A Bijection for Unicellular Partitioned Bicolored Maps

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We want to compute \( B(m,n,N) \), the cardinality of the set of couples:

\[
(\alpha, \beta) \in \Sigma_N^2, \quad \text{tels que} \quad \begin{cases} 
Z(\alpha) = m \\
Z(\beta) = n \\
\alpha \beta = \gamma = (1 \, 2 \, \ldots \, N) 
\end{cases}
\]
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\[
\alpha = (1)(2\ 4)(3)(5\ 7)(6)(8\ 9)
\]
\[
\beta = (1\ 4\ 7\ 9)(2\ 3)(5\ 6)(8)
\]
Introduction

Solution:

\[
\sum_{m, n \geq 1} B(m, n, N) y^m z^n = N! \sum_{p, q \geq 1} \binom{N - 1}{p - 1} \binom{N - 1}{q - 1} y^p z^q
\]
In Order to prove our claim, we introduce a new class of objects: Partitioned Maps.

\( C_{N,p,q} \)

\( (\pi_1, \pi_2, \alpha) \) such as

\[
\begin{align*}
\alpha & \in \Sigma_N \\
\pi_1 & \mapsto [N] \\
\pi_2 & \mapsto [N]
\end{align*}
\]

\( \pi_1 \) partitions the cycles of \( \alpha \)

\( \pi_2 \) partitions the cycles of \( \beta = \alpha^{-1}\gamma \)
Geometrical Interpretation

\[ \alpha = (1)(2 \ 4)(3) \ (5 \ 7)(6)(8 \ 9) \]

\[ \pi_1 = \{2,4,6\}, \{8,9\} \]

\[ \pi_2 = \{2,3,5,6\}, \{1,4,7,8,9\} \]
Notations

Ordered Bicolored Trees and Partial Permutation

\[ |BT(p,q)| = \frac{p + q - 1}{pq} \left( \frac{(p + q - 2)^2}{p - 1} \right) \]

\[ |PP(X,Y,A)| = \binom{|X|}{A} \binom{|Y|}{A} A! \]

\[ \sigma = \begin{pmatrix} 3 & 4 & 5 & 6 & 8 & 10 \\ 4 & 6 & 3 & 1 & 8 & 7 \end{pmatrix} \]

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Our main result is equivalent to

\[ C_{N,p,q} = BT(p,q) \times PP(N,N-1,N-1-(p+q)) \]
Construction Rules for $T$:

Let $m_i^1 = \max(\pi_i^1)$ (the $i^{th}$ Block of $\pi_i$) and $m_i^2 = \max(\pi_i^2)$

i. The white block that contains the integer 1 is the root

ii. The black block $j$ is the descendant of the white $i$ if $\beta(m_i^2) \in \pi_i$

iii. The white block $i$ is the descendant of the black $j$ if $m_i^1 \in \pi_j^1$

iv. If black $j$ and $k$ are descendant of white $i$, $j$ is on the left of $k$ if $\beta(m_i^2) < \beta(m_j^2)$

v. If white $i$ and $l$ are descendant of black $j$, $i$ is on the left of $l$ if $\beta^{-1}(m_i^1) < \beta^{-1}(m_l^1)$
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Partial Permutation (1/2) Relabeling

Bijection

\( \lambda \rightarrow 611 \)

\( \nu \rightarrow 442 \)

\( 295 \rightarrow 184 \)

\( 326 \rightarrow 858 \)

\( 737 \rightarrow 569 \)
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\[ \sigma = \begin{pmatrix} 1 & 3 & 4 & 7 & 9 \\ 4 & 7 & 1 & 6 & 2 \end{pmatrix} \]

Partial Permutation (2/2)

\[ \{m_1^i\} \text{ and } \{\beta(m_2^j)\} \text{ used for the construction of the tree are not used for the construction of the partial permutation} \]
**First Step : extension of the partial permutation and identification of the partitions’ structure**

\[
\sigma = \begin{pmatrix}
3 & 4 & 5 & 6 & 8 & 10 \\
4 & 6 & 3 & 1 & 8 & 7 \\
\end{pmatrix}
\]

- 1, 2, 7 and 9 are missing on the first line.
- 2, 5, 9 and 10 are missing on the second line.

Using the tree construction rules we find:

\[
\lambda(m_1^1) = 1, \lambda(m_1^2) = 2, \lambda(\beta(m_2^1)) = 7, \lambda(\beta(m_2^2)) = 9
\]

\[
v(m_1^1) = 2, v(\beta^{-1}(m_1^1)) = 5, v(\beta^{-1}(m_1^2)) = 9, v(m_2^2) = 10
\]

There are 2 black blocks and 3 white ones.
First Step: extension of the partial permutation and identification of the partitions' structure

\[ \lambda(\pi_1) = \{\{1\}, \{2\}, \{3, 4, 5, 6, 7, 8, 9, 10\}\} \]
\[ \nu(\pi_2) = \{\{1, 2\}, \{3, 4, 5, 6, 7, 8, 9, 10\}\} \]

Then

\[ \lambda(\pi_2) = \{\{6, 7\}, \{1, 2, 3, 4, 5, 8, 9, 10\}\} \]

\[ \sigma = \begin{pmatrix} 3 & 4 & 5 & 6 & 8 & 10 \\ 4 & 6 & 3 & 1 & 8 & 7 \end{pmatrix} \]

\[ \overline{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 9 & 4 & 6 & 3 & 1 & 2 & 8 & 10 & 7 \end{pmatrix} \]
Proof

Second Step : Reconstruction of relabeling permutations

\[ \lambda(\pi_1) = \{\{1\},\{2\},\{3,4,5,6,7,8,9,10\}\} \]
\[ \nu(\pi_2) = \{\{1,2\},\{3,4,5,6,7,8,9,10\}\} \]
\[ \lambda(\pi_2) = \{\{6,7\},\{1,2,3,4,5,8,9,10\}\} \]
\[ \overline{\sigma} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 5 & 9 & 4 & 6 & 3 & 1 & 2 & 8 & 10 & 7 \end{pmatrix} \]

\( \gamma : 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \)
\( \lambda : 3 \)
\( \nu : \)
\( \gamma : 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \)
\( \lambda : 3 \)
\( \nu : 3 \)
\( \gamma : 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \)
\( \lambda : 3 \ 4 \)
\( \nu : 3 \)
\( \gamma : 1 \ 2 \ 3 \ 4 \ 5 \ 6 \ 7 \ 8 \ 9 \ 10 \)
\( \lambda : 3 \ 4 \ 5 \ 1 \ 6 \ 7 \ 8 \ 9 \ 10 \ 2 \)
\( \nu : 3 \ 4 \ 5 \ 6 \ 1 \ 2 \ 7 \ 8 \ 9 \ 10 \)

3 is the least element of the root block that contains 1 by construction : \( \lambda(1) = 3 \)

\( \lambda(1) = 3 \) belongs to \( \lambda(\pi_2^2) \) so that \( \nu(1) \) is the least element of \( \nu(\pi_2^2) \) i.e 3

\( \overline{\sigma}^{-1}(\nu(1)) = \lambda(\beta(1)) = 5 \) belongs to the same block of \( \lambda(\pi_1) \) as \( \lambda(\alpha\beta(1)) \) (cf. stability of the blocks by \( \alpha \)). Hence, \( \lambda(\alpha\beta(1)) = \lambda(\gamma(1)) = \lambda(2) \) is the least element non yet used of \( \lambda(\pi_1^3) \) i.e 4
Proof

And the map is there!
Questions

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