A Bijection for Unicellular Partitioned Bicolored Maps

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Introduction

Our Problem

Enumerate factorizations of a long cycle in two permutations with a fixed number of cycles

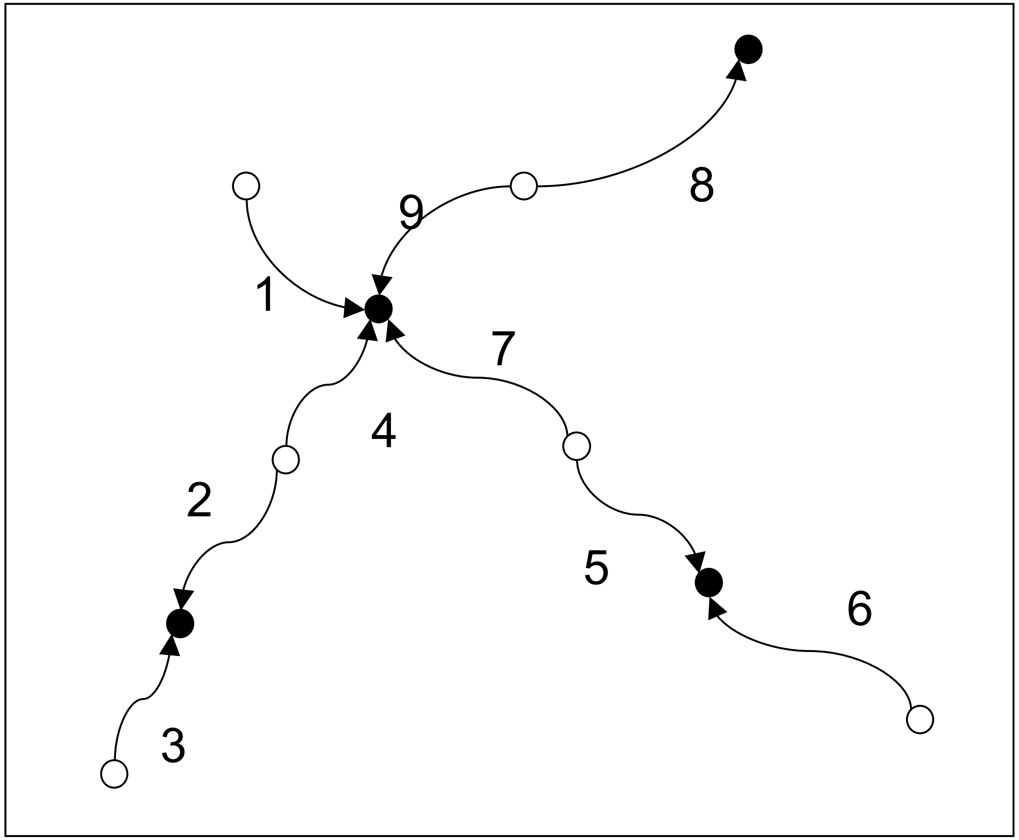
Formally

We want to compute B(m,n,N) the cardinality of the set of couples :

$$(\alpha,\beta)\in\Sigma_N{}^2,\quad \textit{tels que}\ \begin{cases} Z(\alpha)=m \\ Z(\beta)=n \\ \alpha\beta=\gamma=(1\,2\cdots N) \end{cases}$$

Introduction

Geometrical Interpretation



$$\alpha = (1)(2\ 4)(3)(5\ 7)(6)(8\ 9)$$
$$\beta = (1\ 4\ 7\ 9)(2\ 3)(5\ 6)(8)$$

Introduction

Solution :

$$\sum_{m,n\geq 1} B(m,n,N)\, y^m z^n = N! \sum_{p,q\geq 1} \binom{N-1}{p-1,\, q-1} \binom{y}{p} \binom{z}{q}$$

Notations

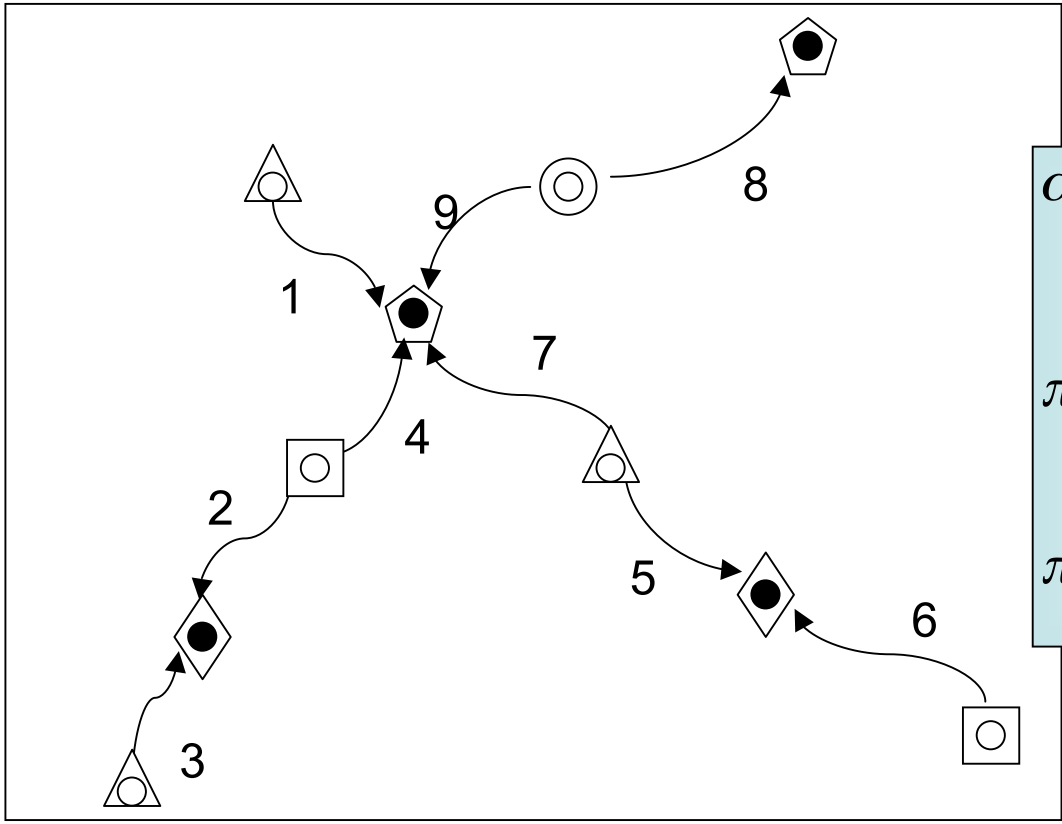
In Order to prove our claim, we introduce a new class of objects : Partitioned Maps

$C_{N,p,q}$

$$(\pi_1, \pi_2, \alpha) \; such \;\; as \begin{cases} \alpha \in \Sigma_N \\ \pi_1 \overset{p}{\mapsto} [N] \\ \pi_2 \overset{q}{\mapsto} [N] \\ \pi_1 \;\; partitions \;\; the \;\; cycles \;\; of \;\; \alpha \\ \pi_2 \;\; partitions \;\; the \;\; cycles \;\; of \;\; \beta = \alpha^{-1}\gamma \end{cases}$$

Notations

Geometrical Interpretation



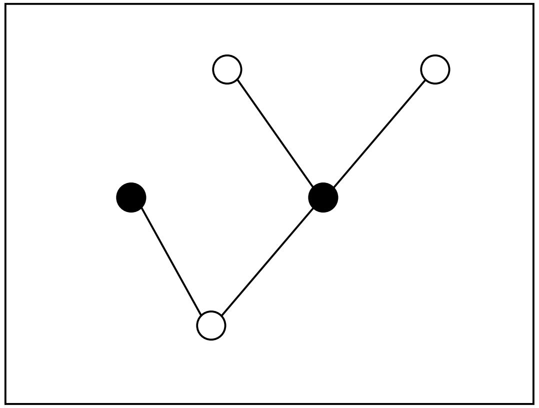
$$\alpha = (1)(2\ 4)(3)(5\ 7)(6)(8\ 9)$$

$$\pi_1 = \left\{ \begin{array}{l} \{1,3,5,7\}, \\ \{2,4,6\}, \\ \{8,9\} \end{array} \right\}$$

$$\pi_2 = \left\{ \begin{array}{l} \{2,3,5,6\}, \\ \{1,4,7,8,9\} \end{array} \right\}$$

Notations

Ordered Bicolored Trees and Partial Permutation



$$\left|BT(p,q)\right| = \frac{p+q-1}{pq}\binom{p+q-2}{p-1}^2$$

$$\sigma = \begin{pmatrix} 3 & 4 & 5 & 6 & 8 & 10 \\ 4 & 6 & 3 & 1 & 8 & 7 \end{pmatrix}$$

$$\left|PP(X,Y,A)\right| = \binom{|X|}{A}\binom{|Y|}{A}A!$$

Bijection

Link between partitioned and non-partitioned unicellular bicolored maps

$$\boxed{\sum_{m,n\geq 1} B(m,n,N) y^m z^n = \sum_{p,q\geq 1} \left|C_{N,p,q}\right| (y)_p (z)_q}$$

Our main result is equivalent to

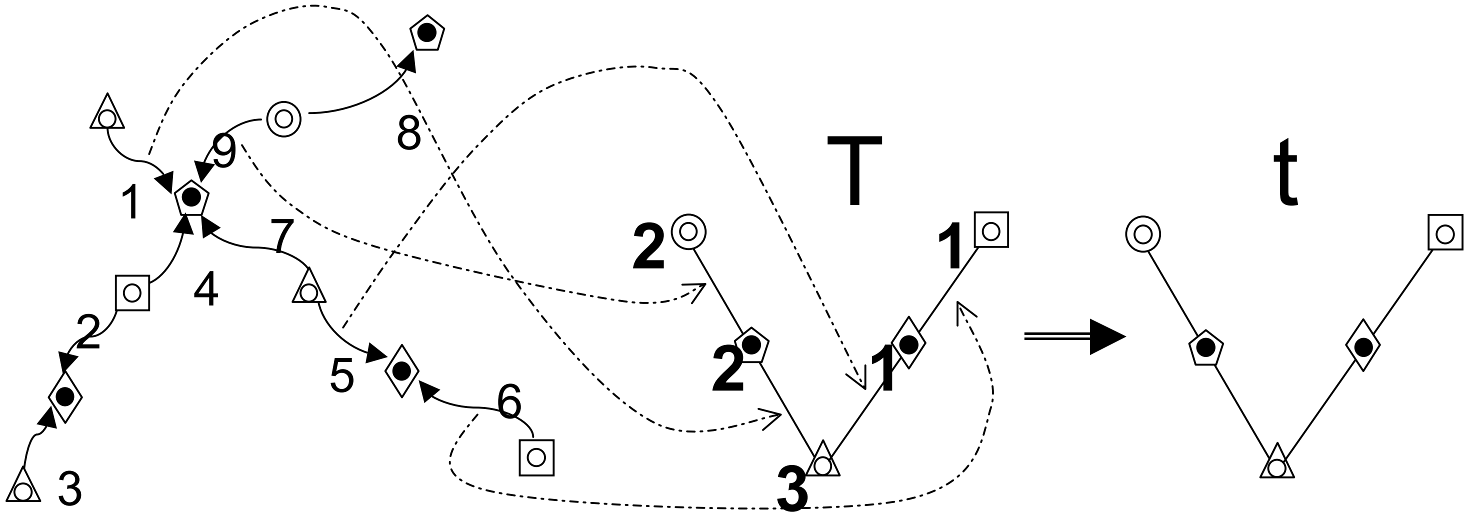
$$\boxed{\left|C_{N,p,q}\right| = \left|BT(p,q)\right| . \left|PP\left(N, N-1, N-1-(p+q)\right)\right|}$$

In order to proof our main result we only need to show

$$\boxed{C_{N,p,q} \cong BT(p,q) \times PP\left(N, N-1, N-1-(p+q)\right)}$$

Bijection

Construction of the Bijection -- Bicolored Tree



Construction Rules for **T** :

Let $m_1^i = \max(\pi_1^i)$ (the i[th] Block of π_1) and $m_2^j = \max(\pi_2^j)$

i. The white block that contains the integer 1 is the root

ii. The black block j is the descendant of the white i if $\beta(m_2^j) \in \pi_1^i$

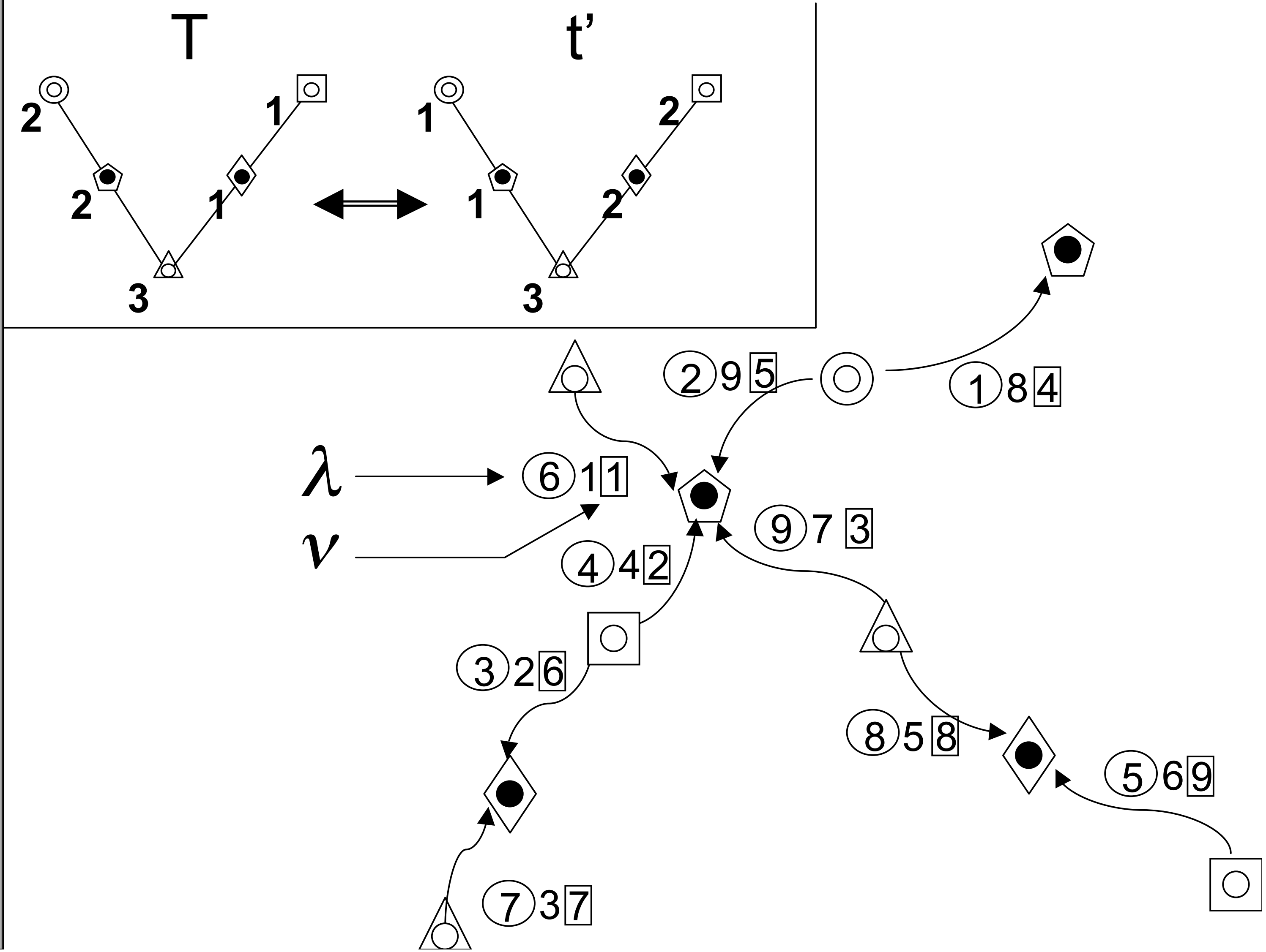
iii. The white block i is the descendant of the black j if $m_1^i \in \pi_2^j$

iv. If black j and k are descendant of white i, j is on the left of k if $\beta(m_2^j) < \beta(m_2^k)$

v. If white i and l are descendent of black j, i is on the left of l if $\beta^{-1}(m_1^i) < \beta^{-1}(m_1^l)$

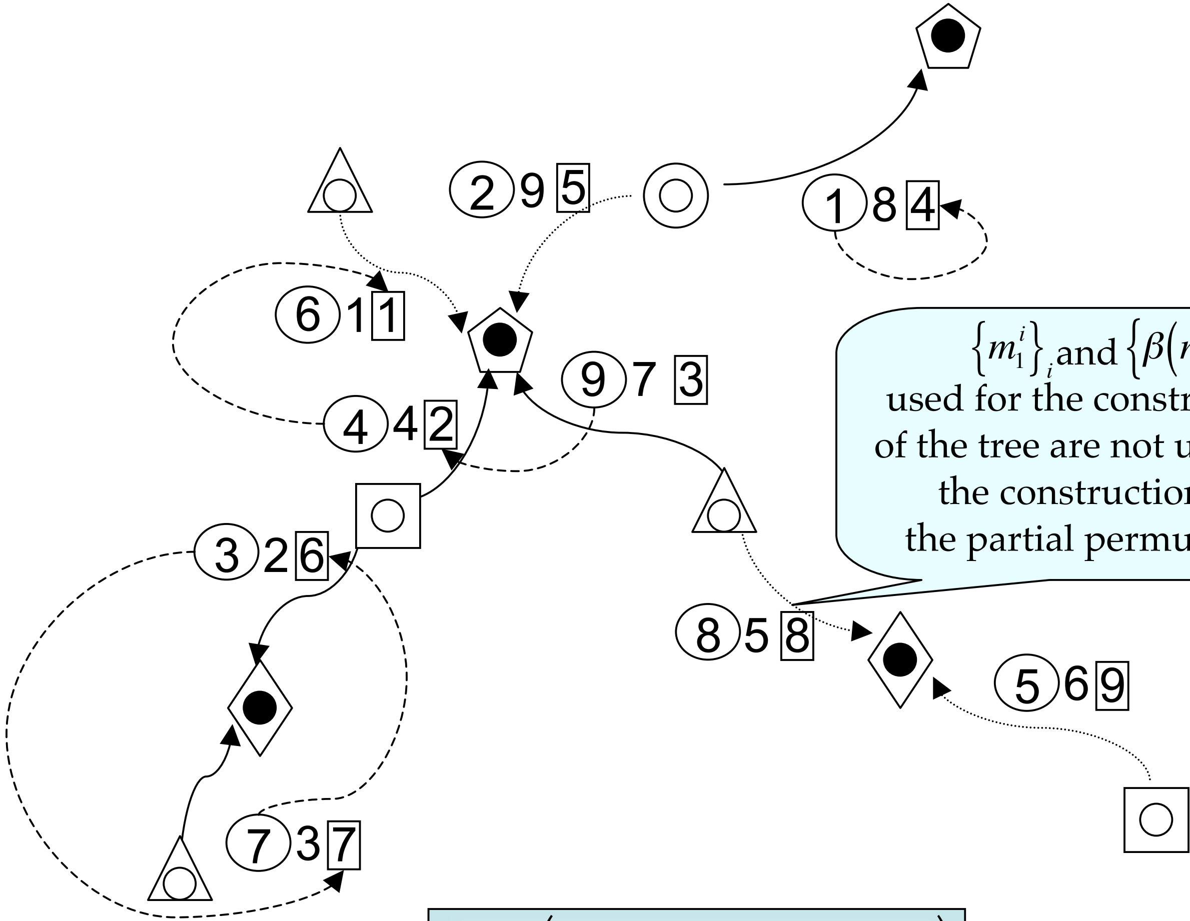
Bijection

Partial Permutation (1/2) Relabeling



Bijection

Partial Permutation (2/2)

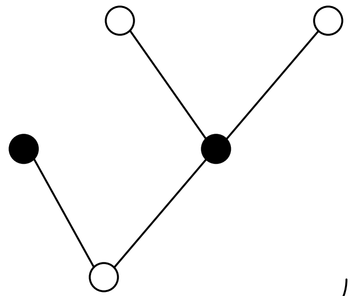


$\{m_1^i\}_i$ and $\{\beta(m_2^j)\}_j$ used for the construction of the tree are not used for the construction of the partial permutation

$$\sigma = \begin{pmatrix} 1 & 3 & 4 & 7 & 9 \\ 4 & 7 & 1 & 6 & 2 \end{pmatrix}$$

Proof

First Step : extension of the partial permutation and identification of the partitions' structure



There are 2 black blocks and 3 white ones

$$\sigma = \begin{pmatrix} 3 & 4 & 5 & 6 & 8 & 10 \\ 4 & 6 & 3 & 1 & 8 & 7 \end{pmatrix}$$

1, 2, 7 and 9 are missing on the first line
2, 5, 9 and 10 are missing on the second line

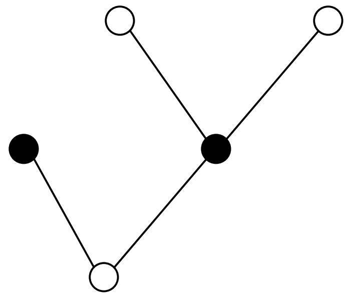
Using the tree construction rules we find

$$\lambda\left(m_1^1\right) = 1,\ \lambda\left(m_1^2\right) = 2,\ \lambda\left(\beta\left(m_2^1\right)\right) = 7,\ \lambda\left(\beta\left(m_2^1\right)\right) = 9$$

$$\upsilon\left(m_2^1\right) = 2,\ \upsilon\left(\beta^{-1}\left(m_1^1\right)\right) = 5,\ \upsilon\left(\beta^{-1}\left(m_1^2\right)\right) = 9,\ \upsilon\left(m_2^2\right) = 10$$

Proof

First Step : extension of the partial permutation and identification of the partitions' structure



$$\sigma = \begin{pmatrix} 3 & 4 & 5 & 6 & 8 & 10 \\ 4 & 6 & 3 & 1 & 8 & 7 \end{pmatrix}$$

Then

$$\lambda(\pi_1) = \{\{1\},\{2\},\{3,4,5,6,7,8,9,10\}\}$$
$$\nu(\pi_2) = \{\{1,2\},\{3,4,5,6,7,8,9,10\}\}$$

$$\overline{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 9 & 4 & 6 & 3 & 1 & 2 & 8 & 10 & 7 \end{pmatrix}$$

$$\lambda(\pi_2) = \{\{6,7\},\{1,2,3,4,5,8,9,10\}\}$$

Proof

Second Step : Reconstruction of relabeling permutations

$$\lambda(\pi_1) = \{\{1\},\{2\},\{3,4,5,6,7,8,9,10\}\}$$
$$\nu(\pi_2) = \{\{1,2\},\{3,4,5,6,7,8,9,10\}\}$$
$$\lambda(\pi_2) = \{\{6,7\},\{1,2,3,4,5,8,9,10\}\}$$

$$\bar{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 9 & 4 & 6 & 3 & 1 & 2 & 8 & 10 & 7 \end{pmatrix}$$

1

$$\begin{array}{lcllllllllll} \gamma & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \lambda & : & 3 \\ \nu & : \end{array}$$

3 is the least element of the root block that contains 1 by construction : $\lambda(1) = 3$

2

$$\begin{array}{lcllllllllll} \gamma & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \lambda & : & 3 \\ \nu & : & 3 \end{array}$$

$\lambda(1) = 3$ belongs to $\lambda(\pi_2^2)$ so that $\nu(1)$ is the least element of $\nu(\pi_2^2)$ i.e 3

3

$$\begin{array}{lcllllllllll} \gamma & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \lambda & : & 3 & 4 \\ \nu & : & 3 \end{array}$$

$\bar{\sigma}^{-1}(\nu(1)) = \lambda(\beta(1)) = 5$ belongs to the same block of $\lambda(\pi_1)$ as $\lambda(\alpha\beta(1))$ (cf. stability of the blocks by α). Hence, $\lambda(\alpha\beta(1)) = \lambda(\gamma(1)) = \lambda(2)$ is the least element non yet used of $\lambda(\pi_1^3)$ i.e 4

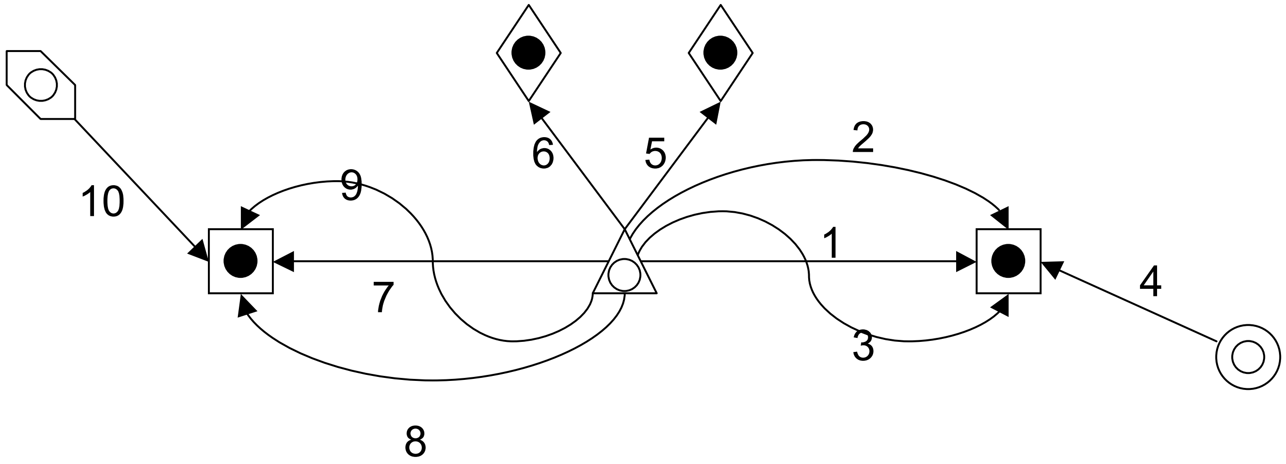
⋮

n

$$\begin{array}{lcllllllllll} \gamma & : & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ \lambda & : & 3 & 4 & 5 & 1 & 6 & 7 & 8 & 9 & 10 & 2 \\ \nu & : & 3 & 4 & 5 & 6 & 1 & 2 & 7 & 8 & 9 & 10 \end{array}$$

Proof

And the map is there !



Questions ?