Poset Topology and Permutation Statistics

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Based on joint work with John Shareshian

Classical Permutation Statistics

For σ in symmetric group $\mathfrak{S}_n$

- Eulerian statistics

Descent set: $\mathrm{DES}(\sigma) := \{i \in [n] : \sigma(i) > \sigma(i+1)\}$
Excedance set: $\mathrm{EXC}(\sigma) := \{i \in [n] : \sigma(i) > i\}$

$$\mathrm{des}(\sigma) := |\mathrm{DES}(\sigma)| \qquad \mathrm{exc}(\sigma) := |\mathrm{EXC}(\sigma)|$$

$$\sigma = 3.25.4.1 \qquad \mathrm{DES}(\sigma) = \{1,3,4\} \qquad \mathrm{des}(\sigma) = 3$$
$$\sigma = 32541 \qquad \mathrm{EXC}(\sigma) = \{1,3\} \qquad \mathrm{exc}(\sigma) = 2$$

- Mahonian statistics

$$\mathrm{inv}(\sigma) := |\{(i,j) : 1 \le i < j \le n \quad \& \quad \sigma(i) > \sigma(j)\}|$$
$$\mathrm{maj}(\sigma) := \sum_{i \in \mathrm{DES}(\sigma)} i$$

$$\mathrm{inv}(32541) = 6 \qquad \mathrm{maj}(32541) = 1 + 3 + 4 = 8$$

Classical Permutation Statistics

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

$\mathfrak{S}_3$	des	exc
123	0	0
132	1	1
213	1	1
231	1	2
312	1	1
321	2	1

$$A_3(t) = 1 + 4t + t^2$$

Classical Permutation Statistics

Eulerian polynomial

$$A_n(t) := \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{des}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} t^{\mathrm{exc}(\sigma)}$$

Basic formula:

$$\frac{A_n(t)}{(1-t)^{n+1}} = \sum_{i \geq 0} (i+1)^n t^i$$

Exponential generating function:

$$\sum_{n \geq 0} A_n(t) \frac{z^n}{n!} = \frac{1-t}{e^{z(t-1)} - t}$$

Classical Permutation Statistics

q-analogs

$$\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} = \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$$

where $[n]_q := 1 + q + \cdots + q^{n-1}$ and $[n]_q! := [n]_q[n-1]_q \cdots [1]_q$

$\mathfrak{S}_3$	inv	maj
123	0	0
132	1	2
213	1	1
231	2	2
312	2	1
321	3	3

$$1 + 2q + 2q^2 + q^3 = (1 + q + q^2)(1 + q)$$

q-Eulerian polynomials

$$A_n^{\mathrm{inv},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{maj},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{des}(\sigma)}$$

$$A_n^{\mathrm{inv},\mathrm{exc}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{exc}(\sigma)}$$

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q-Eulerian polynomials

Theorem (q-exponential generating function, Stanley 1976)

$$\sum_{n\geq 0} A_n^{\mathrm{inv},\mathrm{des}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-t)}{\exp_q(z(t-1))-t}$$

where

$$\exp_q(z) := \sum_{n\geq 0} \frac{z^n}{[n]_q!}$$

Theorem (Gessel 1977)

$$\frac{A_n^{\mathrm{maj},\mathrm{des}}(q,t)}{\prod_{i=0}^{n}(1-tq^i)} = \sum_{i\geq 0}[i+1]_q^n\, t^i$$

Eulerian and Mahonian partners

- $(\mathrm{inv}, \mathrm{des}) \sim (\mathrm{maj}, \mathrm{dmc})$ (Foata 1977)
- $(\mathrm{maj}, \mathrm{des}) \sim (\mathrm{den}, \mathrm{exc})$ (Foata-Zeilberger 1990)
- $(\mathrm{maj}, \mathrm{des}) \sim (\mathrm{inv}, \mathrm{stc})$ (Skandera 2002)
- $(\mathrm{inv}, \mathrm{exc}) \sim (\mathrm{mad}, \mathrm{des})$ (Clarke-Steingrimsson-Zeng 2002)

The maj-exc distribution

$$A_n^{\mathrm{inv},\mathrm{des}}(q,t) := \sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{inv}(\sigma)} t^{\mathrm{des}(\sigma)}$$

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[Clarke-Steingrimsson-Zeng, 1995] It is of course possible to define scores of different families of Eulerian-Mahonian statistics by arbitrarily combining an Eulerian statistic and a Mahonian one. Although some needles are sure to be found in that haystack, most of the possible such statistics seem rather unattractive and unlikely to posses particular interesting properties.

The maj-exc distribution

Conjecture (Shareshian & MW 2005)

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq) - tq\exp_q(z)}$$

$q=1$: reduces to exp. gen. function formula for $A_n(t)$

$$\frac{(1-tq)\exp_q(z)}{\exp_q(ztq) - tq\exp_q(z)} = \frac{(1-t)e^z}{e^{zt}-te^z} = \frac{(1-t)}{e^{z(t-1)}-t}$$

$t=1$: reduces to $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} = [n]_q!$

Computer verification up to $n=9$

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Mahonian partner of des

Definition

Admissible inversion of $\sigma \in \mathfrak{S}_n$ *is a pair* $(\sigma(i), \sigma(j))$ *such that*

- $i < j$
- $\sigma(i) > \sigma(j)$
- *either*
  - $\sigma(j) < \sigma(j+1)$ *or*
  - $\exists k$ *such that* $i < k < j$ *and* $\sigma(k) < \sigma(j)$

Let $\mathrm{ai}(\sigma) := \#$ *admissible inversions of* σ*.*

Define $\mathrm{aid}(\sigma) := \mathrm{ai}(\sigma) + \mathrm{des}(\sigma)$

Admissible inversions of 24153

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Admissible inversions of 24153

$$(2,1), (4,1)$$

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Admissible inversions of 24153

$$(2,1),(4,1),(4,3)$$

So $\mathrm{ai}(24153) = 3$ and $\mathrm{aid}(24.15.3) = 3 + 2 = 5$.

Mahonian partner of des

Theorem (Shareshian & MW 2005)

$$\sum_{n\geq 0} A_n^{\mathrm{aid},\mathrm{des}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq) - tq\exp_q(z)}$$

$t = 1$: reduces to $\sum_{\sigma\in\mathfrak{S}_n} q^{\mathrm{aid}(\sigma)} = [n]_q!$. Hence aid is *Mahonian*.

Mahonian partner of des

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Conjecture (Shareshian & MW)

$(\mathrm{aid},\mathrm{des})$ *and* $(\mathrm{maj},\mathrm{exc})$ *are equidistributed on* $\mathfrak{S}_n$.

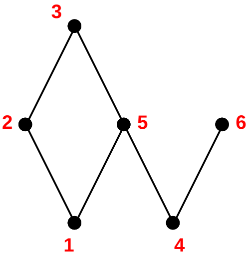
$$A_n^{\mathrm{aid},\mathrm{des}}(q,t) = A_n^{\mathrm{maj},\mathrm{exc}}(q,t)$$

Computer verification up to $n = 9$.

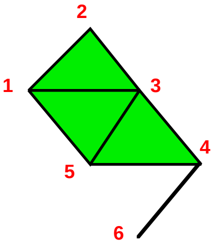
Poset Topology

poset $\longrightarrow$ simplicial complex

Order complex $\Delta(P)$ of a poset P is the simplicial complex whose faces are the chains of P.



P

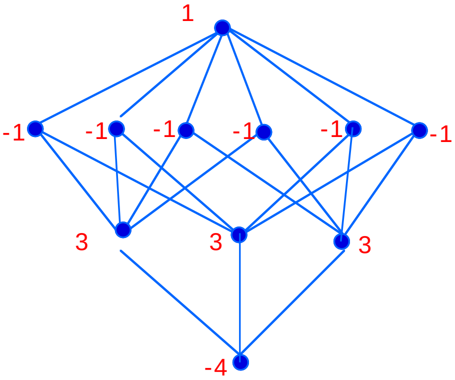


Δ(P)

Poset Topology

Möbius function $\mu : P \times P \to \mathbb{Z}$

$$\mu(x,y) := \begin{cases} 0 & \text{if } x \not\leq y \\ 1 & \text{if } x = y \\ -\sum_{x<z\leq y} \mu(z,y) & \text{if } x < y \end{cases}$$



$\mu(x, \hat{1})$

Poset Topology

Theorem (Ph. Hall)

For all $x < y$ in P

$$\mu(x, y) = \tilde{\chi}(\Delta(x, y)),$$

where $\tilde{\chi}$ is the reduced Euler characteristic.

Euler-Poincaré formula

$$\tilde{\chi}(\Delta) = \sum_{i=0}^{\dim \Delta} (-1)^i \dim \tilde{H}_i(\Delta)$$

If $\tilde{H}_i(\Delta(P)) = 0$ for all $i < \ell(P)$ then

$$\dim \tilde{H}_{\ell(P)}(\Delta(P)) = (-1)^{\ell(P)} \mu(P)$$

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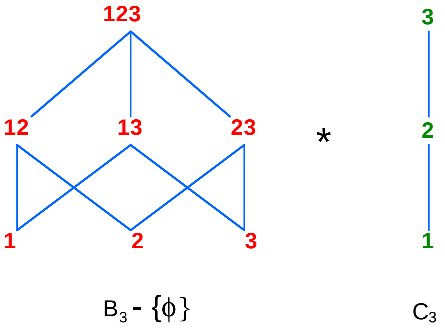
Rees Product-Björner & Welker, 2003

Let P and Q be pure (ranked) posets.

$$P * Q := \{(p, q) \in P \times Q : r(p) \geq r(q)\}$$

$(p_1, q_1) \leq (p_2, q_2)$ if the following holds

- $p_1 \leq_P p_2$
- $q_1 \leq_Q q_2$
- $r(p_2) - r(p_1) \geq r(q_2) - r(q_1)$



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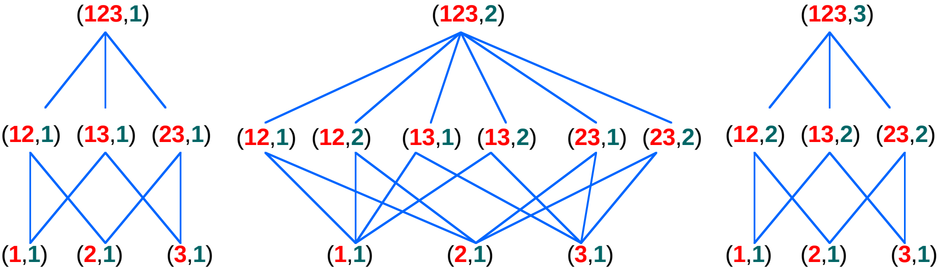


Rees Product-Björner & Welker, 2003

Theorem (Björner & Welker)

The Rees product of two Cohen-Macaulay posets is Cohen-Macaulay. (CM means that homology of each interval vanishes below its top dimension.)

Conjecture (Björner & Welker)

$\dim \tilde{H}_{n-1}((B_n \setminus \{\emptyset\}) * C_n) = \#$ *derangements in* $\mathfrak{S}_n$.

Proved by Jonsson.

Maximal Intervals of Rees Product

Notation:

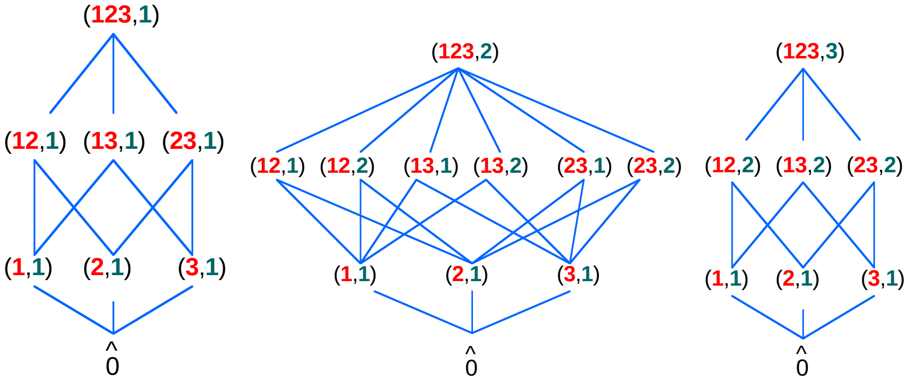
- $[n] := \{1, 2, \ldots, n\}$
- $I_{n,j}$:= the closed interval $[\hat{0}, ([n], j)]$ of $((B_n \setminus \{\emptyset\}) * C_n) \cup \hat{0}$
- $\overline{I}_{n,j}$:= the open interval $(\hat{0}, ([n], j))$
- $a_{n,k}$:= # permutations in $\mathfrak{S}_n$ with k descents.

Theorem (Shareshian & MW)

For all $j = 1, \ldots, n$, the order complex $\Delta(\overline{I}_{n,j})$ has the homotopy type of a wedge of $a_{n,j-1}$ spheres of dimension $n-2$.

The Björner-Welker-Jonsson result follows easily from this.

Maximal Intervals of Rees Product



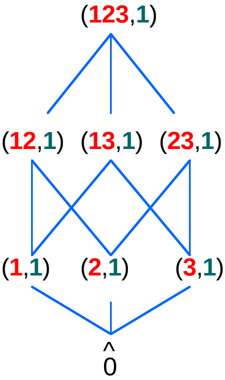
$I_{3,1}$ — $\mu(\hat{0}, \hat{1}) = -1$

$I_{3,2}$ — $\mu(\hat{0}, \hat{1}) = -4$

$I_{3,3}$ — $\mu(\hat{0}, \hat{1}) = -1$

$$A_3(t) = 1 + 4t + t^2$$

Maximal Intervals of Rees Product



$I_{3,1}$

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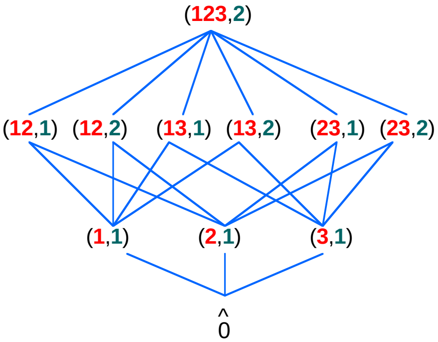

$I_{3,2}$

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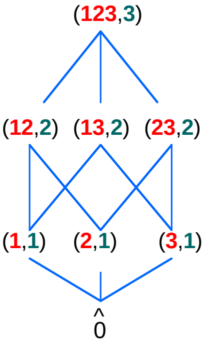


$I_{3,3}$

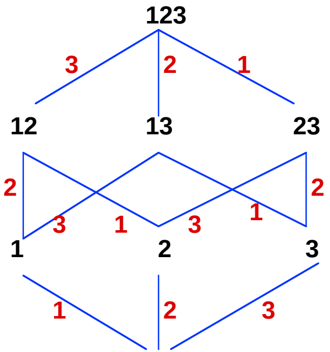
$\mu(\hat{0},\hat{1}) = -1$

$$A_3(t) = 1 + 4t + t^2$$

Lexicographic Shellability

Definition (Björner 1980 - pure case)

Let P be a pure poset with a minimum $\hat{0}$ and a maximum $\hat{1}$. An EL-labeling of P is a labeling of the edges of the Hasse diagram of P so that the lexicographically first maximal chain of each closed interval is the only increasing maximal chain of the interval.

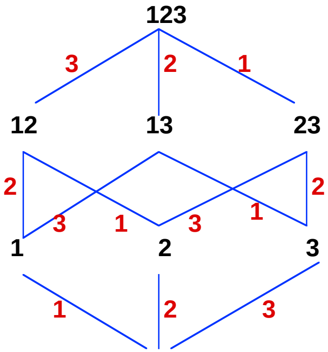


Lexicographic Shellability

Theorem (Björner 1980)

Suppose P is pure and admits an EL-labeling with r decreasing chains. Then $\Delta(\bar{P})$ has the homotopy type of a wedge of r spheres of dimension $(\ell(P)-2)$.

Example: Number of decreasing chains is 1.



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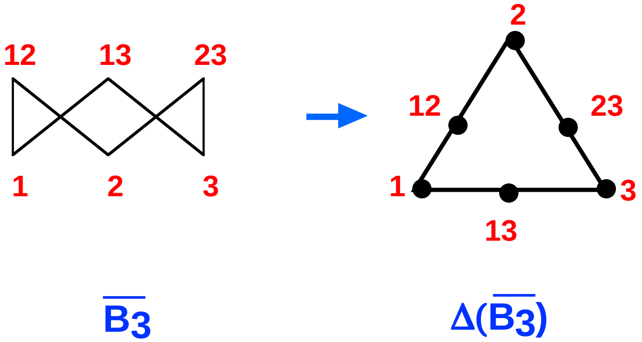


EL-labeling of $I_{n,j}$

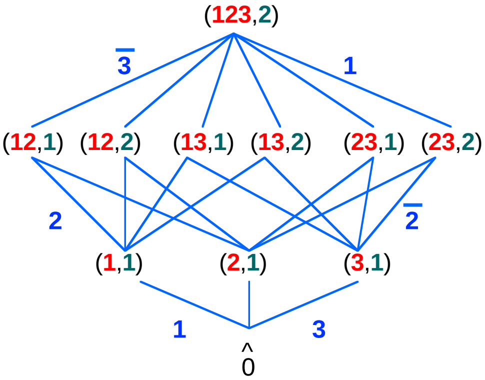
Theorem (Shareshian & MW)

For all $j = 1, \ldots, n$, the interval $I_{n,j}$ admits an EL-labeling with $a_{n,j-1}$ decreasing chains.

Poset of labels: product order on $[n] \times \{0,1\}$

Edge	Label	Code
$(S, i) \longrightarrow (S \cup \{a\}, i)$	$(a, 0)$	a
$(S, i) \longrightarrow (S \cup \{a\}, i+1)$	$(a, 1)$	$\bar{a}$
$\hat{0} \longrightarrow (\{a\}, 1)$	$(a, 0)$	a

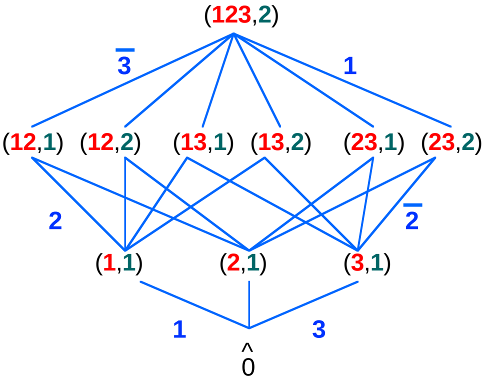
EL-labeling of $I_{n,j}$



Maximal chains of $I_{n,j}$ correspond bijectively to barred permutations with $j-1$ bars in which the first letter is unbarred.

$12\bar{3}$, $3\bar{2}1$, etc.

EL-labeling of $I_{n,j}$



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EL-labeling of $I_{n,j}$

Let $\mathcal{A}_{n,j}$ be the set of barred permutations ω such that

- ω has j bars
- $\omega(1)$ is unbarred
- first entry $\omega(i)$ of each ascent $\omega(i) < \omega(i+1)$ is barred
- second entry $\omega(i+1)$ of each ascent $w(i) < w(i+1)$ is unbarred

Decreasing maximal chains correspond bijectively to the set $\mathcal{A}_{n,j-1}$

Example: $3\bar{1}5\bar{2}64 \in \mathcal{A}_{6,2}$ and $3\bar{1}5\bar{2}6\bar{4} \in \mathcal{A}_{6,3}$

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Decreasing maximal chains correspond bijectively to the set $\mathcal{A}_{n,j-1}$

$\mathcal{A}_{3,0} = \{321\}$, $\mathcal{A}_{3,1} = \{2\bar{1}3, 3\bar{1}2, 3\bar{2}1, 32\bar{1}\}$, $\mathcal{A}_{3,2} = \{3\bar{2}\bar{1}\}$

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Let $\mathcal{A}_{n,j}$ be the set of barred permutations ω such that

- ω has j bars
- $\omega(1)$ is unbarred
- first entry $\omega(i)$ of each ascent $\omega(i) < \omega(i+1)$ is barred
- second entry $\omega(i+1)$ of each ascent $w(i) < w(i+1)$ is unbarred

Decreasing maximal chains correspond bijectively to the set $\mathcal{A}_{n,j-1}$

Let $\mathcal{A}_n := \mathcal{A}_{n,0} \uplus \cdots \uplus \mathcal{A}_{n,n-1}$.
Bijection: $\varphi : \mathcal{A}_n \to \mathfrak{S}_n$

$$\begin{aligned} \varphi(\alpha\bar{1}\beta) &= \varphi(\alpha)1\varphi(\beta) \\ \varphi(\alpha 1) &= 1\varphi(\alpha) \end{aligned}$$

Claim: $\#\,\text{bars}(\omega) = \operatorname{des}(\varphi(\omega))$

EL-labeling of $I_{n,j}$

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Claim: $\#\,\text{bars}(\omega) = \mathrm{des}(\varphi(\omega))$

Hence $\Delta(\bar{I}_{n,j})$ has the homotopy type of a wedge of $a_{n,j-1}$ spheres of dimension $n-2$.

q-Analog of $I_{n,j}$

$B_n(q) :=$ lattice of subspaces of $\mathbb{F}_q^n$.

$I_{n,j}(q) :=$ the closed interval $[\hat{0}, (\mathbb{F}_q^n, j)]$ of $((B_n(q) \setminus \{\emptyset\}) * C_n) \cup \hat{0}$

Theorem (Simion 1995)

$B_n(q)$ admits an EL-labeling λ such that

- *label sequence of each maximal chain is a permutation in $\mathfrak{S}_n$*
- *for each $\sigma \in \mathfrak{S}_n$ there are $q^{\mathrm{inv}(\sigma)}$ maximal chains labeled by σ*

EL-labeling of $I_{n,j}(q)$

Edge	Label	Code
$(U, i) \longrightarrow (V, i)$	$(\lambda(U, V), 0)$	$\lambda(U, V)$
$(U, i) \longrightarrow (V, i+1)$	$(\lambda(U, V), 1)$	$\overline{\lambda(U, V)}$
$\hat{0} \longrightarrow (V, 1)$	$(\lambda(U, V), 0)$	$\lambda(U, V)$

- label sequence of each decreasing maximal chain is a barred permutation in $\mathcal{A}_{n,j-1}$
- for each $\omega \in \mathcal{A}_{n,j-1}$ there are $q^{\mathrm{inv}(|\omega|)}$ decreasing maximal chains labeled by ω, where $|\omega|$ means drop bars.

EL-labeling of $I_{n,j}(q)$

$$\#\text{ decreasing maximal chains of } I_{n,j}(q) = \sum_{\omega \in \mathcal{A}_{n,j-1}} q^{\mathrm{inv}(|\omega|)},$$

EL-labeling of $I_{n,j}(q)$

$$\#\text{ decreasing maximal chains of } I_{n,j}(q) = \sum_{\omega \in \mathcal{A}_{n,j-1}} q^{\mathrm{inv}(|\omega|)},$$

Recall bijection $\varphi : \mathcal{A}_n \to \mathfrak{S}_n$, $\#\text{ bars}(\omega) = \mathrm{des}(\varphi(\omega))$

Claim: $\mathrm{inv}(|w|) = \mathrm{ai}(\varphi(w))$

Therefore

$$\#\text{ decreasing maximal chains of } I_{n,j}(q) = \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{des}(\sigma) = j-1}} q^{\mathrm{ai}(\sigma)}.$$

EL-labeling of $I_{n,j}(q)$

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Theorem (Shareshian & MW)

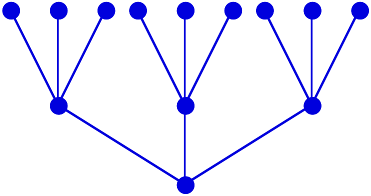
$\Delta(\bar{I}_{n,j}(q))$ has the homotopy type of a wedge of $\sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{des}(\sigma) = j-1}} q^{\mathrm{ai}(\sigma)}$ *spheres of dimension $n-2$.*

Tree Lemma

Let P be a pure poset of length n with min $\hat{0}_P$ and max $\hat{1}_P$

Notation:

- P^* is the dual of P
- $I_j(P)$ is the interval $[\hat{0}, (\hat{1}_P, j)]$ of $((P \setminus \{\hat{0}_P\}) * C_n) \cup \{\hat{0}\}$
- $T_{n,t}$ is the poset whose Hasse diagram is the full t-ary tree of height n with root at the bottom.



$T_{2,3}$

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- $T_{n,t}$ is the poset whose Hasse diagram is the full t-ary tree of height n with root at the bottom.

Lemma

$$\sum_{j=1}^{n} \mu(I_j(P))t^j = -\mu((P^* * T_{n,t}) \cup \{\hat{1}\})$$

Tree Lemma - Left Hand Side

For $P = B_n(q)$

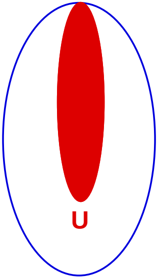
$$\begin{aligned}
\text{LHS} &= -\sum_{j=1}^{n} \mu(I(B_n(q))t^j \\
&= \sum_{j=1}^{n} \mu(I_{n,j}(q))t^j \\
&= \sum_{j=1}^{n} (-1)^n \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{des}(\sigma) = j-1}} q^{\mathrm{ai}(\sigma)} \; t^j \\
&= (-1)^n t \, A_n^{\mathrm{ai,des}}(q,t)
\end{aligned}$$

Tree Lemma - Right Hand Side

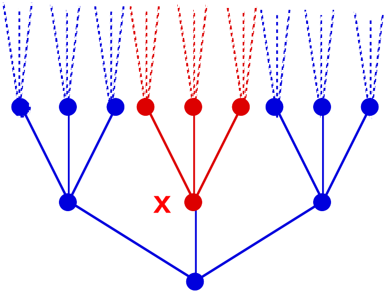
Let $P_{n,t}(q) := (B_n(q) * T_{n,t}) \cup \{\hat{1}\}$.

Upper interval

$$[(U,x),\hat{1}] \cong P_{n-dimU,t}(q)$$



$B_n(q)$



$T_{n,t}$

Tree Lemma - Right Hand Side

Let $P_{n,t}(q) := (B_n(q) * T_{n,t}) \cup \{\hat{1}\}$.

Upper interval

$$[(U,x),\hat{1}] \cong P_{n-dimU,t}(q)$$

Möbius recurrence:

$$\begin{aligned} 0 &= 1 + \sum_{(U,x)<\hat{1}} \mu((U,x),\hat{1}) \\ &= 1 + \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q (1 + t + \cdots + t^k)\, \mu(P_{n-k,t}(q)) \end{aligned}$$

Tree Lemma - Right Hand Side

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$$\Longrightarrow \sum_{n\geq 0} \mu(P_{n,t}(q)) \frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{t\exp_q(zt) - \exp_q(z)}$$

Putting it all together

Tree Lemma:

$$\sum_{j=1}^{n} \mu(I_j(P))\, t^j = -\mu((P^* * T_{n,t}) \cup \{\hat{1}\})$$

$$\sum_{j=1}^{n} \mu(I_{n,j}(q))\, t^j = (-1)^n t\, A_n^{\mathrm{ai,des}}(q,t)$$

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$$\sum_{n\geq 0} A_n^{\mathrm{ai}\quad,\mathrm{des}}(q,t) \frac{z^n}{[n]_q!} = \frac{(1- \quad t)\exp_q(z)}{\exp_q(z \quad t) - \quad t\exp_q(z)}$$

Putting it all together

Tree Lemma:

$$\sum_{j=1}^{n} \mu(I_j(P))\, t^j = -\mu((P^* * T_{n,t}) \cup \{\hat{1}\})$$

$$\sum_{j=1}^{n} \mu(I_{n,j}(q))\, t^j = (-1)^n t\, A_n^{\mathrm{ai},\mathrm{des}}(q,t)$$

$$\sum_{n\geq 0} \mu(P_{n,t}(q)) \frac{z^n}{[n]_q!} = \frac{(1-t)\exp_q(z)}{t\exp_q(zt) - \exp_q(z)}$$

$$\sum_{n\geq 0} A_n^{\mathrm{ai}d,\mathrm{des}}(q,t) \frac{z^n}{[n]_q!} = \frac{(1-qt)\exp_q(z)}{\exp_q(zqt) - qt\exp_q(z)}$$

Equivariant version

Action of $\mathfrak{S}_n$ on B_n induces an action of $\mathfrak{S}_n$ on $I_{n,j}$ which induces a representation of $\mathfrak{S}_n$ on $\tilde{H}_{n-2}(\bar{I}_{n,j})$

Theorem (Shareshian and MW)

$$1 + \sum_{n\geq 1}\sum_{j=1}^{n} \mathrm{ch}\tilde{H}_{n-2}(\bar{I}_{n,j})\, t^{j-1} z^n = \frac{(1-t)E(z)}{E(zt) - tE(z)},$$

where ch *is the Frobenius characteristic,* $E(z) = \sum_{n\geq 0} e_n z^n$ *and* e_n *is the nth elementary symmetric function.*

Proof involves

- Equivariant Tree Lemma
- Whitney homology technique of Sundaram

Connection with Toric Varieties

Theorem (Procesi, Stanley 1989)

Let X_n be the toric variety associated with the Coxeter complex of $\mathfrak{S}_n$. The action of $\mathfrak{S}_n$ on X_n induces a representation of $\mathfrak{S}_n$ on $H^{2j}(X_n)$.

$$\sum_{n\geq 0}\sum_{j=0}^{n-1} \mathrm{ch}\, H^{2j}(X_n)\, t^j z^n = \frac{(1-t)H(z)}{H(zt)-tH(z)},$$

where $H(z)=\sum_{n\geq 0} h_n z^n$ and h_n is the nth complete homogeneous symmetric function.

Corollary (Shareshian and MW)

$$H^{2j}(X_n) \cong_{\mathfrak{S}_n} \tilde{H}_{n-2}(\bar{I}_{n,j+1}) \otimes \mathrm{sgn}$$

Stembridge 1992: nice characterization of $\mathfrak{S}_n$-module stucture of $H^{2j}(X_n)$

Symmetric Function Generalization of Conjecture

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} = \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}$$

Symmetric Function Generalization of Conjecture

$$\frac{(1-t)H(z)}{H(zt)-tH(z)}$$

$$\Big\downarrow \quad \begin{aligned} x_i &:= q^{i-1} \\ z &:= z(1-q) \\ t &:= qt \end{aligned}$$

$$\sum_{n\geq 0} A_n^{\mathrm{maj,exc}}(q,t)\frac{z^n}{[n]_q!} \quad = \quad \frac{(1-tq)\exp_q(z)}{\exp_q(ztq)-tq\exp_q(z)}$$

Symmetric Function Generalization of Conjecture

$$? \quad = \quad \frac{(1-t)H(z)}{H(zt) - tH(z)}$$

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Symmetric Function Generalization of Conjecture

For $\sigma \in \mathfrak{S}_n$, let $\bar{\sigma}$ be obtained by placing bars above each excedance.

$$\bar{5}\bar{3}14\bar{6}2$$

View $\bar{\sigma}$ as a word over ordered alphabet

$$\{\bar{1} < \bar{2} < \cdots < \bar{n} < 1 < 2 < \cdots < n\}.$$

Define

$$\mathrm{EXD}(\sigma) := \mathrm{DES}(\bar{\sigma})$$

$$\mathrm{EXD}(531462) = \mathrm{DES}(\bar{5}.\bar{3}14.\bar{6}2) = \{1, 4\}$$

Claim: $\displaystyle\sum_{i \in \mathrm{EXD}(\sigma)} i = \mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)$

Symmetric Function Generalization of Conjecture

For $S \subseteq [n-1]$, quasisymmetric function

$$F_S(x_1, x_2, \dots) := \sum_{\substack{i_1 \geq \cdots \geq i_n \\ j \in S \Rightarrow i_j > i_{j+1}}} x_{i_1} \dots x_{i_n}$$

From theory of quasisymmetric functions we have

$$F_S(1, q, q^2, \dots) = \frac{q^{\sum S}}{(1-q)(1-q^2)\dots(1-q^n)}$$

Hence

$$F_{\mathrm{EXD}(\sigma)}(1, q, q^2, \dots) = \frac{q^{\mathrm{maj}(\sigma) - \mathrm{exc}(\sigma)}}{(1-q)(1-q^2)\dots(1-q^n)}$$

Symmetric Function Generalization of Conjecture

By setting $x_i := q^{i-1}$ and $z := z(1-q)$ in

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} F_{\mathrm{EXD}(\sigma)} t^{\mathrm{exc}(\sigma)} z^n$$

we get

$$\sum_{n \geq 0} \sum_{\sigma \in \mathfrak{S}_n} q^{maj(\sigma) - \mathrm{exc}(\sigma)} t^{\mathrm{exc}(\sigma)} \frac{z^n}{[n]_q!}$$

Now set $t := qt$ to get

$$\sum_{n \geq 0} A_n^{\mathrm{maj},\mathrm{exc}}(q, t) \frac{z^n}{[n]_q!}$$

Symmetric Function Generalization of Conjecture

Conjecture (Shareshian and MW)

Let

$$Q_{n,j} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j}} F_{\mathrm{EXD}(\sigma)}$$

Then

$$\sum_{n \geq 0} \sum_{j=0}^{n-1} Q_{n,j}\, t^j z^n = \frac{(1-t)H(z)}{H(zt) - tH(z)}$$

Equivalently,

$$Q_{n,j} = \mathrm{ch}(\tilde{H}_{n-2}(\bar{I}_{n,j+1}) \otimes \mathrm{sgn}) = \mathrm{ch}(H^{2j}(X_n))$$

Computer verification up to $n = 9$

Quasisymmetric function conjecture

Conjecture (Shareshian and MW)

- *The quasisymmetric function*

$$Q_{n,j,\lambda} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \mathrm{exc}(\sigma) = j \\ type(\sigma) = \lambda}} F_{\mathrm{EXD}(\sigma)}$$

  is symmetric and Schur positive for all $\lambda \vdash n$ *and* $j = 0, 1, \ldots, n-1$.
- *Similarities with Gessel-Reutenauer quasisymmetric functions.*

Closed form formula

Conjecture (Shareshian and MW)

$$\sum_{\sigma \in \mathfrak{S}_n} q^{\mathrm{maj}(\sigma)} t^{\mathrm{exc}(\sigma)} r^{\mathrm{fix}(\sigma)} =$$

$$\sum_{m=0}^{\lfloor \frac{n}{2} \rfloor} (tq)^m \sum_{\substack{k_0 \geq 0 \\ k_1, \dots, k_m \geq 2 \\ \sum k_i = n}} \begin{bmatrix} n \\ k_0, \dots, k_m \end{bmatrix}_q r^{k_0} \prod_{i=1}^{m} [k_i - 1]_{tq},$$

where

$$\begin{bmatrix} n \\ k_0, \dots, k_m \end{bmatrix}_q = \frac{[n]_q!}{[k_0]_q! [k_1]_q! \cdots [k_m]_q!}.$$

Equivalent to the earlier conjectures.