HOPF ALGEBROIDS for COMBINATORIALISTS

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OVERVIEW

Aims: **(i)** *to describe basic ideas underlying Hopf algebroids,* **(ii)** *to make the case for combinatorial models,* **and (iii)** *to provide simple examples, including origins in algebraic topology.*

1. FROM GRAPHS TO COGROUPOIDS

2. HOPF ALGEBRAS

3. HOPF ALGEBROIDS

4. COMBINATORIAL MODELS

The scheme of things ...

$$\begin{tikzcd}
\begin{array}{c}\text{TOPOLOGY}\\ \text{and}\\ \text{GEOMETRY}\end{array} \arrow[r] \arrow[d] & \text{COMBINATORICS} \arrow[d] \\
\text{ALGEBRA} \arrow[r] \arrow[d] & \begin{array}{c}\text{ALGEBRAIC}\\ \text{COMBINATORICS}\end{array} \\
\begin{array}{c}\text{ALGEBRAIC}\\ \text{GEOMETRY}\end{array} &
\end{tikzcd}$$

1. GRAPHS TO COGROUPOIDS

A **digraph** (V, E) has a set of vertices V and a set of edges E, and **source** and **target** functions s, $t\colon E \to V$. We assume that each vertex is endowed with a loop, given by a **unit** $i\colon V \to E$. So

$$s \cdot i = t \cdot i = 1_V.$$

We also have graphs of groups, graphs of topological spaces, graphs of algebras . . . .

By turning the arrows around, we obtain a **cograph** (V', E'). This has **cosource** and **cotarget** functions s', $t'\colon V' \to E'$, and a **counit** $i'\colon E' \to V'$. They obey

$$i' \cdot s' = i' \cdot t' = 1_{V'}.$$

A **group** $(\{*\}, F)$ is a digraph with a single vertex $*$, whose edges F may be **composed** using a function $\mu\colon F \times F \to F$. This is associative, with unit $i(*)$; so we write

$$i(*) \;=\; 1.$$

There is also an **inverse** $\chi\colon F \to F$, for which

$$\mu(h, \chi h) \;=\; \mu(\chi h, h) = 1.$$

We express these conditions as commutative diagrams of sets and functions. For example

$$\begin{array}{ccc} F & \xrightarrow{\delta} & F \times F \\ {\scriptstyle *}\big\downarrow & & \big\downarrow{\scriptstyle 1\times\chi} \\ F & \xleftarrow[\mu]{} & F \times F \end{array} \quad .$$

A group of topological spaces is nothing more than a topological group!

A **groupoid** (V, G) is a digraph with vertices V, and a function $\mu \colon G \times_V G \to G$ which allows the edges G to be **partially composed**. Here

$$G \times_V G \;=\; \{(g, h) : t(g) = s(h)\} \subseteq G \times G.$$

This is associative whenever possible, with units $\{i(v) = 1_v : v \in V\}$.

There is also an **inverse** $\chi \colon G \to G$, such that

$$s(\chi g) \;=\; t(g) \quad \text{and} \quad t(\chi g) \;=\; s(g),$$

with $\chi^2 = 1$ and

$$\mu(\chi g, g) = 1_{t(g)} \quad \text{and} \quad \mu(g, \chi g) \;=\; 1_{s(g)}.$$

So there is a commutative diagram

$$\begin{tikzcd} G \arrow[r, "(1{,}\chi)"] \arrow[d, "s"'] & G \times_V G \arrow[d, "\mu"] \\ V \arrow[r, "u"'] & G \end{tikzcd},$$

for example.

A classic example is the **pair groupoid**

$$(W, W \times W)$$

for any set (or C^∞-manifold, or algebra ..) W.

The functions $s = \pi_1$ and $t = \pi_2$ are the **projections**, and $i = \Delta$ is the **diagonal**. Since

$$(W \times W) \times_W (W \times W) \;\cong\; W \times W \times W$$

we may define μ by

$$\mu(x, y, z) = (x, z).$$

Then $\chi(x, y) = (y, x)$.

Another famous groupoid is $(X, \Pi_1(X))$, the **fundamental groupoid** of the topological space X. The elements $p \in \Pi_1(X)$ with

$$s(p) = x \quad \text{and} \quad t(p) = y$$

are homotopy classes of paths from x to y.

A classic example of a **cogroupoid** is given by

$$(Y, Y \sqcup Y)$$

for any set (or space, or algebra ...) Y.

The functions $s' = \iota_1$ and $t' = \iota_2$ are the **inclusions**, and $i' = \phi$ is the **fold**. Since

$$(Y \sqcup Y) \sqcup_Y (Y \sqcup Y) \ \cong\ Y \sqcup Y \sqcup Y,$$

we may define **cocomposition** $\mu' = \delta$ as

$$Y \sqcup Y \cong Y \sqcup \varnothing \sqcup Y \longrightarrow Y \sqcup Y \sqcup Y.$$

Then χ' interchanges the summands of $Y \sqcup Y$.

For any commutative ring R there is an isomorphism of function algebras

$$R^{Y \sqcup Y} \ \cong\ R^Y \times R^Y,$$

and the cogroupoid maps induce the pair groupoid $(R^Y, R^Y \times R^Y)$ of rings.

There is a **multiplication cogroupoid**

$$(M, M \times M),$$

for any abelian group or monoid M!

The cosource and cotarget maps are

$$s'(a) = (a, 1) \quad \text{and} \quad t'(a) = (1, a),$$

and the counit i' is the multiplication m. Cocomposition takes values in

$$(M \times M) \times (M \times M)/\sim,$$

where

$$(a, b), (c, d) \sim (a, bc), (1, d) \sim (a, 1), (bc, d);$$

it is given by

$$\delta\colon M \times M \longrightarrow M \times M \times M,$$

with $\delta(a, b) = (a, 1, b)$. Also, $\chi'(a, b) = (b, a)$.

A cogroupoid with $V' = \{*\}$ is a **cogroup**; in this case, s' and t' coincide.

2. HOPF ALGEBRAS

A **Hopf algebra** over a commutative ring R is a cogroup (R, H) of R-algebras. In this context V' is the trivial R-algebra R, and the diagonal is an R-algebra homomorphism

$$\delta \colon H \longrightarrow H \otimes_R H.$$

The counit $i' \colon H \to R$ is an augmentation, and χ' is the antipode.

The condition that δ be a morphism of R-algebras is also the condition that multiplication in H be a morphism of R-coalgebras!

In many interesting cases, R and H are also **graded** by dimension. Sign conventions then apply to commutativity formulae.

Two important Hopf algebras are the

polynomial Steenrod coalgebra $(\mathbb{F}_p, A_*)$

Landweber-Novikov coalgebra $(\mathbb{Z}, B_*)$.

Here A_* is the polynomial algebra

$$\mathbb{F}_p[\xi_1, \ldots, \xi_n, \ldots], \quad \text{with } \dim \xi_n = 2(p^n - 1)$$

and B_* is the polynomial algebra

$$\mathbb{Z}[b_1, \ldots, b_n, \ldots], \quad \text{with } \dim b_n = 2n.$$

The diagonal on A_* is specified by

$$\delta_p(\xi_n) \;=\; \sum_{i=0}^{n} \xi_{n-i}^{p^i} \otimes \xi_i.$$

The diagonal on B_* is given by writing

$$b(x) \;=\; \sum_{n \geq 0} b_n \, x^{n+1}$$

and equating coefficients in

$$\delta(b)(x) \;=\; b'(b''(x)), \qquad (\$)$$

with $b'_k b''_j = b_j \otimes b_k$.

Formula ($) also holds for δ_p over $\mathbb{F}_p$! But algebraic and topological imperatives suggest that we should work p-**locally** instead.

In particular, we consider p-**typical** formal power series

$$\ell(x) \;=\; \sum_{n\geq 0} \ell_n\, x^{p^n}$$

over $\mathbb{Z}_{(p)}$-algebras, where $\dim \ell_n = 2(p^n - 1)$.

The set of such power series is no longer a group, because it fails to be closed under composition or reversion.

At best, we can define a groupoid; so there is no precise p-local analogue of the Hopf algebra B_*.

3. HOPF ALGEBROIDS

A **Hopf algebroid** over a commutative ring R is a cogroupoid (D, Γ) of *commutative* R-algebras.

We write:

the cosource as the **left unit** $\eta_L \colon D \to \Gamma$

the cotarget as the **right unit** $\eta_R \colon D \to \Gamma$

the counit as $\epsilon \colon \Gamma \to D$.

So Γ has distinct left and right D-module structures, induced by η_L and η_R.

Coproduct is the diagonal homomorphism

$$\delta \colon \Gamma \longrightarrow \Gamma \otimes_D \Gamma$$

of commutative R-algebras, where $\otimes_D$ uses both module structures. We write the inverse χ' as **conjugation** $c \colon \Gamma \to \Gamma$.

How can such objects possibly arise?

A **split** Hopf algebroid may be constructed from a commutative Hopf algebra (R, H) whenever H coacts on the right of some commutative R-algebra D. Then we take

$$(D, D \otimes_R H),$$

where η_L includes the left copy of D and η_R is the given coaction.

The coproduct is the D-module map

$$1 \otimes \delta \colon D \otimes_R H \longrightarrow (D \otimes_R H) \otimes_D (D \otimes_R H)$$

induced by the diagonal on H.

Conjugation is generated by $1 \otimes \chi'$ on $1 \otimes H$, and η_R on $D \otimes 1$.

The original examples arose in algebraic topology, from homotopy commutative **ring spectra** E. Up to homotopy, the product defines the multiplication cogroupoid

$$(E, E \wedge E) \qquad (\pounds)$$

of spectra. The homotopy ring $\pi_*(E) = E_*$ is the graded commutative **coefficient ring**.

Then E defines a **homology functor**

$$E_*(-)\colon \left\{ \begin{array}{c} \text{topological} \\ \text{spaces} \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} E_*\text{-} \\ \text{modules} \end{array} \right\}$$

by $E_*(X) = \pi_*(E \wedge X)$.

Now apply $\pi_*(-)$ to $(\pounds)$ to get a cogroupoid

$$(E_*, E_*(E))$$

of E_*-modules! For example, the left unit

$$\eta_L\colon E_* \longrightarrow E_*(E)$$

is induced by $s'\colon E \simeq E \wedge S^0 \to E \wedge E$.

For the **complex cobordism** spectrum MU, the corresponding Hopf algebroid has

$$MU_* \;\cong\; \mathbb{Z}[x_1, \ldots, x_n, \ldots],$$

where dim $x_n = 2n$, and

$$MU_*(MU) \cong MU_*[b_1, \ldots, b_n, \ldots] \cong MU_* \otimes_{\mathbb{Z}} B_*.$$

In this case the Hopf algebroid is split, because conjugation is so well behaved; it is precisely $1 \otimes \chi'$ on $1 \otimes B_*$.

This is not true for the p-typical version, given by the spectrum BP, even though the Hopf algebroid $(BP_*, BP_*(BP))$ has

$$BP_* \;\cong\; \mathbb{Z}_{(p)}[v_1, \ldots, v_n, \ldots],$$

where dim $v_n = 2(p^n - 1)$, and

$$BP_*(BP) \;\cong\; BP_*[t_1, \ldots, t_n, \ldots],$$

where dim $v_n = 2(p^n - 1)$.

4. COMBINATORIAL MODELS

Combinatorial models for coalgebraic structures were probably first studied seriously in the early 1970s.

Here is a model for B_*.

Let $LP(n)$ be the Boolean algebra of **linear partitions** of $\{1,\ldots,n\}$. The finest partition $(1|2|\ldots|n)$ is initial, and the coarsest partition $(1,2,\ldots,n)$ is final.

For example, $LP(3)$ looks like

$$\begin{array}{ccc} (1|2|3) & \xrightarrow{\leq} & (1,2|3) \\ {\scriptstyle <}\downarrow & & \downarrow{\scriptstyle <} \\ (1|2,3) & \xrightarrow[<]{} & (1,2,3) \end{array} \qquad (¥)$$

When λ is $(\lambda_1|\ldots|\lambda_j)$, we write $|\lambda| = j$.

Let $\mathscr{Z}(n)$ be the free abelian group generated by all finite intervals in $LP(\infty)$, and impose the coproduct

$$\delta_\infty[\lambda,\mu] \;=\; \sum_{\lambda\le\theta\le\mu}[\lambda,\theta]\otimes[\theta,\mu].$$

This is the **incidence coalgebra** of $LP(\infty)$.

Now **reduce** $\mathscr{Z}(n)$ by identifying all intervals of equal **type** T. Here

$$T\,[(1|2|\ldots|n),\,\lambda] \;=\; b_{i_1-1}\ldots b_{i_j-1}$$

when the blocks of λ have size i_1, ..., i_j, and

$$T\,[\lambda,\mu] \;=\; [(1|2|\ldots|j),\,\mu/\lambda],$$

when $j=|\lambda|$. As always, $b_0=1$.

Notice that singletons are invisible, and that δ_∞ is compatible with the identification $\sim$.

It follows that $\mathscr{Z}(n)/\!\sim$ is B_*!

In (¥), for example, we have

$$T\,[(1|2|3),(1,2,3)] = b_2$$

and

$$T\,[(1|2|3),(1,2|3)] = T\,[(1,2|3),(1,2,3)] = b_1.$$

So

$$\delta(b_2) = 1 \otimes b_2 + 2b_1 \otimes b_1 + b_2 \otimes 1,$$

from (¥), as required.

The challenge is to find a combinatorial model for MU_* that makes the structure of $(MU_*, MU_* \otimes_{\mathbb{Z}} B_*)$ transparent ...
... and then to realise $(BP_*, BP_*(BP))$!

And to move on to comodules over both!

Haiman has noted that A_* is modeled by equivalence classes of rooted trees. We aim to adapt this to p-typical partitions over $\mathbb{Z}_{(p)}$.