From Alternating Sign Matrices

To Orbital Varieties

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Plan of the talk

- Definition of the Temperley–Lieb model of loops
- Relation to Alternating Sign Matrices
- Quantum Knizhnik–Zamolodchikov Equation
- Relation to $sl(N)$ Orbital Varieties
- Generalization to other orbital varieties / other boundary conditions

(see also: DF+ZJ math-ph/0410061, math-ph/0508059)
The two types of plaquettes are chosen randomly with probabilities $p, 1 - p$.

Question: how do the external vertices connect to each other?
Temperley–Lieb model of loops cont’d

It is convenient to encode the probabilities as a vector $\Psi$ indexed by link patterns, and to normalize it so that the smallest entry is 1.

Conjectures [de Gier, Nienhuis ’01]

(1) The components can be chosen to be integers, the smallest being 1.

(2) The sum of components is the number of alternating sign matrices of size $n$:

$$A_n = \frac{1!4!7! \cdots (3n-2)!}{n!(n+1)!(n+2)! \cdots (2n-1)!}$$

now a Theorem [PDF, PZJ oct ’04]

(3) The largest component is $A_{n-1}$.

[Razumov, Stroganov ’01] formulated a much more general conjecture that interprets combinatorially each individual component. [still unproven]
ASM enumeration: Izergin’s determinant formula

Associate to each horizontal line of the grid a parameter $x_i$ and to each vertical line a parameter $y_i$.

The weight $w(x, y)$ at a vertex depends on the parameters $x, y$ of the lines and is equal to:

$$+ \cdots 0 + 0 \cdots + +
\vdots \quad \text{or} \quad \vdots \quad \text{or} \quad \vdots
+ 0 \cdots + + + \cdots 0$$

$$a(x, y) = q^{1/2}x - q^{-1/2}y$$
$$b(x, y) = q^{1/2}y - q^{-1/2}x$$
$$c(x, y) = (q - q^{-1})(xy)^{1/2}$$

$$A_n(x_1, \ldots, x_n; y_1, \ldots, y_n) \equiv \sum_{6v \text{ DWBC configs}} \prod_{i,j=1}^n w(x_i, y_j)$$

Korepin wrote recursion relations that fix entirely $A_n$ (in terms of $A_{n-1}$). Using them Izergin showed

$$A_n(x_1, \ldots, x_n; y_1, \ldots, y_n) = \frac{\prod_{i,j=1}^n a(x_i, y_j)b(x_i, y_j)}{\prod_{i<j}(x_i - x_j)(y_i - y_j)} \det_{i,j=1}^{n} \left( \begin{array}{c} c(x_i, y_j) \\ a(x_i, y_j)b(x_i, y_j) \end{array} \right)$$

NB: $A_n(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is a symmetric function of the $x_i$, and of the $y_i$.

Kuperberg ('98): set $q = e^{2i\pi/3}$ and $x_i = y_i = 1 \Rightarrow$ recover Zeilberger’s formula for $A_n$. 
### 6 Vertex Model with DWBC at $q = e^{2i\pi/3}$: Okada formula

In the next 2 slides, set $q = e^{2i\pi/3}$.

Okada ('02): $A_n(x_1, \ldots, x_n; y_1, \ldots, y_n)$ is a symmetric function of the full set of parameters $x_i, y_i$.

$$z_i \equiv x_i \quad z_{i+n} \equiv y_i \quad i = 1 \ldots n$$

It is a Schur function: (up to a prefactor)

$$A_n(z_1, \ldots, z_{2n}) = s_Y(z_1, \ldots, z_{2n})$$

It is entirely characterized by the following properties: (Stroganov, '04)

(i) It is a symmetric [homogeneous] polynomial of the $z_i$, of degree $n - 1$ in each variable.

(ii) It satisfies the recursion relation

$$A_n(z_1, \ldots, z_{2n}) \bigg|_{z_j=qz_i} = \prod_{\substack{k=1 \\ k \neq i,j}}^{2n} (q^2 z_i - z_k) A_{n-1}(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_{j-1}, z_{j+1}, \ldots, z_{2n}) .$$
Inhomogeneous T–L model of loops [PDF, PZJ '04]

Introduce local probabilities dependent on the column $i$ via a parameter $z_i$ respecting integrability of the model (i.e. satisfying Yang–Baxter equation). Form the new vector $\Psi(z_1, \ldots, z_{2n})$ of probabilities, normalized so that its components are coprime polynomials.

* Polynomiaity. The components of $\Psi(z_1, \ldots, z_{2n})$ are homogenous polynomials of total degree $n(n-1)$ and of partial degree at most $n-1$ in each $z_i$, with coefficients in $\mathbb{Z}[q]$, $q = e^{2i\pi/3}$.

* Factorization and symmetry. (...)

The sum of components is a symmetric polynomial of all $z_i$.

* Recursion relations. The set of components $\Psi_\pi(z_1, \ldots, z_{2n})$ satisfies linear recursion relations when $z_j = q^2 z_i$; in particular, the sum satisfies the Korepin/Stroganov recursion relation, and therefore

$$\sum_{\pi} \Psi_\pi(z_1, \ldots, z_{2n}) = A_n(z_1, \ldots, z_{2n})$$
qKZ and Affine Hecke representation [Pasquier]

Consider the following set of equations: (level 1 $q$KZ)

$$
\Psi(z_1, \ldots, z_{i+1}, z_i, \ldots, z_{2n}) = \tilde{R}_i(z_{i+1}/z_i)\Psi(z_1, \ldots, z_{2n}), \quad i = 1, 2, \ldots, 2n - 1
$$

$$
\Psi(z_2, z_3, \ldots, z_{2n}, q^6 z_1) = c \sigma^{-1}\Psi(z_1, \ldots, z_{2n})
$$

where $\Psi$ is a vector-valued polynomial of degree $n(n-1)$, $\sigma$ is rotation of link patterns and

$$
\tilde{R}_i(z) = \frac{(q^{-1} - qz) + (1 - z)e_i}{q^{-1}z - q}
$$

$e_i$ = \begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\bigcirc
\end{array}
\end{array}
\end{array}$

= generator of Temperley–Lieb algebra $TL(\beta = -q - q^{-1})$ acting on link patterns. For $q = e^{\pm 2i\pi/3}$, one recovers the previous eigenvector $\Psi$.

Rewrite Eqs. (1) by separating the action on link patterns and that on polynomials:

$$
(q^{-1}z_{i+1} - qz_i)\partial_i \Psi = (e_i + q + q^{-1})\Psi
$$

(1')

where $\partial \equiv \frac{1}{z_{i+1} - z_i}(\tau_i - 1)$ and $\tau_i$ switches $z_i$ and $z_{i+1}$. The operators $(q^{-1}z_{i+1} - qz_i)\partial_i$ acting on polynomials form a representation of the Hecke algebra. Together with the cyclic shift of spectral parameters they generate a representation of affine Hecke.
Rational limit and Hotta’s construction

Consider $q = -e^{-\hbar a/2}$, $z_i = e^{-\hbar w_i}$, $\hbar \to 0$. In this limit the $e_i$ form a representation of $\mathbb{T}L(\beta = 2)$ which is a quotient of the symmetric group. The $e_i$ generate the Joseph representation on orbital varieties, and Eq. (1’) is related to Hotta’s construction of this representation. Each $\Psi_\pi$ is the multidegree of an orbital variety. NB: $\Psi_\pi(z_i = 0, a = 1) = \text{degree}$, $\Psi(a = 0) = \text{Joseph polynomial}$.

Here the orbital varieties are the irreducible components of the scheme of upper triangular $N \times N$ matrices that square to zero, $N = 2n$. Torus action = conjugation by diagonal matrices and scaling.

**Example:** $N = 4$. Two components:

\[
\begin{align*}
O & = \left\{ M = \begin{pmatrix} 0 & m_{13} & m_{14} \\ m_{23} & m_{24} & 0 \end{pmatrix} \right\} \\
\Psi & = (a + z_1 - z_2)(a + z_3 - z_4)
\end{align*}
\]

\[
\begin{align*}
O & = \left\{ M = \begin{pmatrix} m_{12} & m_{13} & m_{14} \\ 0 & m_{24} & m_{34} \end{pmatrix} : m_{12}m_{24} + m_{13}m_{34} = 0 \right\} \\
\Psi & = (a + z_2 - z_3)(2a + z_1 - z_4)
\end{align*}
\]
Other orbital varieties/boundary conditions

**B-type orbital varieties:** consider \((2r + 1) \times (2r + 1)\) matrices such that \(M^T J + JM = 0\) where \(J\) is the antidiagonal matrix with 1’s on the antidiagonal, and \(M^2 = 0\).

The multidegrees of irreducible components of this scheme satisfy B-type \(q\)KZ equation at \(q = -1\).

\(q\)-deform and set \(q = e^{2i\pi/3}, z_i = 1\).

Results for \(r\) even:

Theorem [DF ’05]: if one normalizes the solution of \(q\)KZ equation so that its smallest entry is 1, then the sum of components is \(A_V(r)\), the number of Vertically Symmetric Alternating Sign Matrices of size \(r + 1\).

Conjecture: the largest component is the number of Cyclically SymmetricTranspose Complement Plane Partitions in a hexagon of size \(r \times r \times r\).
The $O(1)$ loop model: closed boundary conditions

The components are the (unnormalized) probabilities of the following model on a strip:
Other orbital varieties/boundary conditions

**C-type orbital varieties:** consider \((2r) \times (2r)\) matrices such that \(M^T J + JM = 0\) where \(J\) is the antidiagonal matrix with 1’s (resp. –1’s) in the upper (resp. lower) triangle. and \(M^2 = 0\).

Take its multidegrees, \(q\)-deform them, and set \(q = e^{2i\pi/3}\), \(z_i = 1\). Conjectures: \((r\ \text{even})\)

◊ With the normalization that the smallest component is 1, the sum of components is the number of Cyclically Symmetric Self-Complementary Plane Partitions in a hexagon of size \(r \times r \times r\).

◊ The largest entry is the sum of components at size \(r - 1\).

**D-type orbital varieties:** consider \((2r) \times (2r)\) matrices such that \(M^T J + JM = 0\) where \(J\) is the antidiagonal matrix with 1’s on the antidiagonal, and \(M^2 = 0\).

Take its multidegrees, \(q\)-deform them, and set \(q = e^{2i\pi/3}\), \(z_i = 1\). Conjectures:

◊ With the normalization that the smallest component is 1, the sum of components is the number of Half-Turn Symmetric Alternating Sign Matrices of size \(r\).

◊ The largest entry is the sum of components of the \(C\)-type solution at size \(r - 1\).