Chapter 1, GROUP REPRESENTATIONS

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1 Groups and their actions.

1.1 In our treatment groups will appear always as transformation groups, the main point being that, given a set X the set of all bijective mappings of X into X is a group under composition. We will denote this group $S(X)$ and call it *the symmetric group* of X.

In practice the full symmetric group is used only for X finite, in this case it is usually more convenient to fix as X the interval $[1, n]$ formed by the first n integers (for a given value of n); in this case the corresponding symmetric group has $n!$ elements and it is indicated by S_n, its elements are called *permutations.*

In general the groups wich appear are subgroups of the full symmetric group, defined by special properties of the set X arising from some extra structure (like a topology or the structure of a linear space etc.), the groups of our interest will usually be symmetry groups of the structure under consideration.

To illustrate this concept we start:

Definition. *A partition of a set X is a family of non empty disjoint subsets A_i with union X.*

A partition of a number n is a (non increasing) sequence of positive numbers:

$$m_1 \geq m_2 \geq \ldots \geq m_k > 0 \textit{ with } \sum_{j=1}^{k} m_j = n.$$

Remark. To a partition of the set $[1, 2, \ldots, n]$ we can associate the partition of n given by the cardinality of the sets.

We usually will denote a partition by a greek letter $\lambda := m_1 \geq m_2 \geq \ldots \geq m_k$ and write $\lambda \vdash n$ to mean that it is a partition on n.

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We represent graphically such a partition by a **Young diagram**. The numbers m_i appear then as the lengths of the rows (cf. Chapter 3), e.g. λ =(8,5,5,2):

$$\lambda = \begin{array}{llllllll} \bullet & \bullet & & & & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & & & \\ \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \end{array} \qquad \tilde{\lambda} = \begin{array}{llll} \bullet & & & \\ \bullet & & & \\ \bullet & & & \\ \bullet & \bullet & & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \\ \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & \bullet \end{array}$$

Sometimes it is useful to relax the condition and call partition of n any sequence $m_1 \geq m_2 \geq \ldots \geq m_k \geq 0$ with $\sum_{j=1}^{k} m_j = n$. We then call **heigth** of λ and denote it by $ht(\lambda)$ the number of non 0 elements in the sequence m_i i.e. the number or rows of the diagram.

We can also consider the columns of the diagram which will be thought as rows of the **dual partition**, the dual partition of λ will be denoted by $\tilde{\lambda}$, for instance for $\lambda = (8, 5, 5, 2)$ we have $\tilde{\lambda} = (4, 4, 3, 3, 3, 1, 1, 1)$.

If $X = \cup A_i$ is a partition, the set:

$$G := \{\sigma \in S_n | \sigma(A_i) = A_i, \ \forall i\},$$

is a subgroup of $S(X)$, isomorphic to the product $\prod S(A_i)$ of the symmetric groups on the sets A_i.

1.2 It is useful at this stage to procede in a formal way. We set:

Definition. *An action of a group G on a set X is a mapping $\pi : G \times X \to X$, denoted by $gx := \pi(g, x)$ satisfying the following conditions:*

$$1x = x, \quad h(kx) = (hk)x \tag{1.2.1}$$

for all $h, k \in G$ and $x \in X$.

The reader will note that the definition just given can be reformulated as follows:

i) The map $\varrho(h) := x \to hx$ from X to X is bijective for all $h \in G$.

ii) The map $\varrho : G \to S(X)$ is a *group homomorphism.*

In our theory we will usually fix our attention on a given group G and consider different actions of the group, it is then convenient to refer to a given action on a set X as to a *G-set.*

Examples. a) The action of G as left multiplications on itself.

b) For a given subgroup H of G, the action of G on the set $G/H := \{gH | g \in G\}$ of cosets is given by:

$$a(bH) := abH. \tag{1.2.2}$$

c) The action of $G \times G$ on G given by $(a, b)c = acb^{-1}$.

d) The action of G by conjugation on itself.

e) The action of a subgroup of G induced by restricting an action of G.

It is useful right from the start to use a *categorical* language:

Definition. *Given two $G-$sets X, Y, a $G-$equivariant mapping, or more simply a morphism, is a map $f : X \to Y$ such that for all $g \in G$ and $x \in X$ we have:*

$$f(gx) = gf(x).$$

In this case we also say thet f *intertwines* the two actions. Of course if f is bijective we speak of an *isomorphism* of the 2 actions.

The class of $G-$sets and equivariant maps is clearly a **Category**.

This is particularly important when G is the homotopy group of a space X and the $G-$sets correspond to covering spaces of X.

Example. The equivariant maps of the action of G on itself by left multiplication are the right multiplications. They form a group isomorphic to the *opposite* of G (but also to G).

More generally:

Proposition. *The invertible equivariant maps of the action of G on G/H by left multiplication are induced by the right multiplications with elements of the normalizer $N_G(H)$ of H (cf. 1.4). They form a group Γ isomorphic to $N_G(H)/H$.*

Proof. Let $\sigma : G/H \to G/H$ be such a map, hence for all $a, b \in G$ we have $\sigma(a.bH) = a\sigma(bH)$. In particular if $\sigma(H) = uH$ we must have that:

$$\sigma(H) = uH = \sigma(hH) = huH, \forall h \in H, \implies Hu \subset uH.$$

If we assume σ to be invertible we see that also $Hu^{-1} \subset u^{-1}H$ hence $uH = Hu$ and $u \in N_G(H)$. Conversely if $u \in N_G(H)$ the map $\sigma(u) : aH \to auH = aHu$ is well defined and in Γ. The map $u \to \sigma(u^{-1})$ is clearly a surjective homomorphism from $N_G(H)$ to Γ with kernel H.

Exercise Describe the set of equivariant maps $G/H \to G/K$ for 2 subgroups.

2 Orbits, invariants and equivariant maps.

2.1 The first important notion in this setting is given by the following:

Proposition. *The binary relation R in X given by: xRy if and only if there exists $g \in G$ with $gx = y$, is an equivalence relation.*

Definition. *The equivalence classes under the previous equivalence are called G-orbits (or simply orbits), the orbit of a given element x is formed by the elements gx with $g \in G$ and is denoted Gx. The mapping $G \to Gx$ given by $g \to gx$ is called the orbit map.*

The orbit map is equivariant (with respect to the left action of G). The set X is partitioned in its orbits, and the set of all orbits (quotient set) is denoted by X/G.

In particular we say that the action of G is *transitive* or that X is a *homogeneous space* if there is a unique orbit.

More generally we say that a subset Y of X is G *stable* if it is a union of orbits. In this case G induces naturally an action on Y. Of course the complement $\mathcal{C}(Y)$ of Y in X is also G stable and X is decomposed as $Y \cup \mathcal{C}(Y)$ in 2 stable subsets.

The finest decomposition into stable subsets is the decomposition into orbits.

Basic examples

i Let $\sigma \in S_n$ be a permutation and A the cyclic group which it generates, then the orbits of A on the set $[1, n]$ are the *cycles* of the permutation.

ii Let G be a group and H, K be subgroups, we have the action of $H \times K$ on G induced by the left and right action. The orbits are the *double cosets*. In particular if either H or K is 1 we have left or right cosets.

iii Consider G/H, the set of left cosets gH, with the action given by 1.2.2. Given a subgroup K on G/H the K orbits in G/H are in bijictive correspondence with the double coset KgH.

iv The action of G on itself by conjugation $(g, h) \to ghg^{-1}$. Its orbits are the *conjugacy classes*.

v An action of the additive group $\mathbb{R}_+$ of real numbers on a set X is called a *1-parameter group of transformations* or in a more physical language a *reversible dynamical system*.

In this case the parameter t is thought as *time* and an orbit is seen as the time evolution of a physical state. The hypotheses of the group action mean that the evolution is reversible (i.e. all the group transformations are invertible) and the *forces* do not vary with time so that the evolution of a state depends only on the time lapse (group homomorphism property).

The previous examples lead to single out the following general fact:

Remark. Let G be a group and K a normal subgroup in G, if we have an action of G on a set X we see that G acts also on the set of K orbits X/K, since $gKx = Kgx$, moreover we have $(X/K)/G = X/G$.

2.2 The study of group actions should start with the elementary analysis of a single orbit. The next main concept is that of *stabilizer*:

Definition. *Given a point $x \in X$ we set $G_x := \{g \in G | gx = x\}$. G_x is called the* **stabilizer** *(or* **little group***) of x.*

Proposition. *G_x is a subgroup and the action of G on the orbit Gx is isomorphic to the action on the coset space G/G_x.*

Proof. The fact that G_x is a subgroup is clear. Given two elements $h, k \in G$ we have that $hx = kx$ if and only if $k^{-1}hx = x$ or $k^{-1}h \in G_x$.

The mapping between G/G_x and Gx which assigns to a coset hG_x the element hx is thus well defined and bijective, it is also clearly $G-$equivariant and so the claim follows.

Example. Consider the action of $G \times G$ on G by left right translation. G is a single orbit and the stabilizer of 1 is the subgroup $\Delta := \{(g, g) | g \in G\}$ isomorphic to G embedded in $G \times G$ diagonally.

Example. In the case of a 1-parameter subgroup acting continuously on a topological space, the stabilizer is a closed subgroup of $\mathbb{R}$. If it is not the full group it is the set of integral multiples $ma, m \in \mathbb{Z}$ of a positive number a. The number a is to be considered as the first time in which the orbit returns to the starting point. This is the case of a **periodic orbit**.

Remark. Given two different elements in the same orbit their stabilizers are conjugate, in fact $G_{hx} = hG_x h^{-1}$. In particular when we identify an orbit to a coset space G/H this implicitely means that we have made the choice of a point for which the stabilizer is H.

Remark. The orbit cycle decomposition of a permutation can be interpreted in the previous language. To give a permutation on a set S is equivalent to give an action of the group of integers $\mathbb{Z}$ on S.

If S is finite this induces an action of a finite cyclic group isomorphic to $Z/(n)$.

To study a single orbit we only remark that a subgroup will be of the form $Z/(m)$ with m a divisor of n. The corresponding coset space is $Z/(m)$ and the generator $\overline{1}$ of $Z/(n)$ acts on $Z/(m)$ as the cycle $\overline{x} \to \overline{x} + \overline{1}$.

Consider the set of all subgroups of a group G, on this set G acts by conjugation. The orbits of this action are the *conjugacy classes of subgroups*, let us denote by $[H]$ the conjugacy class of a subgroup H.

The stabilizer of a subgroup H under this action is called its *normalizer*. It should not be confused with the *centralizer* which for a given subset A of G is the stabilizer under conjugation of all the elements of A.

Given a group G and an action on X it is useful to introduce the following notions.

For an orbit in X the conjugacy class of the stabilizers of its elements is well defined. We say that two orbits are of the same *orbit type* if the associated stabilizer class is the same. This is equivalent to say that the two orbits are isomorphic as $G-$spaces. It is often useful to partition the orbits according to the orbit types.

Exercise Determine the points in G/H with stabilizer H.

Exercise The group of symmetries of the G action permutes transitively orbits of the same type.

Suppose that G and X are finite and assume that we have n_i orbits of type $[H_i]$ then we have, from the partition into orbits, the formula:

$$\frac{|X|}{|G|} = \sum_i \frac{n_i}{|H_i|}$$

we denote in general by $|A|$ the cardinality of a finite set A.

Exercise Let G be a group with $p^m n$ elements, p a prime number not dividing n. Deduce the theorems of Sylow by considering the action of G by left multiplication on the set of all subsets of G with p^m elements (Wielandt).

Exercise Given two subgroups H, K of G describe the orbits of H acting on G/K, in particular give a criterion for G/K to be a single H orbit.

Discuss the special case $[G : H] = 2$.

2.3 From all the elements of X we may single out the ones for which the stabilizer is the full group G.

These are the *fixed points* of the action or *invariant points*, i.e. the points whose orbit consists of the point alone. These points will be usually denoted by X^G.

$$X^G := \{\textit{fixed points} \text{ or } \textit{invariant points}\}.$$

We have thus introduced in a very general sense the notion of *Invariant* but its full meaning for the moment is completely obscure, we have first to procede with the formal theory.

2.4 One of the main features of set theory consists in the fact that it allows us to perform constructions, out of given sets we construct new ones.

This is also the case of $G-$sets. Let us point out at least 2 constructions:

(1) Given 2 $G-$sets X, Y we give the structure of a $G-$set to their disjoint sum $X \sqcup Y$ by acting separately on the two sets and to their product $X \times Y$ setting:

$$g(x, y) := (gx, gy), \tag{2.4.1}$$

(i.e. once the group acts on the elements it acts also on the pairs.)

(2) Consider now the set Y^X of all maps from X to Y, we can act with G (verify it) setting:

$$(gf)(x) := gf(g^{-1}x). \tag{2.4.2}$$

Notice that in the second definition we have used again twice the action of G, the particular formula given is justified by the fact that it is really the only way to get a group action using the two actions.[1]

We want to explicit immediately a rather important consequence of our formalism:

[1] It reflects a general fact well known in category theory, that maps between two objects X, Y are a covariant functor in Y and controvariant in X.

Proposition. *A map $f : X \to Y$ between two $G-$sets is equivariant (cf. 1.2) if and only if it is a fixed point under the $G-$action on the maps.*

Proof. This statement is really a tautology, nevertheless deserves to be clearly understood. The proof is trivial following the definitions. Equivariance means that $f(gx) = gf(x)$. This, if we substitute x with $g^{-1}x$, reads $f(x) = gf(g^{-1}x)$ which in the functional language means that the function f equals the function gf, i.e. it is invariant.

Exercise

i) Show that the orbits of G on $G/H \times G/K$ are in canonical 1-1 correspondence with the double cosets HgK of G.

ii) Given a G equivariant map $\pi : X \to G/H$ show that:
 a) $\pi^{-1}(H)$ is stable under the action of H.
 b) The set of G orbits on X is in 1-1 correspondence with the H orbits on $\pi^{-1}(H)$.
 c) Study the case in which $X = G/K$ is also homogeneous.

2.5 We will often consider a special case of the previous section, the case of the trivial action of G on Y. In this case of course the action of G on the functions is simply:

$$(gf)(x) = f(g^{-1}x) \tag{2.5.1}$$

A mapping is equivariant if and only if it is constant on the orbits. In this case we will always speak of *Invariant function.* In view of the particular role of this idea in our treatment we repeat the formal definition.

Definition. *A function f on a G set X is called an invariant if $f(g^{-1}x) = f(x)$ for all $x \in X$ and $g \in G$.*

As we have just remarked a function is invariant if and only if it is constant on the orbits. Formally we may thus say that the quotient mapping $\pi := X \to X/G$ is an invariant map and any other invariant function factors as :

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\pi"'] & & Y \\ & X/G \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

We want to explicit the previous remark in a case of importance.

Let X be a finite G set. Consider a field F (a ring would suffice) and the set F^X of functions on X with values in F.

An element $x \in X$ can be identified to the characteristic function of $\{x\}$, in this way X becomes a basis of F^X as vector space.

The induced group action of G on F^X is by linear transformation and by permuting the basis elements.

F^X is called a **permutation representation** and we will see its role in the next sections. Since a function is invariant if and only if it is constant on orbits we deduce:

Proposition. *The invariants of G on F^X form the subspace of F^X having as basis the characteristic functions of the orbits.*

In other words given an orbit $\mathcal{O}$ consider $u_{\mathcal{O}} := \sum_{x\in\mathcal{O}} x$. The elements $u_{\mathcal{O}}$ form a basis of $(F^X)^G$.

We finish this section with two examples which will be useful in the theory of symmetric functions.

Consider the set $[1, n] := \{1, 2, \dots, n\}$ with its canonical action of the symmetric group.

The maps from $\{1, 2, \dots, n\}$ to the field $\mathbb{R}$ of real numbers form the standard vector space $\mathbb{R}^n$. The symmentric group then acts by permuting the coordinates and in every orbit there is a unique vector $(a_1, a_2, \dots, a_n)$ with $a_1 \geq a_2 \geq \dots \geq a_n$.

The set of these vectors can thus be identified to the orbit space. It is a convex cone with boundary the elements in which at least two coordinates are equal.

Exercise Discuss the orbit types of the previous example.

Definition. *A function $M : \{1, 2, \dots, n\} \to \mathbb{N}$ (to the natural numbers) is called a* **monomial**. *The set of monomials is a semigroup by addition of values and we indicate by x_i the monomial which is the characteristic function of $\{i\}$.*

As already seen the symmetric group acts on these functions by $(\sigma f)(k) := f(\sigma^{-1}(k))$ and the action is compatible with the addition, moreover $\sigma(x_i) = x_{\sigma(i)}$.

Remark. It is customary to write the semigroup law of monomials multiplicatively. Given a monomial M if $M(i) = h_i$ we have that $M = x_1^{h_1} x_2^{h_2} \dots x_n^{h_n}$. The number $\sum_i h_i$ is the **degree** of the monomial.

Representing a monomial as a vector $(h_1, h_2, \dots, h_n)$ we see that every monomial is equivalent, under the symmetric group, to a unique vector in which the coordinates are non increasing. The non zero coordinates of such a vector form thus a partition, with at most n parts, of the degree of the monomial.

The permutation representation with coefficients in a commutative ring F, associated to the monomials is the *polynomial ring* $F[x_1, \dots, x_n]$ in the given variables. The invariant elements are called **symmetric polynomials**.

Exercise To a monomial M we can also associate a partition of the set $\{1, 2, \dots, n\}$ by the equivalence $i \cong j$ iff $M(i) = M(j)$, show that the stabilizer of M is the group of permutations which preserve the sets of the partition (cf. 1.1) and determine a basis of invariant symmetric polynomials.

2.6 It is time to develop some other examples. First of all consider the set $[1, n]$ and a ring A (in most applications the integers or the real or complex numbers).

A function f from $[1, n]$ to A may be thought as a vector and displayed for instance as a row with the notation $(a_1, a_2, \dots, a_n)$ where $a_i := f(i)$. The set of all functions is thus denoted by A^n. The symmetric group acts on such functions according to the general formula 2.5.1.

$$\sigma(a_1, a_2, \ldots, a_n) = (a_{\sigma^{-1}1}, a_{\sigma^{-1}2}, \ldots, a_{\sigma^{-1}n}).$$

In this simple example we already see that the group action is linear. We will refer to this action as the **standard permutation action**.

Remark that if $\underline{e}_i$ denotes the canonical basis vector with coordinates 0 except 1 in the i^{th} position, we have $\sigma(\underline{e}_i) = \underline{e}_{\sigma(i)}$. This formula allows us to describe the matrix of σ in the given basis, it is the matrix $\delta_{\sigma^{-1}(j),i}$. These matrices are called *permutation matrices.*

This is a general fact, if we consider a $G-$set X and a ring A, the set of functions on X with values in A form also a ring under pointwise sum and multiplication and we have that:

Remark. The group G acts on the functions with values in A as a group of ring automorphisms.

In this particular example it is important to procede further. Once we have the action of S_n on A^n we may continue and act on the functions on A^n! In fact let us consider the *coordinate functions*: $x_i : (a_1, a_2, \ldots, a_n) \to a_i$, it is clear from the general formulas that the symmetric group permutes the cordinate functions and $\sigma(x_i) = x_{\sigma(i)}$. The reader may note the fact that the inverse has now disappeared.

If we have a ring R and an action of a group G on R as ring automorphisms it is clear that:

Proposition. *The invariant elements form a subring of R.*

Thus we can speak of *the ring of invariants R^G.*

2.7 We need another generality. Suppose that we have two group actions on the same set X i.e. assume that we have two groups G and H acting on the same set X.

We say that the two actions commute if $gh(x) = hg(x)$ for all $x \in X$, $g \in G$ and $h \in H$.

This means that every element of G gives rise to an H equivariant map (or we can reverse the roles of G and H) it also means that we really have an action of the product group $G \times H$ on X given by $(g, h)x = ghx$.

In this case we easily see that if a function f is G invariant and $h \in H$ then hf is also G invariant. Hence H acts on the set of G invariant functions.

More generally suppose that we are given a G action on X and a normal subgroup K of G, then it easily seen that the quotient group G/K acts on the set of K invariant functions and a function is G invariant if and only if it is K and G/K invariant.

Example. The right and left action of G on itself commute (Example 1.2 c).

3 Linear actions, groups of automorphisms, commuting groups.

3.1 In 2.5, given an action of a group G on a set X and a field F, we have deduced an action over the set F^X of functions from X to F, which is linear, i.e. given by linear operators.

In general the groups G and the sets X on which they act may have further structures, as in the case of a topological or differentiable or algebraic action. In these cases it will be important to restrict the set of functions to the ones compatible with the structure under consideration, we will do it systematically.

If X is finite the vector space of functions on X with values in F has, as a possible basis, the characteristic functions of the elements. It is convenient to identify an element x with its characteristic function and thus say that our vector space has X as a basis (cf. 2.5).

A function f is thus written as $\sum_{x \in X} f(x)x$, the linear action of G on F^X induces on this basis the action from which we started, we call such an action a *permutation representation.*

In the algebraic theory we may in any case consider the set of all functions which are finite sums of the characteristic functions of points, i.e. the functions which are 0 outside a finite set.

These are usually called **functions with finite support**, we will often denote these functions by the symbol $F[X]$, which is supposed to remind us that its elements are linear combinations of elements of X.

In particular for the left action of G on itself we have the **algebraic regular representation** of G on $F[G]$. We shall see that this representation is particularly important.

Let us stress a feature of this representation.

We have two actions of G on G, the left and the right action, which commute with each other, or in other words we have an action of $G \times G$ on G, given by $(h, k)g = hgk^{-1}$ (for which $G = G \times G/\Delta$ where $\Delta = G$ embedded diagonally cf. 1.2c and 2.2).

Thus we have the corresponding two actions on $F[G]$ by $(h, k)f(g) = f(h^{-1}gk)$ and we may view the right action as symmetries of the left action and conversely.

Sometimes it is convenient to denote by ${}^h f^k = (h, k)f$ to stress the left and right actions.

After these basic examples we give a general definition:

Definition. *Given a vector space V over a field F (or more generally a module) we say that an action of a group G on V is linear if every element of G induces a linear transformation on V, a linear action of a group is also called a linear representation*[2] *or also a $G-$module.*

In a different language let us consider the set of all linear invertible transformations of V, this is a group under composition (i.e. it is a subgroup of the group of all invertible transformations) and will be called the:

General linear group of V, indicated with the symbol $GL(V)$.

[2]sometimes we drop the term *linear* and just speak of *a representation*

In case we take $V = F^n$ (or equivalently in case V is finite dimensional and we identify V with F^n by choosing a basis) we can identify $GL(V)$ with the group of $n \times n$ invertible matrices, denoted $GL(n, F)$.

According to our general principles a linear action is thus a homomorphism ϱ of G in $GL(V)$ (or in $GL(n, F)$).

When we are dealing with linear representations we usually consider also equivariant linear maps between them, thus obtaining a category and a notion of isomorphism.

Exercise Two linear representations $\rho_1, \rho_2 : G \to GL(n, F)$ are isomorphic if and only if there is an invertible matrix $X \in GL(n, F)$ such that $X\rho_1(g)X^{-1} = \rho_2(g)$ for all $g \in G$.

Before we procede any further we should remark an important feature of the theory.

Given 2 linear representations U, V we can form their direct sum $U \oplus V$ which is a representation by setting $g(u, v) = (gu, gv)$. If $X = A \cup B$ is a G set, disjoint union of 2 G stable subsets, we clearly have $F^{A\cup B} = F^A \oplus F^B$ thus the decomposition in direct sum is a generalization of the decomposition of a space in G stable sets.

If X is an orbit it cannot be further decomposed as set while F^X might be decomposable. The simplest example is $G = \{1, \tau = (12)\}$ the group with 2 elements of permutations of $[1, 2]$, the space F^X decomposes ($\mathrm{char} F \neq 2$), setting:

$$u_1 := \frac{e_1 + e_2}{2}, \ u_2 := \frac{e_1 - e_2}{2}$$

we have $\tau e_1 = e_1, \ \tau(e_2) = -e_2$.

We have implicitely used the following ideas:

Definition. *i) Given a linear representation V a subspace U of V is a* **subrepresentation** *if it is stable under G.*

ii) V is a **decomposable representation** *if we can find a decomposition $V = U_1 \oplus U_2$ with the U_i proper subrepresentations, otherwise it is called* **indecomposable**.

iii) V is an **irreducible** *representation if the only subrepresentations of V are V and 0.*

We will study in detail some of the deep connections between these notions.

We will stress in a moment the analogy with the abstract theory of modules over a ring A. First 2 basic examples:

Example. Let A, B be the group of all, resp. of upper triangular (i.e. 0 below the diagonal) invertible $n \times n$ matrices over a field F.

Exercise The vector space F^n is irreducible as an A module, indecomposable but not irreducible as a B module.

Definition. *Given 2 linear representations U, V of a group G, the space of G equivariant linear maps is denoted $Hom_G(U, V)$ and called: Space of intertwining operators.*

In these notes unless specified otherwise our vector spaces will always be assumed to be finite dimensional.

It is quite useful to rephrase the theory of linear representations in a different way:

Consider the space $F[G]$:

Theorem. *i) The group multiplication extends to a bilinear product on $F[G]$ for which $F[G]$ is an associative algebra with 1, called* **the group algebra**.

ii) Linear representations of G are the same as $F[G]$ modules.

Proof. The first part is immediate. As for the second given a linear representation of G we have the module action $(\sum_{g\in G} a_g g)v := \sum_{g\in G} a_g(gv)$. The converse is clear.

It is useful to view the product of elements of $F[G]$ as *convolution of functions*:

$$(ab)(g) = \sum_{h,k\in G\,|\,hk=g} a(h)b(k) = \sum_{h\in G} a(h)b(h^{-1}g) = \sum_{h\in G} a(gh)b(h^{-1})$$

this definition extends to L^1-functions on locally compact groups endowed with Haar measure.

Remark. 1) Consider the left and right action on the functions $F[G]$.

Let $h, k, g \in G$ and identify g with the characteristic function of the set $\{g\}$ then ${}^h g^k = hgk^{-1}$ (as functions).

The space $F[G]$ as $G \times G$ module is the permutation representation associated to $G = G \times G/\Delta$ with its $G \times G$ action (3.1).

Thus a space of functions on G is stable under left (res. right) action if and only if it is a left (resp. right) ideal of the group algebra $F[G]$.

2) Notice that the direct sum of representations is the same as the direct sum as modules, also a G linear map between two representations is the same as a module homomorphism.

Example. Let us consider a finite group G, a subgroup K and the linear space $F[G/K]$, which as we have seen is a permutation representation.

We can identify the functions on G/K as the functions

$$F[G]^K = \{a \in F[G] | ah = a, \ \forall h \in K\}. \tag{3.1.1}$$

on G which are invariant under the right action of K. In this way the element $gK \in G/K$ is identified to the characteristic function of the coset gK and $F[G/K]$ is identified to the left ideal of the group algebra $F[G]$ having as basis the characteristic functions χ_{gK} of the cosets.

If we denote by $u := \chi_K$ the characteristic function of the subgroup K we see that $\chi_{gK} = gu$ and u generates this module over $F[G]$.

Given 2 subgroups H, K and the linear spaces $F[G/H]$, $F[G/K] \subset F[G]$ we want to determine their intertwiners.

For an intertwiner f, and $u := \chi_H$ as before, let $f(u) = a \in F[G/K]$. We have $hu = u, \ \forall h \in H$ and so, since f is an intertwiner $a = f(u) = f(hu) = ha$. Thus we must have that a is also left invariant under H. Conversely given such an a the map $b \to \frac{ba}{|H|}$ is an intertwiner mapping u to a. Since u generates $F[G/H]$ as a module we see that:

Proposition. *The space $Hom_G(F[G/H], F[G/K])$ of intertwiners can be identified with the (left) H invariants of $F[G/K]$, or to the H–K invariants ${}^H F[G]^K$ of $F[G]$. It has as basis the characteristic funtions of the double cosets HgK.*

In particular for $H = K$ we have that the functions which are biinvariants under H form under convolution the endomorphism algebra of $F[G/H]$.

These functions have as basis the characteristic funtions of the double cosets HgH, one usually indicates by $T_g = T_{HgH}$ the corresponding operator. In this way we have the *Hecke algebra* and *Hecke operators*, the multiplication rule between such operators depends on the multiplication on cosets $HgHHkH = \cup Hh_iH$ and each double coset appearing in this product appears with a positive integer multiplicity so that $T_gT_h = \sum n_iT_{h_i}$.[3]

Similar results when we have 3 subgroups H, K, L and compose:

$$Hom_G(F[G/H], F[G/K]) \times Hom_G(F[G/K], F[G/L]) \xrightarrow{\circ} Hom_G(F[G/H], F[G/L])$$

The notion of permutation representation is a special case of that of **induced representation**, if M is a representation of a subgroup H of a group G we consider the space of functions $f : G \to M$ with the constrain:

$$Ind_H^G M := \{f : G \to M | f(gh^{-1}) = hf(g), \ \forall h \in H, \ g \in G\}.$$

On this space of functions define a G action by $(gf)(x) := f(g^{-1}x)$. It is easy to see that this is a well defined action. Moreover we can identify $m \in M$ with the function f_m such that $f_m(x) = 0$ if $x \notin H$ and $f_m(h) = h^{-1}m$ if $h \in H$.

Exercise   Verify that, choosing a set of representatives of the cosets G/H we have as vector space the decomposition

$$Ind_H^G M := \oplus_{g \in G/H} \ gM.$$

3.2 Let V be a $G-$module. Given a linear function $f \in V^*$ on V, by definition the function gf is given by $(gf)(v) = f(g^{-1}v)$ and hence it is again a linear function.

Thus G acts dually on the space V^* of linear functions and it is clear that this is a linear action which is called the *contragredient* action.

In matrix notations, using dual bases, the contragredient action of an operator T is given by the inverse transpose of the matrix of T.

We will use the notation $< \varphi | v >$ for the value of a linear form on a vector and thus have the identity:

$$< g\varphi | v > = < \varphi | g^{-1} v > . \tag{3.2.1}$$

[3] It is important in fact to use these concepts in a much more general way as done by Hecke in the theory of modular forms. Hecke studies the action of $Sl(2, Z)$ on $M_2(\mathcal{Q})$ the 2×2 rational matrices. In this case one has also double cosets, a product structure on $M_2(\mathcal{Q})$ and the fact that a double coset is a finite union of right or left cosets. These properties suffice to develop the Hecke algebra. In this case this algebra acts on a different space of functions, the modular forms (cf. Ogg).

Alternatively it may be convenient to define on V^* a *right action* by the more symmetric formula:

$$< \varphi g|v >=< \varphi|gv > . \tag{3.2.2}$$

Exercise. Prove that the dual of a permutation representation is isomorphic to the same permutation representation. In particular one can apply this to the dual of the group algebra.

In the set of all functions on a finite dimensional vector space V a special role play the polynomial functions. By definition a polynomial fuction is an element of the subalgebra (of the algebra of all functions with values in F) generated by the linear functions.

If we choose a basis and consider the coordinate functions $x_1, x_2, \ldots, x_n$ with respect to the chosen basis, a polynomial function is a usual polynomial in the x_i. If F is infinite, the expression as a polynomial is unique and we can consider the x_i as given variables.

The ring of polynomial functions on V will be denoted by $P[V]$ the ring of formal polymomials by $F[x_1, x_2, \ldots, x_n]$.

Choosing a basis we have always a surjective homomorphism $F[x_1, x_2, \ldots, x_n] \to P[V]$ which is an isomorphism if F is infinite.

Exercise If F is a finite field with q elements prove that $P[V]$ has dimension q^n over F, and that the kernel of the map $F[x_1, x_2, \ldots, x_n] \to P[V]$ is the ideal generated by the elements $x_i^q - x_i$.

Since the linear funtions are preserved under a given group action we have:

Proposition. *Given a linear action of a group G on a vector space V, G acts on the polynomial functions $P[V]$ by the rule $(gf)(v) = f(g^{-1}v)$ as a group of ring automorphisms.*

Of course the full linear group acts on the polynomial functions. In the language of coordinates we may view the action as linear changes of coordinates.

Exercise Show that we always have a linear action of $GL(n, F)$ on the formal polynomial ring $F[x_1, x_2, \ldots, x_n]$.

3.3 We assume the base field to be infinite for simplicity although the reader can see easily what happens for finite fields. One trivial but important remark is that the group action on $P[V]$ preserves the degree.

Recall that a function f is homogeneous of degree k if $f(\alpha v) = \alpha^k f(v)$ for all $\alpha' s$ and $v' s$.

The set $P[V]_q$ of homogeneous polynomials of degree q is a subspace, called in classical language the space of *quantics*. If $dim(V) = n$ one speaks of $n-ary$ quantics.

In general a direct sum of vector spaces $U = \oplus_{k=0}^{\infty} U_k$ is called a *graded vector space.* A subspace W of U is called *homogeneous*, if, setting $W_i := W \cap U_i$, we have $W = \oplus_{k=0}^{\infty} W_k$.

The space of polynomials is thus a graded vector space $P[V] = \oplus_{k=0}^{\infty} P[V]_k$. One has immediately $(gf)(\alpha v) = f(\alpha g^{-1}v) = \alpha^k (gf)(v)$ which has an important consequence:

Theorem. *If a polynomial f is an invariant (under some linear group action) then also its homogeneous components are invariant.*

Proof. Let $f = \sum f_i$ be the decomposition of f in homogeneous components, $gf = \sum gf_i$ is the decomposition in homogeneous components of gf. If f is invariant $f = gf$ and then $f_i = gf_i$ for each i since the decomposition into homogeneous components is unique.

In order to summarize the analysis done up to now let us also recall that an algebra A is called a *graded algebra* if it is a graded vector space, $A = \oplus_{k=0}^{\infty} A_k$ and, for all h, k we have $A_h A_k \subset A_{h+k}$.

Proposition. *The spaces $P[V]_k$ are subrepresentations. The set $P[V]^G$ of invariant polynomials is a graded subalgebra.*

3.4 To some extent the previous theorem may be viewed as a special case of the more general setting of commuting actions.

Let thus be given two representations $\varrho_i : G \to GL(V_i)$, $i = (1, 2)$, consider the linear transformations between V_1 and V_2 which are G equivariant, it is clear that they form a linear subspace of the space of all linear maps between V_1 and V_2.

The space of all linear maps will be denoted by $hom(V_1, V_2)$ while the space of equivariant maps will be denoted $hom_G(V_1, V_2)$. In particular when the two spaces coincide we write $End(V)$ or $End_G(V)$ instead of $hom(V, V)$ or $hom_G(V, V)$.

These spaces are in fact now algebras, under composition of operators. Choosing bases we have that $End_G(V)$ is the set of all matrices which commute with all the matrices coming from the group G.

Consider now the set of invertible elements of $End_G(V)$, i.e. the group H of all linear operators which commute with G.

By the remarks of 3.3, H preserves the degrees of the polynomials and maps the algebra of G invariant functions in itself thus:

Remark. H induces a group of automorphisms of the graded algebra $P[V]^G$.

We view this remark as a generalization of Proposition 3.3 since the group of scalar multiplications commutes (by definition of linear transformation) with all linear operators. Moreover it is easy to prove:

Exercise Given a graded vector space $U = \oplus_{k=0}^{\infty} U_k$ define an action ϱ of the multiplicative group F^* of F setting $\varrho(\alpha)(v) := \alpha^k v$ if $v \in U_k$. Prove that a subspace is stable under this action if and only if it is a graded subspace (F is assumed to be infinite).

4 Semisimple algebras.

4.1 One of the main themes of our theory will be related to completely reducible representations, it is thus important to establish these notions in full detail and generality.

Definition.

i) A set S of operators on a vector space U is *irreducible* or *simple* if the only subspaces of U which are stable under S are 0 and U.

ii) A set S of operators on a vector space U is *completely reducible* or *semisimple* if U decomposes as a direct sum of stable irreducible subspaces.

iii) A set S of operators on a vector space U is *indecomposable* if the space U cannot be decomposed in the direct sum of two non trivial stable subspaces.

Of course a space is irreducible if and only if it is completely reducible and indecomposable.

A typical example of completely reducible sets of operators is the following. Let $U = \mathbb{C}^n$ and S a set of matrices. For a matrix A denote by $A^* = \overline{A}^t$ its adjoint, it is characterized by the fact that$(Au, v) = (u, A^*v)$ for the standard Hermitian product $\sum z_i \overline{w}_i$.

Lemma. *If a subspace M of $\mathbb{C}^n$ is stable under A then $M^\perp$ (the orthogonal under the Hermitian product) is stable under A^*.*

Proof. If $m \in M, u \in M^\perp$ we have $(m, A^*u) = (Am, u) = 0$ since M is A stable. Thus $A^*u \in M^\perp$.

Proposition. *If $S = S^*$ then $\mathbb{C}^n$ is the orthogonal sum of irreducible submodules, in particular it is semisimple.*

Proof. Take an S stable subspace M of $\mathbb{C}^n$ of minimal dimension, it is then necessarily irreducible.

Consider its orthogonal complement $M^\perp$, by adjonction and the previous lemma we get that $M^\perp$ is S stable and $\mathbb{C}^n = M \oplus M^\perp$.

We then procede in the same way on $M^\perp$.

A special case is when S is a group of unitary operators.

More generally we say that S is *unitarizable* if there is an Hermitian product for which the operators of S are unitary. If we consider a matrix mapping the standard basis of $\mathbb{C}^n$ to a basis orthonormal for some given Hermitian product we see that

Lemma. *A set of matrices is unitarizable if and only if it is conjugate to a set of unitary matrices.*

These ideas have an important consequence.

Theorem. *A finite group G of linear operators on a finite dimensional complex space U is unitarizable and hence the module is semisimple.*

Proof. We fix an arbirary positive Hermitian product (u,v) on U. Define a new Hermitian product as:

$$(4.1.1) \qquad < u, v > := \frac{1}{|G|}\sum_{g\in G}(gu, gv).$$

Then $< hu, hv >= \sum_{g\in G}(ghu, ghv) = \sum_{g\in G}(gu, gv) =< u, v >$ and G is unitary for this new product. If G was already unitary the new product coincides with the initial one.[4]

4.2 It is usually more convenient to use the language of modules since the irreducibility or complete reducibility of a space U under a set S of operators is clearly equivalent to the same property under the subalgebra of operators generated by S.

Let us recall that, in 3.1, given a group G, one can form its *group algebra* $F[G]$. Every linear representation of G extends by linearity to an $F[G]$ module and conversely. A map between $F[G]$ modules is a (module) homomorphism if and only if it is G equivariant. Thus from the point of view of representation theory it is equivalent to study the category of G representations or that of $F[G]$ modules.

We consider thus a ring R and its modules, using the same definitions for reducible, irreducible modules. We define $R^\vee$ to be the set of (isomorphism classes) of irreducible modules of R, the *spectrum* of R.

Given an irreducible module N we will say that it is of *type* α if α indicates its isomorphism class.

Given a set S of operators on U we set $S' := \{A \in End(U) | As = sA,\ \forall s \in S\}$, S' is called the *centralizer* of S, equivalently it should be thought as the set of all S linear endomorphisms. One immediately verifies:

Proposition.

i) S' *is an algebra.*

ii) $S \subset S''$

iii) $S' = S'''$

The centralizer of the operators induced by R in a module M is also usually indicated by $End_R(M, M)$ or $End_R(M)$ and called the *endomorphism ring.*

Any ring R can be considered as a module on itself by left multiplication (and as a module on the opposite of R by right multiplication), this module is usually called *the regular representation*; of course in this case a submodule is the same as a left ideal, an irreducible submodule is also referred to as a *minimal left ideal.*

A trivial but useful fact on the regular representation is:

[4]The previous theorem has a far reaching generalization, by replacing the average given by the sum with an integral.

Proposition. *The ring of endomorphisms of the regular representation is the opposite of R acting by right multiplications.*

Proof. Let $f \in End_R(R)$ we have $f(a) = f(a1) = af(1)$ by linearity, thus f is the right multiplication by $f(1)$.

Given two homomorphisms f, g we have $fg(1) = f(g(1)) = g(1)f(1)$ and so the mapping $f \to f(1)$ is an isomorphism between $End_R(R)$ and R^0.

One can generalize the previous considerations as follows: Let R be a ring.

Definiton. *A cyclic module is a module generated by a single element.*

A cyclic module should be thought as the linear analogue of a single orbit.

The structure of cyclic modules is quite simple, if M is generated by an element m we have the map $\varphi : R \to M$ given by $\varphi(r) = rm$ (analogue of the orbit map).

By hypothesis φ is surjective, its kernel is a left ideal J and so M is identified to R/J.

Thus a module is cyclic if and only if it is a quotient of the regular representation.

As done for groups in Chap.1, 3.1 given 2 cyclic modules R/J, R/I we can compute $Hom_R(R/J, R/I)$ as follows.

If $f : R/J \to R/I$ is a homomorphism and $\overline{1} \in R/J$ is the class of 1 we have $f(\overline{1}) = \overline{x}$, $f : r\overline{1} \to r\overline{x}$ for some $x \in R$. We must have then $J\overline{x} = f(J\overline{1}) = 0$ hence $Jx \subset I$. Conversely if $Jx \subset I$ the map $f : r\overline{1} \to r\overline{x}$ is a well defined homomorphism.

Thus if we define the set $(I : J) := \{x \in R | Jx \subset I\}$, we have

$$I \subset (I : J) := \{x \in R | Jx \subset I\}, \quad Hom_R(R/J, R/I) = (I : J)/I$$

In particular for $J = I$ we have the *idealizer* $\mathcal{I}(I)$ of I , $\mathcal{I}(I) := \{x \in R | Ix \subset I\}$. The idealizer is the maximal subring of R in which I is a two sided ideal, then $\mathcal{I}(I)/I$ is the ring $Hom_R(R/I, R/I) = End_R(R/I, R/I)$.

Exercise In a ring R consider an idempotent, $e \in R$, $e^2 = e$. Set $f := 1 - e$. We have the decomposition:

$$R = eRe \oplus eRf \oplus fRe \oplus fRf$$

which presents R as matrices:

$$R = \begin{vmatrix} eRe & eRf \\ fRe & fRf \end{vmatrix}.$$

Prove that $End_R(Re) = eRe$.

Example Consider for A the full ring of $n \times n$ matrices over a field F. As a module we take F^n and in it the basis element e_1.

Its annihilator is the left ideal I_1 of matrices with the first column 0. In this case though we have a more precise picture.

Let J_1 denote the left ideal of matrices having 0 in all columns except the first. Then $M_n(F) = J_1 \oplus I_1$ and the map $a \to ae_1$ restricted to J_1 is an isomorphism of modules between J_1 and F^n.

In fact we can define in the same way J_i (the matrices with 0 outside the i^{th} column).

Then $M_n(F) = \oplus_{i=1}^n J_i$ is a direct sum of the algebra $M_n(F)$ into irreducible left ideals all isomorphic, as modules, to the representation F^n.[5]

Theorem. *The regular representation of $M_n(F)$ is the direct sum of n copies of the standard module F^n.*

The example of matrices suggests the following:

Definition. We say that a ring R is *semisimple* if it is semisimple as a left module on itself.

This definition is a priori not symmetric although it will be proved to be so from the structure theorem of semisimple rings.

REMARK Let us decompose a semisimple ring R as direct sum of irreducible left ideals, since 1 generates R and it must be in a finite sum of the given sum we see:

Proposition. *A semisimple ring is a direct sum of finitely many minimal left ideals.*

Corollary. *If F is a field[6] then $M_m(F)$ is semisimple.*

4.3 We wish to collect some examples of semisimple rings.

First of alll from the results in 4.1 and 4.2 we deduce:

Theorem. *The group algebra $\mathbb{C}[G]$ of a finite group is semisimple.*[7]

Next we have the obvious fact:

Proposition. *The direct sum of two semisimple rings is semisimple.*

In fact we let the following simple exercise to the reader.

EXERCISE To decompose a ring A in a direct sum of two rings is equivalent to give an element $e \in A$ such that:

$$e^2 = e, \; ea = ae, \; \forall a \in A. \; e \text{ is called a central idempotent.}$$

Having a central idempotent e, every A module M decomposes canonically as

$$eM \oplus (1-e)M.$$

Thus if $A = A_1 \oplus A_2$ the module theory of A reduces to the one of A_1, A_2.

From the previous paragraph and these remarks we deduce:

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[5] REMARK The same proof, with small variations, applies with a division algebra D in place of F.

[6] also if it is a division ring

[7] Remark In fact it is not difficult to generalize to an arbitrary field. The general statement is:

The group algebra $F[G]$ of a finite group over a field F is semisimple if and only if the characteristic of F does not divide the order of G.

Theorem. *A ring $A := \oplus_i M_{n_i}(F)$, F a field is semisimple.*[8]

4.4 Our next task will be to show that also the converse to theorem 4.3 is true, i.e. that every semisimple ring is a finite direct sum of rings of type $M_m(D)$, D division ring.

We will limit the proof to finite dimensional algebras over $\mathbb{C}$.For the moment we collect one further remark, let us recall that:

Definition. A ring R is called *simple* if it does not possess any non trivial two sided ideals.

Equivalently it is irreducible as a module over $R \otimes R^0$ under the left and right action $(a \otimes b)r := arb$.

This definition is slightly confusing since a simple ring is by no means semisimple, unless it satisfies further properties (the d.c.c. on left ideals). A classical example of an infinite dimensional simple algebra is the algebra of differential operators $F < x_i, \dfrac{\partial}{\partial x_i} >$ (F a field of characteristic 0).

We have:

Proposition. *If D is a division ring $M_m(D)$ is simple.*

Proof. Let I be a non trivial two sided ideal, $a \in I$ a non zero element. We write a as a linear combination of elementary matrices $a = \sum a_{ij} e_{ij}$, thus $e_{ii} a e_{jj} = a_{ij} e_{ij}$ and at least one of these elements must be non zero. Multiplying it by a scalar matrix we can obtain an element e_{ij} in the ideal I, then we have $e_{hk} = e_{hi} e_{ij} e_{jk}$ and we see that the ideal coincides with the full ring of matrices.

Exercise The same argument shows more generally that for any ring A the ideals of the ring $M_m(A)$ are all of the form $M_m(I)$ for I an ideal of A.

4.5 We start now with the general theory and with the following basic facts:

Theorem. *(Schur's lemma) The centralizer $\Delta := End_R(M, M)$ of an irreducible module M is a division algebra.*

Proof. Let $a : U \to U$ be a non zero R linear endomorphism. Its kernel and image are submodules of M. Since M is irreducible and $a \neq 0$ we must have $Ker(a) = 0$, $Im(a) = M$ hence a is an isomorphism and so it is invertible. This means that every non 0 element in Δ is invertible, this is the definition of a division algebra.

This lemma has several variations, the same proof shows that:

Corollary. *If $a : M \to N$ is a homomorphism between two irreducible modules then either $a = 0$ or a is an isomorphism.*

4.6 A particularly important case is when U is a finite dimensional vector space over $\mathbb{C}$, in this case since the only division algebras over $\mathbb{C}$ is $\mathbb{C}$ itself we have that:

[8] more generally a ring $A := \oplus_i M_{n_i}(D_i)$, with the D_i division algebras is semisimple.

Theorem. *Given an irreducible set S of operators on a finite dimensional space over $\mathbb{C}$ then its centralizer S' is formed by the scalars $\mathbb{C}$.*

Proof. Rather than applying the structure theorem of finite dimensional dvision algebras one can argue that, given an element $x \in S'$ and an eigenvalue α of x the space of eigenvectors of x for this eigenvalue is stable under S and so, by irreducibility, it is the whole space hence $x = \alpha$.

Remarks. 1. If the base field is the field of real numbers we have (according to the theorem of Frobenius (cf. [ ])) 3 possibilities for Δ: $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ the algebra of quaternions.

2. It is not necessary to assume that U is finite dimensional, it is enough to assume that it is of countable dimension.

In fact U is also a vector space over Δ and so Δ, being isomorphic to a $\mathbb{C}$ (or $\mathbb{R}$) subspace of U is also countably dimensional.

This implies that every element of Δ is algebraic over $\mathbb{R}$. Otherwise Δ would contain a field isomorphic to the rational function field $\mathbb{R}(t)$ which is impossible, since this field contains the uncountably many linearly independent elements $\frac{1}{t-r}$, $r \in \mathbb{R}$.

Now one can prove that a division algebras over $\mathbb{R}$ in which every element is algebraic is necessarily finite dimensional[9] and thus the theorem of Frobenius applies.

In order to understand semisimple algebras from this point of view we make a general remark about matrices.

Let $M = M_1 \oplus M_2 \oplus M_3 \oplus \ldots \oplus M_k$ be an R module decomposed in a direct sum.

For each i, j consider $A(j,i) := hom_R(M_i, M_j)$. For 3 indices we have the composition map $A(k,j) \times A(j,i) \to A(k,i)$.

The groups $A(j,i)$ together with the composition maps allow us to recover the full endomorphism algebra of M as **block matrices**:

$$A = (a_{ji}),\ a_{ji} \in A(j,i).$$

(One can give a formal abstract construction starting from the associativity properties).

In more concrete form let $e_i \in End(M)$ be the projection on the summand M_i with kernel $\oplus_{j \neq i} M_j$. The elements e_i are *a complete set of orthogonal idempotents in $End(M)$* i.e. they satisfy the properties

$$e_i^2 = e_i,\ e_i e_j = e_j e_i = 0,\ i \neq j,\ \text{and}\ \sum_{i=1}^{k} e_i = 1.$$

When we have in a ring S such a set of idempotents we decompose S as

$$S = (\sum_{i=1}^{k} e_i) S (\sum_{i=1}^{k} e_i) = \oplus_{i,j} e_i S e_j.$$

[9]this depends on the fact that every element algebraic over $\mathbb{R}$ satisfies a quadratic polynomial.

This sum is direct by the orthogonality of the idempotents.

We have $e_iSe_je_jSe_k \subset e_iSe_k,\ e_iSe_je_hSe_k = 0,\ j \neq h$. In our case $S = End_R(M)$ and $e_iSe_j = hom_R(M_j, M_i)$.

In particular assume that the M_i are all isomorphic to a module N and let $A := End_R(N)$ then:

$$End_R(N^{\oplus k}) = M_k(A).$$

Assume now we have two modules N, P such that $hom_R(N, P) = hom_R(P, N) = 0$, let $A := End_R(N)$, $B := End_R(P)$ then:

$$End_R(N^{\oplus k} \oplus P^{\oplus h}) = M_k(A) \oplus M_h(B).$$

Clearly we have a similar statement for several modules.

We can add together all these remarks in the case in which a module is a finite direct sum of irreducibles.

Assume $N_1, N_2, \dots, N_k$ are the distinct irreducible which appear with multiplicities $h_1, h_2, \dots, h_k$, let $D_i = End_R(N_i)$ (a division ring) then:

(4.6.1) $$End_R(\oplus_{i=1}^k N_i^{h_i}) = \oplus_{i=1}^k M_{h_i}(D_i).$$

4.7 We are now ready to characterize semisimple rings. If R is semisimple we have that $R = \oplus_{i=1}^k N_i^{m_i}$ as in the previous section as a left R module, then:

$$R^0 = End_R(R) = \oplus_{i\in I} N_i^{m_i} = \oplus_{i\in I} M_{m_i}(\Delta_i).$$

We deduce that $R = R^{00} = \oplus_{i\in I} M_{m_i}(\Delta_i)^0$.

The opposite of the matrix ring over a ring A is the matrices over the opposite ring (use transposition) and so we deduce finally:

Theorem. *A semisimple ring is isomorphic to the direct sum of matrix rings R_i over division algebras.*

Some comments are in order.

1. We have seen that the various blocks of this sum are simple rings, they are thus distinct irreducible representations of the ring $R \otimes R^0$ acting by the left and right action.

We deduce that the matrix blocks are minimal 2 sided ideals. From the theory of isotypic components which we will discuss it follows that the only ideals of R are direct sums of these minimal ideals.

2. We have now the left right symmetry, if R is semisimple so is also R^0.

Since any irreducible module N is cyclic there is a surjective map $R \to N$. This map restricted to one of the N_i must be non zero hence:

Corollary. *Each irreducible R module is isomorphic to one of the N_i (appearing in the regular representation).*

§1 Characters.

2.1 We want to deduce some of the basic theory of characters of finite groups with some comments to compact groups.

Definition. *Given a linear representation $\rho : G \to GL(V)$ of a group G, where V is a finite dimensional vector space over a field F we define its* **character** *to be the following function on G.*

$$\chi_\rho(g) := tr(\rho(g)).$$

Here tr is the usual trace.

We say that a character is irreducible if it comes from an irreducible representation.

Some properties are immediate.

Proposition. *1) $\chi_\rho(g) = \chi_\rho(aga^{-1}),\ \forall a, g \in G.$*

This means that the character is constant on conjugacy classes, such a function is called a **class function***.*

2) Given two representations ρ_1, ρ_2 we have:

$$\chi_{\rho_1 \oplus \rho_2} = \chi_{\rho_1} + \chi_{\rho_2},\ \chi_{\rho_1 \otimes \rho_2} = \chi_{\rho_1}\chi_{\rho_2},$$

3) Furthermore if ρ is unitarizable we have that the character of the dual representation ρ^* is the conjugate of χ_ρ:

$$\rho^* = \overline{\rho}.$$

Proof. Let us show 3) since the others are clear. If ρ is unitarizable there is a basis in which the matrices $\rho(g)$ are unitary, hence in the dual representation we obtain the conjugate matrix which is equal to the inverse transposed and $tr(\rho^*(g)) = tr(\overline{\rho(g)}) = \overline{tr(\rho(g))}$.

4.8 To understand the deeper properties of characters we state the following for finite groups.

Proposition. *Let $\rho : G \to GL(V)$ be a complex finite dimensional representation of a finite group G, then:*

$$dim_{\mathbb{C}} V^G = \frac{1}{|G|} \sum \chi_{\rho(g)}.$$

Proof. Let us consider the operator $\pi := \frac{1}{|G|} \sum \rho(g)$, we claim that it is the projection operator on V^G. In fact if $v \in V^G$:

$$\pi(v) = \frac{1}{|G|} \sum \rho(g)(v) = \frac{1}{|G|} \sum v = v.$$

Otherwise:

$$\rho(h)\pi(v) = \frac{1}{|G|}\sum \rho(h)\rho(g)v = \frac{1}{|G|}\sum \rho(hg)v = \pi(v).$$

Then $dim_{\mathbb{C}} V^G = tr(\pi) = tr(\frac{1}{|G|}\sum \rho(g)) = \frac{1}{|G|}\sum tr(\rho(g)) = \frac{1}{|G|}\sum \chi_{\rho(g)}$ by linearity of the Trace.

The previous proposition has an important consequence.

Theorem (Orthogonality of characters). *Let χ_1, χ_2 be the characters of two irreducible representations ρ_1, ρ_2 then:*

$$\frac{1}{|G|}\sum \chi_1(g)\overline{\chi}_2(g) = \begin{cases} 0 & \textit{if } \rho_1 \neq \rho_2 \\ 1 & \textit{if } \rho_1 = \rho_2 \end{cases}.$$

Proof. Let $V_1,\ V_2$ the the spaces of the two representations consider $hom(V_2, V_1) = V_1 \otimes V_2^*$.

As representation it has character $\chi_1(g)\overline{\chi}_2(g)$ from 2.1.

We have seen that $hom_G(V_2, V_1) = (V_1 \otimes V_2^*)^G$ hence

$$dim_{\mathbb{C}} hom_G(V_2, V_1) = \frac{1}{|G|}\sum \chi_1(g)\overline{\chi}_2(g)$$

from the previous proposition.

Finally by Schur's lemma and the fact that $V_1,\ V_2$ are irreducible, it follows the theorem.

In fact a more precise theorem holds. Let us consider the Hilbert space $\mathbb{C}[G]$ of functions on G, the Hilbert scalar product being for two functions a, b on G:

$$(a, b) := \frac{1}{|G|}\sum_{g\in G} a(g)\overline{b(g)}.$$

inside we consider the subspace $\mathbb{C}_c(G)$ of class functions then.

Theorem. *The irreducible characters are an orthonormal basis of $\mathbb{C}_c(G)$.*

For a finite group G decompose the group algebra in matrix blocks according to 1.8 as $\mathbb{C}[G] = \oplus_i^m M_{h_i}(\mathbb{C})$.

The m blocks correspond to the m irreducible representations and the m irreducible characters are the composition of the projection to a factor $M_{h_i}(\mathbb{C})$ followed by the ordinary trace.

A function $f = \sum_{g\in G} f(g)g \in \mathbb{C}[G]$ is a class function if and only if $f(ga) = f(ag)$ for all $a, g \in G$. This means that f lies in the center of the group algebra.

The space of class functions is identified to the center of $\mathbb{C}[G]$.

The center of a matrix algebra $M_h(\mathbb{C})$ is formed by the scalar matrices thus the center of $\oplus_i^m M_{h_i}(\mathbb{C})$ equals $\mathbb{C}^{\oplus m}$.

It follows that the number of irreducible characters equals the dimension of the space of class functions. Since the irreducible characters are orthonormal they are a basis.

As corollary we have:

Corollary. *The number of irreducible representations of a finite group G equals the number of conjugacy classes in G.*

If $h_1, \dots, h_r$ are the dimensions of the distinct irreducible representations of G we have $|G| = \sum_i h_i^2$.

There is a deeper result on the dimensions of irreducible representations (see.):

Theorem. *The dimension h of an irreducible representations of a finite group G divides the order of G.*

The previous informations allow us to compute a priori the dimensions h_i in some simple cases but in general are only a small piece of information.

Example-exercise. Let us consider again the action of a finite group G by permutations on G itseld by left multiplication. The corresponding permutation representation is $\mathbb{C}[G]$ the group algebra. Its character is the *regular character* and it is easily computed, an element g permutes the basis elements without fixed points if it is different from 1 thus its trace is 0 while $tr(1) = |G|$.

Let us denote by χ_r this regular character. By taking the scalar product of the regular character with an irreducible character χ_1, of some irreducible module N we have:

$$\frac{1}{|G|} \sum_{g \in G} \chi_r(g) \chi_1(g) = \chi_1(1) = \dim N$$

we thus recover the fact that each irreducible module appears in the regular representation with multiplicity equal to its dimension.

4.9 We make now a computation on induced characters which will be useful when we discuss the symmetric group.

Let G be a finite group H a subgroup and V a representation of H we want to compute the character χ of $Ind_H^G(V) = \oplus_{x \in G/H} xV$. An element $g \in G$ induces a transformation on $\oplus_{x \in G/H} xV$ which can be thought of as a matrix in block form. Its trace comes only from the contributions of the blocks xV for which $gxV = xV$. The character $\chi(g)$, is thus a sum of contributions from the fixed points $(G/H)^g$ of g on G/H, i.e. the cosets xH such that $gxH = xH$ or $x^{-1}gx \in H$.

If $gxV = xV$ the map g on xV has the same trace as the map $x^{-1}gx$ on V thus:

$$\chi(g) = \sum_{(G/H)^g} \chi_V(x^{-1}gx)$$

It is useful to transform the previous formula, $gxH = xH$ if and only if $x^{-1}gx \in H$, so let $X := \{x \in G | x^{-1}gx \in H\}$.

The set X is a union of right cosets $G(g)x$ where $G(g)$ is the centralizer of g in G and the map $x \to x^{-1}gx$ is a bijection between the set of such cosets and the intersection of the conjugacy class C_g of g with H. Decompose then $C_g \cap H = \cup_i O_i$ into H conjugacy classes, fix an element $g_i \in O_i$ in each class and let $H(g_i)$ be the centralizer of g_i in H, then $|O_i| = |H|/|H(g_i)|$ and finally.

$$\chi(g) = \frac{1}{|H|}\sum_{x\in X}\chi_V(x^{-1}gx) = \frac{1}{|H|}\sum_i\sum_{a\in O_i}|G(a)|\chi_V(a) = \sum_i \frac{|G(g)|}{|H(g_i)|}\chi_V(g_i). \tag{4.9.1}$$

In particular one can apply this to the case $V = 1$ so that we have the permutation representation of G on G/H.

Proposition. *The number of fixed points of g on G/H equals the character of the permutation representation and is:*

$$\chi(g) = \frac{|C_g \cap H||G(g)|}{|H|} = \sum_i \frac{|G(g)|}{|H(g_i)|}. \tag{4.9.2}$$

5 Induction and restriction.

5.1 We collect now some general facts about representations of groups.

First of all let H be a group, $\phi : H \to H$ an automorphism and $\rho : G \to GL(V)$ a linear representation.

Composing with ϕ we get a new representation V^ϕ given by $G \xrightarrow{\phi} G \xrightarrow{\rho} GL(V)$, it is immediately verified that, if ϕ is an inner automorphism V^ϕ is equivalent to ϕ.

Let now $H \subset G$ be a normal subgroup, every element $g \in G$ induces by inner conjugation in G an automorphism ϕ_g of H.

Given a representation M of G and an H submodule $N \subset M$ we clearly have that $gN \subset M$ is again an H submodule and canonically isomorphic to N^{ϕ_g}, it depends only on the coset gH of G.

In particular assume that, M is irreducible as G module and N is irreducible as H module, then all the submodules gN are irreducible H modules and $\sum_{g\in G/H} gN$ is a G submodule hence $\sum_{g\in G/H} gN = M$.

We want in particular apply this when H has index 2 in $G = H \cup uH$, we shall then use the canonical sign representation ϵ of $\mathbb{Z}/(2) = G/H$.

Theorem. *1) Given an irreducible representation N of H it extends to a representation of G if and only if N is isomorphic to N^{ϕ_u}, in this case it extends in two ways up to the sign representation.*

2) An irreducible representation M of G restricted to H either remains irreducible or splits into 2 irreducible representations $N \oplus N^{\phi_u}$ according to wether M is not or is isomorphic to $M \otimes \epsilon$.

Proof. Let $h_0 = u^2 \in H$. If N is also a G representation the map $u : N \to N$ is an isomorphism with N^{ϕ_u}, conversely let $t : N \to N = N^{\phi_u}$ be an isomorphism so that $tht^{-1} = \phi_u(h)$, then $t^2ht^{-2} = h_0hh_0^{-1}$.

Since N is irreducible we must have $h_0^{-1}t^2 = \lambda$ is a scalar.

We can substitute t with $t\sqrt{\lambda}^{-1}$ and can thus assume that $t^2 = h_0$ (on N).

It follows that mapping $u \to t$ one has the required extension of the representation it also is clear that the choice $-t$ is the other possible choice changing the sign of the representation.

2) From our previous discussion if $N \subset M$ is an irreducible H submodule then $M = N + N^{\phi_u}$ and we clearly have two cases $M = N$ or $M = N \oplus N^{\phi_u}$.

In the first case tensoring by the sign representation changes the representation while in the second we can represent M as the set $N \oplus N$ of pairs (n_1, n_2) over which H acts diagonally while $u(n_1, n_2) := (h_0n_2, n_1)$.

Similarly $M \otimes \epsilon$ is $N \oplus N$ of pairs (n_1, n_2) over which H acts diagonally while $u(n_1, n_2) := -(h_0n_2, n_1)$.

Then it is immediately seen that the map $(n_1, n_2) \to (n_1, -n_2)$ is an isomorphism of the two structures. $\square$

One should compare this property of the possible splitting of irreducible representations with the similar feature for conjugacy classes.

With the same notations as before.

Exercise. Given a conjugacy class C of G contained in H it is either a unique conjugacy class in H or it splits into 2 conjugacy classes permuted by exterior conjugation by u. The second case occurs if and only if the stabilizer of an element in the conjugacy class is contained in H.

5.2 Let now G be a group H a subgroup and N a representation of H (over some field k).

One considers $k[G]$ as a rigth $k[H]$ module and forms $k[G] \otimes_{k[H]} N$ which is a representation under G by the left action of G on $k[G]$.

Definition. $k[G] \otimes_{k[H]} N$ is called the representation induced from N from H to G, it is also denoted by $Ind_H^G N$.

Exercise. If $G \supset H \supset K$ are groups and N is a K module we have

$$Ind_H^G(Ind_K^H N)) = Ind_K^G N$$

The representation $Ind_H^G N$ is in a natural way described by $\oplus_{g \in G/H} gN$ where by $g \in G/H$ we mean that g runs over a choice of representatives of cosets. The action of G on such a sum is easily described.

There is a similar construction by forming $\hom_{k[H]}(k[G], N)$ where now $k[G]$ is considered as a left $k[H]$ module by the right action and $\hom_{k[H]}(k[G], N)$ is a representation under G by the action of G deduced from the left action on $k[G]$.

$$\hom_{k[H]}(k[G], N) := \{f : G \to N|\, f(gh) = hf(g)\},\ (gf)(k) := f(g^{-1}k).$$

If G is a finite group one has a $G \times G$ isomorphism between $k[G]$ and its dual (3.1) and we obtain an isomorphism

$$k[G] \otimes_{k[H]} N = k[G]^* \otimes_{k[H]} N = \hom_{k[H]}(k[G], N)$$

for algebraic groups and rational representations it is better to take the point of view of the representation $\hom_{k[H]}(k[G], N)$.

If H is a closed subgroup of G one can define $\hom_{k[H]}(k[G], N)$ as the set of regular maps $G \to N$ which are $H-$ equivariant.

The regular maps from an affine algebraic variety V to a vector space U can be identified to $A(V) \otimes U$ where $A(V)$ is the ring of regular functions on V hence if V has an action under an algebraic group H and U is a rational representation f H the space of H equivariant maps $V \to U$ is identified to the space of invariants $(A(V) \otimes U)^H$.

Assume now that G is linearly reductive and let us invoke the decomposition 1.2.1. $k[G] = \oplus_i U_i^* \otimes U_i$ hence

$$\hom_{k[H]}(k[G], N) = (k[G] \otimes N)^H = \oplus_i U_i^* \otimes (U_i \otimes N)^H$$

finally in order to compute $(U_i \otimes N)^H$ remark that $(U_i \otimes N)^H = \hom_H(U_i^*, N)$.

Assume then that N is irreducible and that H is also linearly reductive, it follows from Schur's Lemma that the dimension of the space $\hom_H(U_i^*, N)$ equals the multiplicity of N in the representation U_i^*. We deduce thus

Theorem Frobenius reciprocity. *The multilicity with which an irreducible representation V of G appears in* $\hom_{k[H]}(k[G], N)$ *equals the multiplicity with which N appears in V as representation of H.*