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b)$ and $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - a_{ii}$ if $i \in I^{\mathrm{im}}$,

vii) $^\forall i \in I$ and $b \in B$ s.t. $\varphi_i(b) = -\infty$, we have $\tilde{e}_i b = \tilde{f}_i b = 0$.

Definition *Morphism of Crystals*

Let B_1 and B_2 be crystals. A *morphism of crystals* $\psi \colon B_1 \to B_2$ is a map $\psi \colon B_1 \to B_2$ s.t.

i) for $b \in B_1$ we have

$$\mathrm{wt}(\psi(b)) = \mathrm{wt}(b) \text{ and } \varepsilon_i(\psi(b)) = \varepsilon_i(b),$$
$$\varphi_i(\psi(b)) = \varphi_i(b) \text{ for all } i \in I$$

ii) if $b \in B_1$ and $i \in I$ satisfy $\tilde{f}_i b \in B_1$, then

$$\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$$

Definition *Morphism of Crystals*

Let $\psi\colon B_1 \to B_2$ be a morphism of crystals

i) ψ is called a *strict morphism* if

$$\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b),\ \ \psi(\tilde{f}_i b) = \tilde{f}_i \psi(b) \quad \text{for all } i \in I \text{ and } b \in B_1$$

where $\psi(0) = 0$

ii) ψ is called an *embedding* if the underlying map $\psi\colon B_1 \to B_2$ is injective. we say that B_1 is a *subcrystal* of B_2. If ψ is a strict embedding, we say that B_1 is a *full subcrystal* of B_2.

Example

1. $B = B(\lambda),\ \lambda \in P^+,\ b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} u_\lambda + qL(\lambda)$

$$wt(b) \stackrel{\text{def}}{=} \lambda - (\alpha_{i_1} + \cdots \alpha_{i_r})$$

$$\varepsilon_i(b) = \begin{cases} \max\left\{k \geq 0\,;\ \tilde{e}_i^k b \neq 0\right\} & \text{for } i \in I^{\mathrm{re}}, \\ 0 & \text{for } i \in I^{\mathrm{im}}, \end{cases}$$

$$\varphi_i(b) = \begin{cases} \max\left\{k \geq 0\,;\ \tilde{f}_i^k b \neq 0\right\} & \text{for } i \in I^{\mathrm{re}}, \\ \mathrm{wt}_i(b) & \text{for } i \in I^{\mathrm{im}}, \end{cases}$$

Example

2. $B = B(\infty)$ and $b = \tilde{f}_{i_1} \cdots \tilde{f}_{i_r} \mathbf{1} + qL(\infty)$

$$\mathrm{wt}(b) = -(\alpha_{i_1} + \cdots + \alpha_{i_r})$$

$$\varepsilon_i(b) = \begin{cases} \max\{k \geq 0\,;\, \tilde{e}_i^k b \neq 0\} & \text{for } i \in I^{\mathrm{re}}, \\ 0 & \text{for } i \in I^{\mathrm{im}}, \end{cases}$$

$$\varphi_i(b) = \varepsilon_i(b) + \mathrm{wt}_i(b) \quad (i \in I).$$

Definition *Tensor Product of Crystals*

Let $B_1 \bigotimes B_2 = \{b_1 \bigotimes b_2 | b_1 \in B_1, b_2 \in B_2\}$, and define the maps $\mathrm{wt}, \varepsilon_i, \varphi_i$ as follows.

$$\begin{aligned} \mathrm{wt}(b \otimes b') &= \mathrm{wt}(b) + \mathrm{wt}(b'), \\ \varepsilon_i(b \otimes b') &= \max(\varepsilon_i(b), \varepsilon_i(b') - \mathrm{wt}_i(b)), \\ \varphi_i(b \otimes b') &= \max(\varphi_i(b) + \mathrm{wt}_i(b'), \varphi_i(b')). \end{aligned}$$

For $i \in I$, we define

$$\tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b'), \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'), \end{cases}$$

Definition *Tensor Product of Crystals*

For $i \in I^{\mathrm{re}}$, we define

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b'), \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b'), \end{cases}$$

and, for $i \in I^{\mathrm{im}}$, we define

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') - a_{ii}, \\ 0 & \text{if } \varepsilon_i(b') < \varphi_i(b) \leq \varepsilon_i(b') - a_{ii}, \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b'). \end{cases}$$

Proposition

Proposition

1. $B_1 \bigotimes B_2$ is a crystal.

Proposition

1. $B_1 \bigotimes B_2$ is a crystal.
2. $$\begin{array}{ccc} (B_1 \bigotimes B_2) \bigotimes B_3 & \xrightarrow{\sim} & B_1 \bigotimes (B_2 \bigotimes B_3) \\ (b_1 \bigotimes b_2) \bigotimes b_3 & \longmapsto & b_1 \bigotimes (b_2 \bigotimes b_3) \end{array}$$

Proposition

1. $B_1 \bigotimes B_2$ is a crystal.
2. $$\begin{array}{ccc} (B_1 \bigotimes B_2) \bigotimes B_3 & \xrightarrow{\sim} & B_1 \bigotimes (B_2 \bigotimes B_3) \\ (b_1 \bigotimes b_2) \bigotimes b_3 & \longmapsto & b_1 \bigotimes (b_2 \bigotimes b_3) \end{array}$$

Remark Hence the category of crystals forms a tensor category.

Example

Example

1. For $\lambda \in P$, let $T_\lambda = \{t_\lambda\}$ and define

$$\mathrm{wt}(t_\lambda) = \lambda, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0 \quad \text{for all} \quad i \in I,$$
$$\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty \quad \text{for all} \quad i \in I.$$

$\implies T_\lambda$ is a crystal.

Example

1. For $\lambda \in P$, let $T_\lambda = \{t_\lambda\}$ and define

$$\mathrm{wt}(t_\lambda) = \lambda, \quad \tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0 \quad \text{for all} \quad i \in I,$$
$$\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty \quad \text{for all} \quad i \in I.$$

$\Longrightarrow$ T_λ is a crystal.

2. Let $C = \{c\}$ with

$$\mathrm{wt}(c) = 0 \text{ and } \varepsilon_i(c) = \varphi_i(c) = 0,\ \tilde{f}_i c = \tilde{e}_i c = 0,\ {}^\forall i \in I$$

$\Longrightarrow$ C is a crystal.

Example

Example

3. For each $i \in I$, let $B_i = \{b_i(-n)\,;\ n \geq 0\}$ with

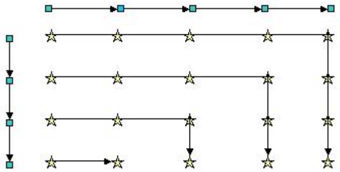
$$\begin{aligned}
&\operatorname{wt} b_i(-n) = -n\alpha_i,\\
&\tilde{e}_i b_i(-n) = b_i(-n+1), \quad \tilde{f}_i b_i(-n) = b_i(-n-1),\\
&\tilde{e}_j b_i(-n) = \tilde{f}_j b_i(-n) = 0 \quad \text{if } \ j \neq i,\\
&\varepsilon_i(b_i(-n)) = n, \quad \varphi_i(b_i(-n)) = -n \quad \text{if } \ i \in I^{\mathrm{re}},\\
&\varepsilon_i(b_i(-n)) = 0, \quad \varphi_i(b_i(-n)) = \operatorname{wt}_i(b_i(-n)) = -na_{ii} \quad \text{if } \ i \in I^{\mathrm{im}},\\
&\varepsilon_j(b_i(-n)) = \varphi_j(b_i(-n)) = -\infty \quad \text{if } j \neq i.
\end{aligned}$$

$\Longrightarrow$ B_i is a crystal. The crystal B_i is called an *elementary crystal*.

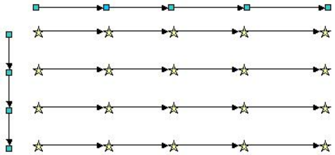
Example

B_1, B_2 : crystals for $U_q(\mathfrak{g})$-modules in O_{int}

i) $a_{ii} = 2$



ii) $a_{ii} \leq 0$



Example

Example

1. $T_\lambda \bigotimes T_\mu \simeq T_{\lambda+\mu}$

Example

1. $T_\lambda \bigotimes T_\mu \simeq T_{\lambda+\mu}$
2. $B(\lambda) \hookrightarrow B(\infty) \bigotimes T_\lambda$

Example

1. $T_\lambda \bigotimes T_\mu \simeq T_{\lambda+\mu}$
2. $B(\lambda) \hookrightarrow B(\infty) \bigotimes T_\lambda$
3. $B(\lambda+\mu) \hookrightarrow B(\lambda) \bigotimes B(\mu)$

Example

1. $T_\lambda \bigotimes T_\mu \simeq T_{\lambda+\mu}$
2. $B(\lambda) \hookrightarrow B(\infty) \bigotimes T_\lambda$
3. $B(\lambda+\mu) \hookrightarrow B(\lambda) \bigotimes B(\mu)$
4. $B \bigotimes C \not\cong B$

Example

$\mathbf{i} = (i_1, i_2, \cdots)$, $i_k \in I$ s.t. every $i \in I$ appear infinitely many times in $\mathbf{i}$

$$B(\mathbf{i}) \stackrel{\text{def}}{=} \{\cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \in \cdots \otimes B_{i_k} \otimes \cdots \otimes B_{i_1} \,;\, x_k \in \mathbf{Z}_{\geq 0}, \text{ and } x_k = 0 \text{ for } k \gg 0\}.$$

Example

$\mathbf{i} = (i_1, i_2, \cdots)$, $i_k \in I$ s.t. every $i \in I$ appear infinitely many times in $\mathbf{i}$

$$B(\mathbf{i}) \stackrel{\text{def}}{=} \{\cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1)$$
$$\in \cdots \otimes B_{i_k} \otimes \cdots \otimes B_{i_1} \,;\, x_k \in \mathbf{Z}_{\geq 0}, \text{ and } x_k = 0 \text{ for } k \gg 0\}.$$

$\Longrightarrow B(\mathbf{i})$ becomes a crystal;

$b = \cdots \otimes b_{i_k}(-x_k) \otimes \cdots \otimes b_{i_1}(-x_1) \in B(\mathbf{i})$

$\mathrm{wt}(b) = -\sum_k x_k \alpha_{i_k}$

for $i \in I^{\mathrm{re}}$, $\varepsilon_i(b) = \max\left\{x_k + \sum_{l>k} \langle h_i, \alpha_{i_l}\rangle x_l \,;\, 1 \leq k,\, i = i_k\right\}$

$\varphi_i(b) = \max\left\{-x_k - \sum_{1\leq l<k} \langle h_i, \alpha_{i_l}\rangle x_l \,;\, 1 \leq k,\, i = i_k\right\}$

for $i \in I^{\mathrm{im}}$, $\quad \varepsilon_i(b) = 0 \quad$ and $\quad \varphi_i(b) = \mathrm{wt}_i(b)$

Example

For $i \in I^{\mathrm{re}}$, we have

$$\tilde{e}_i b = \begin{cases} \cdots \otimes b_{i_{n_e}}(-x_{n_e}+1) \otimes \cdots \otimes b_{i_1}(-x_1) & \text{if } \varepsilon_i(b) > 0, \\ 0 & \text{if } \varepsilon_i(b) \leq 0, \end{cases}$$

$$\tilde{f}_i b = \cdots \otimes b_{i_{n_f}}(-x_{n_f}-1) \otimes \cdots \otimes b_{i_1}(-x_1),$$

where n_e (resp. n_f) is the largest (resp. smallest) $k \geq 1$ such that $i_k = i$ and $x_k + \sum_{l>k} \langle h_i, \alpha_{i_l} \rangle x_l = \varepsilon_i(b)$.

Note such an n_e exists if $\varepsilon_i(b) > 0$.

Example

When $i \in I^{\mathrm{im}}$, let

$$n_f = min\{k | i_k = i \quad \text{and} \quad \sum_{l>k} \langle h_i, \alpha_{i_l} \rangle x_l = 0\}$$

Then

$$\tilde{f}_i b = \cdots \otimes b_{i_{n_f}}(-x_{n_f} - 1) \otimes \cdots \otimes b_{i_1}(-x_1)$$

and

$$\tilde{e}_i b = \begin{cases} \cdots \otimes b_{i_{n_f}}(-x_{n_f} + 1) \otimes \cdots \otimes b_{i_1}(-x_1) \\ \qquad \text{if } x_{n_f} > 0 \text{ and } \sum_{k<l\le n_f} \langle h_i, \alpha_{i_l} \rangle x_l < a_{ii} \text{ for any} \\ \qquad k \text{ such that } 1 \le k < n_f \text{ and } i_k = i, \\ 0 \qquad \text{otherwise.} \end{cases}$$

5.Crystal Embedding Theorem

Theorem *(Jeong-Kashiwara-K-Shin 2006)*

For all $i \in I$, $^{\exists!}$ strict embedding

$$\begin{aligned} \Psi_i : B(\infty) &\longrightarrow B(\infty) \otimes B_i \\ \mathbf{1} &\longmapsto \mathbf{1} \bigotimes b_i(0) \end{aligned}$$

Let $\mathbf{i} = (i_1, i_2, \cdots) \in I^\infty$

Observe : $B(\infty) \hookrightarrow B(\infty) \bigotimes B_{i_1} \hookrightarrow B(\infty) \bigotimes B_{i_2} \bigotimes B_{i_1} \hookrightarrow \cdots$

Let $\mathbf{i} = (i_1, i_2, \cdots) \in I^\infty$

Observe : $B(\infty) \hookrightarrow B(\infty) \bigotimes B_{i_1} \hookrightarrow B(\infty) \bigotimes B_{i_2} \bigotimes B_{i_1} \hookrightarrow \cdots$

Proposition

$$B(\infty) \hookrightarrow B(\mathbf{i}) = \{\cdots \bigotimes b_{i_k}(-x_k) \bigotimes \cdots b_{i_1}(-x_1)\}$$

Let $\mathbf{i} = (i_1, i_2, \cdots) \in I^\infty$

Observe : $B(\infty) \hookrightarrow B(\infty) \bigotimes B_{i_1} \hookrightarrow B(\infty) \bigotimes B_{i_2} \bigotimes B_{i_1} \hookrightarrow \cdots$

Proposition

$$B(\infty) \hookrightarrow B(\mathbf{i}) = \{\cdots \bigotimes b_{i_k}(-x_k) \bigotimes \cdots b_{i_1}(-x_1)\}$$

Corollary

$$\begin{aligned} B(\infty) \simeq \text{connected component of } B(\mathbf{i}) \\ \text{containing } \cdots \bigotimes b_{i_k}(0) \bigotimes \cdots b_{i_1}(0) \end{aligned}$$

Theorem *(Jeong-Kashiwara-K-Shin 2006)*

Theorem *(Jeong-Kashiwara-K-Shin 2006)*

1. Let B be a crystal s.t.

   i) $\mathrm{wt}(B) \subset -Q_+$,
   ii) ${}^\exists b_0 \in B$ s.t. $\mathrm{wt}(b_0) = 0$,
   iii) for any $b \in B$ s.t. $b \neq b_0$, ${}^\exists$ some $i \in I$ s.t. $\tilde{e}_i b \neq 0$,
   iv) for all i, ${}^\exists$ a strict embedding $\Psi_i : B \to B \otimes B_i$.

   Then $B \xrightarrow{\sim} B(\infty)$

Theorem *(Jeong-Kashiwara-K-Shin 2006)*

1. Let B be a crystal s.t.

   i) $\mathrm{wt}(B) \subset -Q_+$,
   ii) $^\exists b_0 \in B$ s.t. $\mathrm{wt}(b_0) = 0$,
   iii) for any $b \in B$ s.t. $b \neq b_0$, $^\exists$ some $i \in I$ s.t. $\tilde{e}_i b \neq 0$,
   iv) for all i, $^\exists$ a strict embedding $\Psi_i \colon B \to B \otimes B_i$.

   Then $B \xrightarrow{\sim} B(\infty)$

2. Let $\lambda \in P^+$. Then

$$B(\lambda) \xrightarrow{\sim} \text{connected component of } B(\infty) \otimes T_\lambda \otimes C$$
$$\text{containing } \mathbf{1} \otimes t_\lambda \otimes c$$

Example

Let $I = \{1, 2\}$ and $\mathbf{i} = (1, 2, 1, 2, \ldots)$ and
$A = \begin{pmatrix} 2 & -a \\ -b & -c \end{pmatrix}$ for some $a, b \in \mathbf{Z}_{>0}$ and $c \in 2\mathbf{Z}_{\geq 0}$.

$B \stackrel{\text{def}}{=} \{\cdots \otimes b_2(-x_{2k}) \otimes b_1(-x_{2k-1}) \otimes \cdots \otimes b_2(-x_2) \otimes b_1(-x_1) |$

i) $ax_{2k} - x_{2k+1} \geq 0$ for all $k \geq 1$,

ii) $^\forall k \geq 2$ with $x_{2k} > 0$, we have $x_{2k-1} > 0$ and $ax_{2k} - x_{2k+1} > 0.\}$

Example

Let $I = \{1, 2\}$ and $\mathbf{i} = (1, 2, 1, 2, \dots)$ and

$$A = \begin{pmatrix} 2 & -a \\ -b & -c \end{pmatrix} \quad \text{for some } a, b \in \mathbf{Z}_{>0} \text{ and } c \in 2\mathbf{Z}_{\geq 0}.$$

$$B \stackrel{\text{def}}{=} \{\cdots \otimes b_2(-x_{2k}) \otimes b_1(-x_{2k-1}) \otimes \cdots \otimes b_2(-x_2) \otimes b_1(-x_1) |$$

i) $ax_{2k} - x_{2k+1} \geq 0$ for all $k \geq 1$,

ii) ${}^{\forall}k \geq 2$ with $x_{2k} > 0$, we have $x_{2k-1} > 0$ and $ax_{2k} - x_{2k+1} > 0.\}$

$$\Longrightarrow B \simeq B(\infty)$$

Example

$$B^{\lambda} \stackrel{\text{def}}{=} \{\cdots \otimes b_2(-x_{2k}) \otimes b_1(-x_{2k-1}) \otimes \cdots \otimes b_2(-x_2) \otimes b_1(-x_1) \otimes t_{\lambda} \otimes c \,|$$

i) $ax_{2k} - x_{2k+1} \geq 0$ for all $k \geq 1$,

ii) $^{\forall}k \geq 2$ with $x_{2k} > 0$, we have $x_{2k-1} > 0$ and $ax_{2k} - x_{2k+1} > 0$,

iii) $0 \leq x_1 \leq \langle h_1, \lambda \rangle$,

iv) if $x_2 > 0$ and $\langle h_2, \lambda \rangle = 0$, then $x_1 > 0$. $\}$

Example

$$B^\lambda \stackrel{\text{def}}{=} \{\cdots \otimes b_2(-x_{2k}) \otimes b_1(-x_{2k-1}) \otimes \cdots \otimes b_2(-x_2) \otimes b_1(-x_1) \otimes t_\lambda \otimes c |$$

i) $ax_{2k} - x_{2k+1} \geq 0$ for all $k \geq 1$,

ii) ${}^\forall k \geq 2$ with $x_{2k} > 0$, we have $x_{2k-1} > 0$ and $ax_{2k} - x_{2k+1} > 0$,

iii) $0 \leq x_1 \leq \langle h_1, \lambda \rangle$,

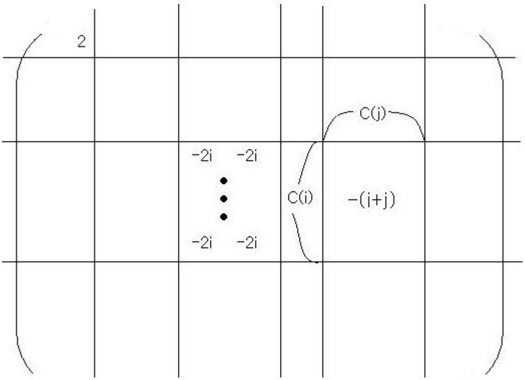
iv) if $x_2 > 0$ and $\langle h_2, \lambda \rangle = 0$, then $x_1 > 0$. $\}$

$$\Longrightarrow B^\lambda \simeq B(\lambda)$$

Example *Quantum Monster Algebra*

$$I = \{(i,t) | i = -1, 1, 2, \cdots \quad 1 \le t \le c(i)\}$$
$$A = (-(i+j))_{(i,t),(j,s)\in I}$$

$=$



Example

$$\begin{aligned}\mathbf{i} = (\mathbf{i}(k))_{k=1}^{\infty} = ((-1,1),(1,1),\dots,(1,c(1));\\ (-1,1),(1,1),\dots,(1,c(1)),(2,1),\dots,(2,c(2));\\ (-1,1),(1,1),\dots,(1,c(1)),(2,1),\dots,(2,c(2)),\\ (3,1),\dots,(3,c(3));(-1,1),\dots)\end{aligned}$$

Note

i) every $(i,t) \in I$ appears infinitely times in $\mathbf{i}$

ii) $(-1,1)$ appears at the $b(n)$-th position for $n \geq 0$, where

$$b(n) = nc(1) + (n-1)c(2) + \cdots + c(n) + n + 1.$$

Example

For $k \in \mathbf{Z}_{>0}$, $k^{(-)}$ = the largest integer $l < k$ s.t. $\mathbf{i}(l) = \mathbf{i}(k)$.

$B \stackrel{\text{def}}{=} \{\cdots \otimes b_{\mathbf{i}(k)}(-x_k) \otimes \cdots \otimes b_{\mathbf{i}(1)}(-x_1) \in B(\mathbf{i}) |$

i) $x_{b(1)} = 0$,

ii) ${}^{\forall} n \geq 1$, $\displaystyle\sum_{b(n)<l<b(n+1)} \langle h_{(-1,1)}, \alpha_{\mathbf{i}(l)} \rangle x_l \geq x_{b(n+1)}$,

iii) if $\mathbf{i}(k) \neq (-1,1)$, $x_k > 0$ and $k^{(-)} > 0$, then
$\displaystyle\sum_{k^{(-)}<l<k} \langle h_{\mathbf{i}(k)}, \alpha_{\mathbf{i}(l)} \rangle x_l < 0$.

If $x_l = 0$ for all $k^{(-)} < l < k$ s.t. $\mathbf{i}(l) \neq (-1,1)$, then
$\displaystyle\sum_{b(n)<l<b(n+1)} \langle h_{(-1,1)}, \alpha_{\mathbf{i}(l)} \rangle x_l > x_{b(n+1)}$ s.t. $k^{(-)} < b(n) < k.\}$

Example

For $k \in \mathbf{Z}_{>0}$, $k^{(-)}$ = the largest integer $l < k$ s.t. $\mathbf{i}(l) = \mathbf{i}(k)$.

$B \stackrel{\text{def}}{=} \{\cdots \otimes b_{\mathbf{i}(k)}(-x_k) \otimes \cdots \otimes b_{\mathbf{i}(1)}(-x_1) \in B(\mathbf{i})|$

i) $x_{b(1)} = 0$,

ii) ${}^\forall n \geq 1$, $\displaystyle\sum_{b(n)<l<b(n+1)} \langle h_{(-1,1)}, \alpha_{\mathbf{i}(l)} \rangle x_l \geq x_{b(n+1)}$,

iii) if $\mathbf{i}(k) \neq (-1,1)$, $x_k > 0$ and $k^{(-)} > 0$, then

$$\sum_{k^{(-)}<l<k} \langle h_{\mathbf{i}(k)}, \alpha_{\mathbf{i}(l)} \rangle x_l < 0.$$

If $x_l = 0$ for all $k^{(-)} < l < k$ s.t. $\mathbf{i}(l) \neq (-1,1)$, then

$$\sum_{b(n)<l<b(n+1)} \langle h_{(-1,1)}, \alpha_{\mathbf{i}(l)} \rangle x_l > x_{b(n+1)} \text{ s.t. } k^{(-)} < b(n) < k.\}$$

$\Longrightarrow B \simeq B(\infty)$

Example

$$B^\lambda \stackrel{\text{def}}{=} \{\cdots \otimes b_{\mathbf{i}(k)}(-x_k) \otimes \cdots \otimes b_{\mathbf{i}(1)}(-x_1) \otimes t_\lambda \otimes c \,|$$

(i)–(iii) in the previous example,

iv) $0 \leq x_1 \leq \langle h_{(-1,1)}, \lambda \rangle$,

v) if $\mathbf{i}(k) \neq (-1,1)$, $\langle h_{\mathbf{i}(k)}, \lambda \rangle = 0$, $x_k > 0$ and $k^{(-)} = 0$,

then ${}^{\exists}l$ s.t. $1 \leq l < k$, $\langle h_{\mathbf{i}(k)}, \alpha_{\mathbf{i}(l)} \rangle < 0$ and $x_l > 0$. $\}$

Example

$$B^{\lambda} \overset{\text{def}}{=} \{ \cdots \otimes b_{\mathbf{i}(k)}(-x_k) \otimes \cdots \otimes b_{\mathbf{i}(1)}(-x_1) \otimes t_{\lambda} \otimes c \,|$$

(i)–(iii) in the previous example,

iv) $0 \leq x_1 \leq \langle h_{(-1,1)}, \lambda \rangle$,

v) if $\mathbf{i}(k) \neq (-1,1)$, $\langle h_{\mathbf{i}(k)}, \lambda \rangle = 0$, $x_k > 0$ and $k^{(-)} = 0$,
then $^{\exists} l$ s.t. $1 \leq l < k$, $\langle h_{\mathbf{i}(k)}, \alpha_{\mathbf{i}(l)} \rangle < 0$ and $x_l > 0$. $\}$

$$\Longrightarrow B^{\lambda} \simeq B(\lambda)$$

Crystal Embedding Theorem

THANK YOU