# Introduction to Symmetric Functions
# Chapter 1

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Abstract. A development of the symmetric functions using the plethystic notation.

CHAPTER 2

# Partitions

Start with $n$ unlabeled objects and partition them (break them into subsets) into non-empty subsets. Since the objects are indistinguishable from each other, a partition is specified by the sizes of the non-empty subsets and we can list these sizes in weakly decreasing order. An object of this type is called a partition of $n$ and we will represent this as a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\lambda_i$ representing the number of objects in the $i^{th}$ subset and $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$ where $k$ is the number of non-empty subsets of the partition. We denote '$\lambda$ is a partition of $n$' by $\lambda \vdash n$. $k$ is known as the length of the partition and we will denote it by $\ell(\lambda) := k$. By defintion, we have $\lambda_1 + \lambda_2 + \cdots + \lambda_{\ell(\lambda)} = n$ and we will adopt the convention that for any $i > \ell(\lambda)$, $\lambda_i = 0$ so that $\sum_{i\geq 1} \lambda_i = n$.

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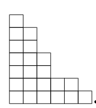
For each partition we will associate a subset of the points in the first quadrant of the cartesian lattice, $D(\lambda) = \{(i, j) : 0 \leq i < \lambda_{j+1}, j \geq 0\}$. The cardinality of this set is $|D(\lambda)| = n$ and we will represent $D(\lambda)$ by drawing a diagram in the coordinate plane with a square for each $(i, j) \in D(\lambda)$ with the bottom left hand corner of the square at the coordinate $(i, j)$. A set of cells $S$ corresponds to a partition if and only if $(i, j) \in S$ implies that $(i', j') \in S$ for all $0 \leq i' \leq i$ and $0 \leq j' \leq j$.

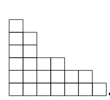
Example 1. Let $\lambda = (6, 5, 3, 3, 2, 2, 1)$ which is a partition of 22 and $D(\lambda)$ is represented by the following diagram

.

Let $n_i(\lambda)$ be the number of $j \leq \ell(\lambda)$ such that $\lambda_j \geq i$ and set $\lambda' = (n_1(\lambda), n_2(\lambda), \ldots, n_{\lambda_1}(\lambda))$. Since $n_i(\lambda) \geq n_{i+1}(\lambda)$, $\lambda'$ is also a partition and we refer to it as the conjugate partition to $\lambda$.

Exercise 1. Show that $(i, j) \in D(\lambda)$ if and only if $(j, i) \in D(\lambda')$ and conclude the diagram associated to $D(\lambda')$ is exactly the diagram for $D(\lambda)$ flipped about the line $x = y$.

Example 2. As in the previous example $\lambda = (6, 5, 3, 3, 2, 2, 1)$, then we can easily calculate each of the $n_i(\lambda)$ and determine $\lambda' = (7, 6, 4, 2, 2, 1)$. $D(\lambda')$ is represented by the diagram

.

From this point forward we will identify $\lambda$ and $D(\lambda)$ in abuse of notation. This will appear in our formulas as expressions like $\lambda \subseteq \mu$ (or the language $\lambda$ is contained in $\mu$) to mean $D(\lambda) \subseteq D(\mu)$ or $s \in \lambda$ in place of $s \in D(\lambda)$. We shall also try to develop notation which is consistent with this identification, for instance the size of the partition $\lambda$ will be $|\lambda|$ which represents $|D(\lambda)|$.

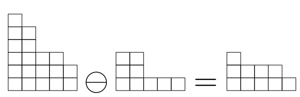
The union of two partitions $\lambda \cup \mu$ is another abuse of notation which will represent the partition whose diagram contains $D(\lambda) \cup D(\mu)$. Intuitively $\lambda \cup \mu$ is the smallest partition which contains both the partition $\lambda$ and the partition $\mu$. An equivalent definition will be the weakly increasing sequence $(max(\lambda_1, \mu_1), max(\lambda_2, \mu_2), \ldots, max(\lambda_m, \mu_m))$ with $m = max(\ell(\lambda), \ell(\mu))$. Similarly $\lambda \cap \mu$ will be the partition which is contained in both the partition $\lambda$ and the partition $\mu$ (that is, $D(\lambda \cap \mu) = D(\lambda) \cap D(\mu)$) so that $\lambda \cap \mu = (min(\lambda_1, \mu_1), min(\lambda_2, \mu_2), \ldots, min(\lambda_m, \mu_m))$.

We will sometime need to consider the partition $\lambda \uplus \mu$ which will be the sequence

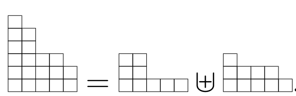
$$(\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)}, \mu_1, \mu_2, \ldots, \mu_{\ell(\mu)})$$

rearranged so that the entries are in decreasing order. The complementary operation to $\uplus$ will be the $\lambda \ominus \mu$ where if there is a subset $S = \{s_1 < s_2 < \ldots < s_{\ell(\lambda)}\}$ such that $(\lambda_{s_1}, \lambda_{s_2}, \ldots, \lambda_{s_{\ell(\lambda)}}) = \mu$ then $\lambda \ominus \mu$ is the ordered sequence of parts $(\lambda_i : i \notin S)$. These operations are complementary in the sense that $\lambda \uplus \mu = \nu$ if and only if $\lambda = \nu \ominus \mu$ (and by symmetry $\mu = \nu \ominus \lambda$). We will have that $|\lambda \uplus \mu| = |\lambda| + |\mu|$ and $\ell(\lambda \uplus \mu) = \ell(\lambda) + \ell(\mu)$.

Example 3. Let $\lambda = (5, 5, 4, 2, 2, 1)$ and $\mu = (5, 2, 2)$ then $\lambda \ominus \mu = (5, 4, 1)$ and $\lambda = \mu \uplus (\lambda \ominus \mu)$. In pictures this is represented as



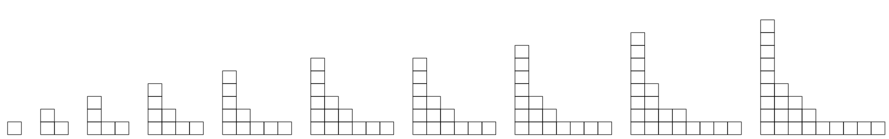
and



Exercise 2. Prove:

$$\left| \bigcup_{\mu \vdash n} \mu \right| = \sum_{k=1}^{n} d(k)$$

where $d(k)$ is the number of divisors of the number $k$. For example, we have



are the first nine partitions $\bigcup_{\mu \vdash n} \mu$ and $\sum_{k=1}^{n} d(k)$ takes on the values $1, 3, 5, 8, 10, 14, \ldots$.

Exercise 3. Show that $(\lambda \uplus \mu)' = (\lambda'_1 + \mu'_1, \lambda'_2 + \mu'_2, \ldots, \lambda'_r + \mu'_r)$ with $r = max(\ell(\lambda), \ell(\mu))$.

Let $m_i(\lambda) = n_i(\lambda) - n_{i-1}(\lambda)$ (with the convention that $n_0(\lambda) = 0$) so that $m_i(\lambda)$ represents the number of parts of size $\lambda$ which are exactly equal to $i$. Another way of representing the partition $\lambda$ will be by specifying the number of parts of each size $i$ (that is we indicate the values of $m_i(\lambda)$ for each $i$). This will be done with the notation $\lambda = (k^{m_k(\lambda)}, (k-1)^{m_{k-1}(\lambda)}, \ldots, 2^{m_2(\lambda)}, 1^{m_1(\lambda)})$ for the value $k = \lambda_1$.

There are several statistics which are associated to a given partition. If $\lambda$ is a partition then we define $z_\lambda := 1^{m_1(\lambda)} m_1(\lambda)! 2^{m_2(\lambda)} m_2(\lambda)! 3^{m_3(\lambda)} m_3(\lambda)! \cdots$. This statistic is of interest because $|\lambda|!/z_\lambda$ is the number of permutations of $|\lambda|$ that have cycle type $\lambda$.

We also will denote $n(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$ which is something of a measure of how tall a partition is since a single row partition will have $n((k)) = 0$ and a single columned partition will have $n((1^k)) = \binom{k}{2}$. This statistic often appears naturally when considering weighted sums of partitions. For instance, the generating function for the number of partitions of size $n$ is $\prod_{i\geq 1}\frac{1}{1-x^i}$ and to look for a $q$-analog of this expression we might consider the generating function

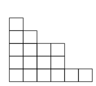
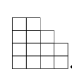
$$\prod_{i\geq 1}\frac{1}{1-qx^i} = \sum_{n\geq 0}\sum_{\lambda\vdash n} q^{n(\lambda)} x^{|\lambda|}. \tag{2.1}$$

If $\mu \subseteq \lambda$, then the symbol $\lambda/\mu$ will represent the collection of cells $D(\lambda) - D(\mu)$ (where the operation $-$ means the difference of sets). $\lambda/\mu$ is called a skew partition. Unless otherwise specified when a skew partition $\lambda/\mu$ is indicated in a formula, it is automatically assumed that $\mu \subseteq \lambda$.

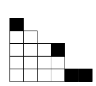
If $\lambda/\mu$ has at most one cell per column then we will say that it is a horizontal strip and indicate this by the notation $\lambda/\mu \in \mathcal{H}$. This condition can also be expressed by the inequalities $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \lambda_3 \geq \cdots$.

If $\lambda/\mu$ has at most one cell per column then it will be called a vertical strip (and indicated by $\lambda/\mu \in \mathcal{V}$). $\lambda/\mu \in \mathcal{V}$ if and only if $\lambda_i \geq \mu_i \geq \lambda_i - 1$ for all $i \geq 1$. Furthermore if $\lambda/\mu \in \mathcal{H}$ (respectively $\lambda/\mu \in \mathcal{V}$) and $|\lambda/\mu| = k$ then we will use the notation $\lambda/\mu \in \mathcal{H}_k$ (resp. $\lambda/\mu \in \mathcal{V}_k$).

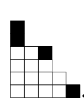
Example 4. Let $\lambda = (6,4,4,2,1)$ and $\mu = (4,4,3,2)$ are represented by the following diagrams

Then the skew diagram $\lambda/\mu$ is a horizontal strip and it is represented by the cells which are colored in the diagram below

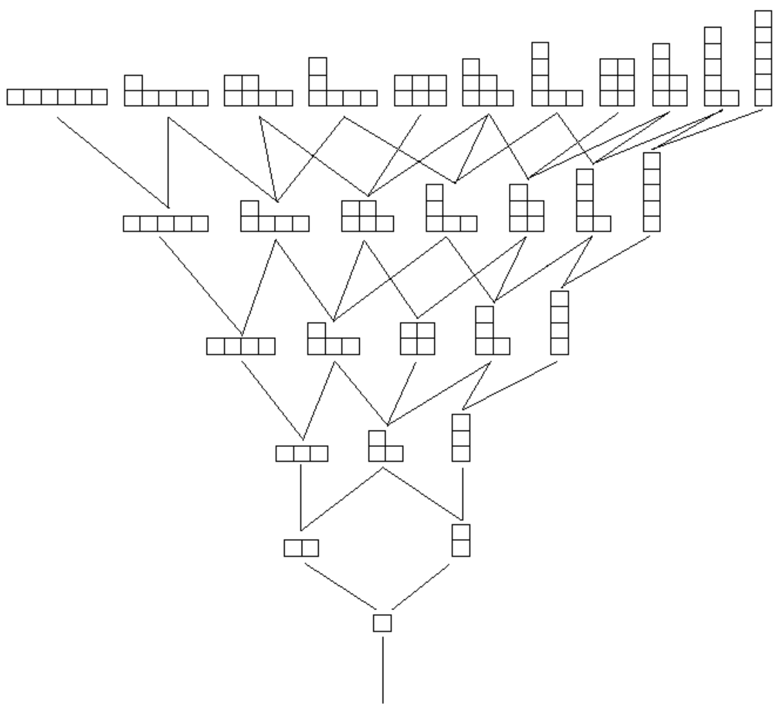


$\lambda' = (5,4,3,3,1,1)$ and $\mu' = (4,4,3,2)$ and $\lambda'/\mu'$ is a vertical strip and the diagram consists of the cells which are colored in the diagram



There are several ways of ordering the set of partitions. The first is to consider one partition $\mu$ is smaller than another $\lambda$ if $D(\mu) \subseteq D(\lambda)$. We will denote this as an abuse of notation by $\mu \subseteq \lambda$. Note that $\mu \subseteq \lambda$ if and only if $\lambda_i \leq \mu_i$ for all $i$. This is a partial order on the set of partitions in that it possible that both $\mu \not\subseteq \lambda$ and $\lambda \not\subseteq \mu$.

EXAMPLE 5. We list the partitions which are size less than or equal to 6 and place a line between two partitions if the smaller is less than the larger in containment order. This partial order defines a lattice structure on the set of partitions.

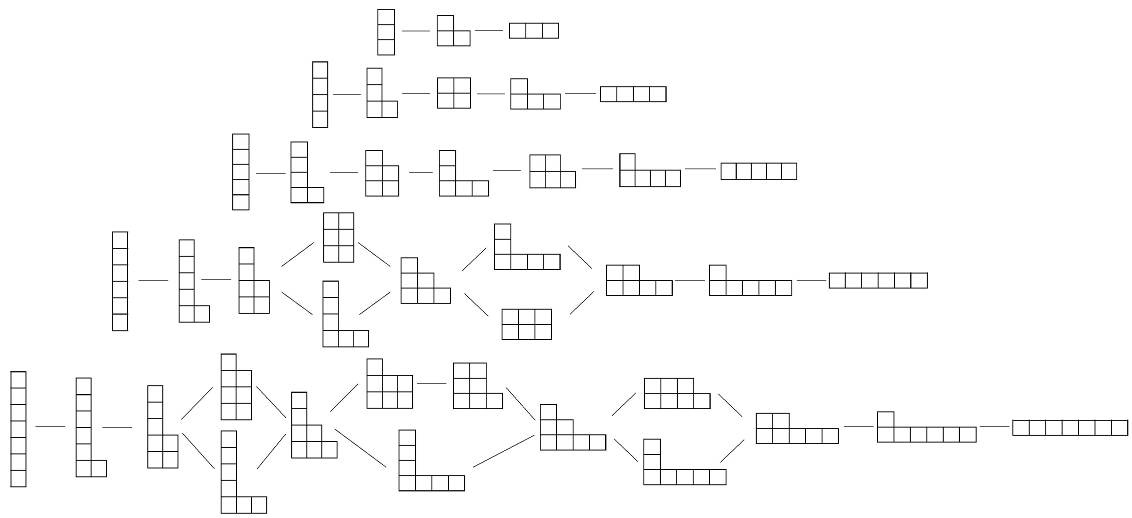


EXERCISE 4. Prove that $|\{\mu : \mu \subseteq \lambda\}| = |\{i : m_i(\lambda) > 0\}| = |\{\mu : \mu \supseteq \lambda\}| - 1$

Another way of ordering the partitions is to compare the entries lexicographically, that is if $\lambda, \mu$ are partitions then $\lambda <_{lex} \mu$ if either $\lambda_1 < \mu_1$ or $\lambda_1 = \mu_1$ and $(\lambda_2, \lambda_3, \ldots, \lambda_{\ell(\lambda)}) <_{lex} (\mu_2, \mu_3, \ldots, \mu_{\ell(\mu)})$. This is a total order on partitions but it is possible that $|\lambda| > |\mu|$ and $\lambda <_{lex} \mu$. Although this is a perfectly natural order on the set of partitions it does not seem to be the one that arises most naturally in the algebra structures that we will study. This and similar orders (such as degree lex or reverse lex) are ones that we might use for studying ideals in a polynomial ring. For the time being this structure is not as important as the third order which we will define.

The last order we will consider is also a partial order on partitions which arises naturally from the algebra structure that we will be looking at. This order will be denoted by $<$ and we will say that $\lambda \leq \mu$ if and only if $\sum_{j=1}^{i} \lambda_j \leq \sum_{j=1}^{i} \mu_j$ for all $i \geq 1$. This partial order is a total order on partitions of size strictly less than 6. At $n = 6$ both the pair $(4, 1, 1)$ and $(3, 3)$ and the pair $(3, 1, 1, 1)$ and $(2, 2, 2)$ are not comparable with respect to this order.

Example 6. Below we show the poset of partitions of size $3, 4, 5, 6$ and $7$. Notice that for partitions of $3, 4$ and $5$ this is a linear order but for the partitions of size $6$ and $7$ it is not since for example $(2, 2, 2)$ and $(3, 1, 1, 1)$ are not comparable in this order.



An intuitive definition of the statement $\mu \leq \lambda$ is that the diagram for $\mu$ is narrower and taller than $\lambda$. We also clearly have for $\mu \leq \lambda$ then $\mu \leq_{lex} \lambda$. The converse of this statement is not true.

Exercise 5. Prove that $\lambda \leq \mu$ if an only if $\mu' \leq \lambda'$.

For any $n$, let $p(n)$ represent the number of partitions of $n$. $\frac{1}{1-q^k} = \sum_{r\geq 0} q^{rk}$ is a generating function for the partitions which only contain parts of size $k$ (that is, there is a rectangular partition at of size $n$ consisting of parts of size only $k$ if and only if $n = rk$ for some integer $r$). Since in a partition the parts may be chosen independently, the generating function for the sequence $p(n)$ is the product of the generating functions $\frac{1}{1-q^i}$ for all $i$, that is

$$\sum_{n\geq 0} q^n p(n) = \prod_{i\geq 1} \frac{1}{1-q^i}. \tag{2.2}$$

We can use this generating function formula to give the first few values of $p(n)$.

$$\sum_{n\geq 0} q^n p(n) = 1 + q + 2\,q^2 + 3\,q^3 + 5\,q^4 + 7\,q^5 + 11\,q^6 + 15\,q^7$$
$$+ 22\,q^8 + 30\,q^9 + 42\,q^{10} + 56\,q^{11} + 77\,q^{12} + \cdots$$

Despite the fact that we can use the generating function to calculate the number of partitions, it does not really seem to be a very satisfying formula because in order to use it we potentially need to consider expanding the product of $n$ terms in order to calculate the value of $p(n)$.

A remarkable fact was noticed by Leonhard Euler in 17** that we will exploit to find a recurrence on the number of partitions of $n$. He calculated the first several terms of the product $\prod_{i\geq 1}(1-q^i)$ and noticed that the coefficient was always equal to $\pm 1$ or 0. In fact he was quite easily able to guess a formula for this denominator and several years later was able to prove it. We will take advantage of its existence to derive a method of calculating the number of partitions of size $n$.

PROPOSITION 2.1. *(Euler's pentagonal number theorem)*

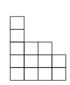
$$\prod_{i\geq 1} 1-q^i = 1+\sum_{m\geq 1}(-1)^m\left(q^{\frac{m(3m-1)}{2}}+q^{\frac{m(3m+1)}{2}}\right) \tag{2.3}$$

There is a clever proof of this proposition that comes from one of the first American mathematicians F. Franklin [**5**]. The proof uses a technique which is fairly ubiquitous in algebraic combinatorics, to show that terms in a sum cancel associate a combinatorial object to each term in the sum and then show that they cancel by producing a map which sends an element with positive weight to a term with negative weight.

There are several other accounts of this proof: [**2**], [**7**], [**8**], [**10**].

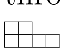
We will need to talk about partitions as a combinatorial object. $\lambda$ is a partition if it is a sequence $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ with $\lambda_1 \geq \lambda_2 \geq \ldots \lambda_{\ell(\lambda)}$ where we use the notation $\ell(\lambda)$ to represent the number of parts of $\lambda$. The symbol $|\lambda|$ will represent the size of the partition so that $|\lambda| = \lambda_1+\lambda_2+\cdots+\lambda_{\ell(\lambda)}$. $\lambda$ is a strict partition if in addition $\lambda_1 > \lambda_2 > \cdots \lambda_{\ell(\lambda)}$.

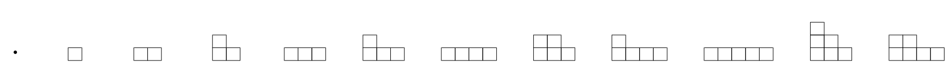
There is a way of graphically representing a partition with rows of boxes. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{\ell(\lambda)})$ is represented by a row of $\lambda_1$ boxes below a row of $\lambda_2$ boxes below a row of $\lambda_3$ boxes etc. Each of these rows of cells will be left justified. For example the partition $(4, 4, 3, 1, 1)$ is represented by the following diagram:



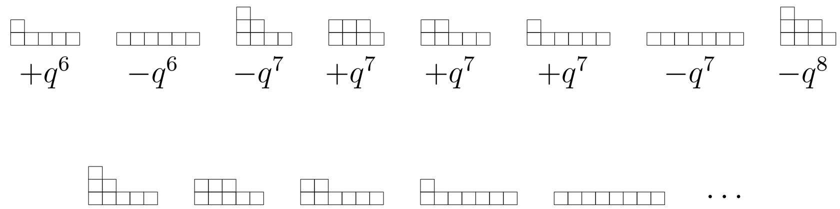
EXAMPLE 7. We note that the left hand side of this equation is the generating function for all strict partitions (partitions where all parts are distinct) weighted with $(-1)^{\ell(\lambda)}q^{|\lambda|}$. That is,

$$\prod_{i\geq 1} 1-q^i = \sum_{\lambda\ strict}(-1)^{\ell(\lambda)}q^{|\lambda|} \tag{2.4}$$

This follows by observing that to determine the coefficient of $q^n$ by expansion of the product on the left we have a contribution of $(-1)^k q^{\lambda_1+\lambda_2+\cdots+\lambda_k}$ for every sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ such that $\lambda_i > \lambda_{i+1}$ for $1 \leq i < k$. Below we expand the terms of this generating function through degree 8. For example, a term of the form $(-q^4)(-q^2)$ is represented by the picture  and we record the weight of $+q^6$ just below the picture.



$1 \quad -q \quad -q^2 \quad +q^3 \quad -q^3 \quad +q^4 \quad -q^4 \quad +q^5 \quad +q^5 \quad -q^5 \quad -q^6 \quad +q^6$



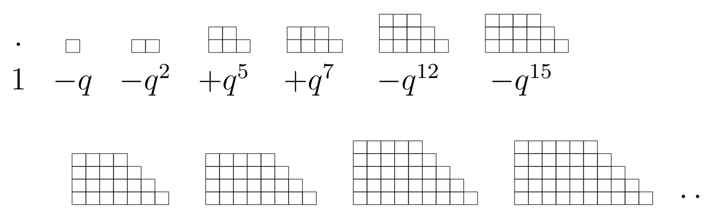
$$+q^6 \quad -q^6 \quad -q^7 \quad +q^7 \quad +q^7 \quad +q^7 \quad -q^7 \quad -q^8$$

$$-q^8 \quad +q^8 \quad +q^8 \quad +q^8 \quad -q^8 \quad + \cdots$$

Now we notice that all of the terms cancel except for the ones stated in the theorem, that is we have

$$\prod_{i\geq 1} 1 - q^i = 1 - q - q^2 + q^5 + q^7 + \cdots$$

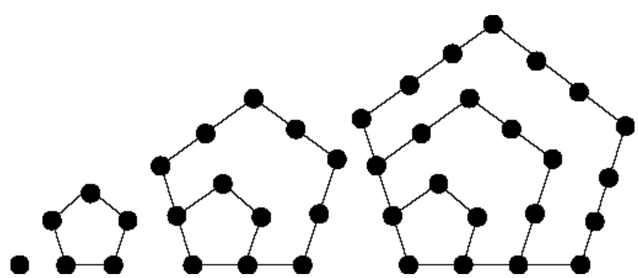
In fact, we will show that one way of looking at this expression is to observe terms which survive are those that correspond to the following pictures:



$$\prod_{i\geq 1} 1 - q^i = \quad 1 \quad -q \quad -q^2 \quad +q^5 \quad +q^7 \quad -q^{12} \quad -q^{15}$$

$$+q^{22} \quad +q^{26} \quad -q^{35} \quad -q^{40} \quad \cdots$$

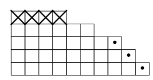
From the image in this example one might think that the theorem would be better named the *trapazoidal* number theorem. There is a reason that the numbers $m(3m-1)/2$ are referred to as pentagonal numbers and if $m \to -m$ then the pentagonal number is transformed to $\to -m(-3m-1)/2 = m(3m+1)/2$. Observe the picture below how a sequence of pentagons have exactly $m(3m-1)/2$ points in them (and this continues for $m > 4$).



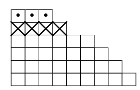
Proof. To show that this proposition holds we show that there is an involution $\phi$ on the strict partitions $\lambda$ of $n$ such that $\phi(\lambda)$ is also a partition of $n$ and the length of $\phi(\lambda)$ will have length either one smaller or one larger than that of $\lambda$. This means that if the weight of a strict partition is $(-1)^{\ell(\lambda)}q^{|\lambda|}$ then the weight of $\phi(\lambda)$ is $-(-1)^{\ell(\lambda)}q^{|\lambda|}$ and so this term corresponding to $\phi(\lambda)$ will cancel with the term corresponding to $\lambda$. This involution will fail to 'work' for the partitions of the form $(2m-1, 2m-2, \ldots, m)$ which are of size $2m^2 - \frac{(m+1)m}{2} = \frac{m(3m-1)}{2}$ and $(2m, 2m-1, \ldots, m+1)$ which are of size $2m^2 - \frac{(m-1)m}{2} = \frac{m(3m+1)}{2}$.

For a strict partition $\lambda$ we will let $r$ equal to the smallest part of $\lambda$ ($r = \lambda_{\ell(\lambda)}$) and let $s$ equal the number of parts which are consecutive at the beginning of the partition. In other words $s$ is the largest integer such that $(\lambda_1, \lambda_2, \ldots, \lambda_s) = (\lambda_1, \lambda_1 - 1, \ldots, \lambda_1 - s + 1)$.

If $s \neq \ell(\lambda)$ and $r > s$ then we will let $\phi(\lambda)$ equal the partition $(\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, \lambda_{s+1}, \ldots, \lambda_{\ell(\lambda)}, s)$. That is, if the diagram for the partition looks something like the following where there is an $\times$ in each of the cells corresponding to $r$ and a dot in the cells corresponding to $s$

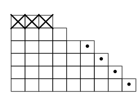


then $\phi(\lambda)$ will be the partition with the diagonal of $s$ cells filled with a dot moved to the top row of the partition.

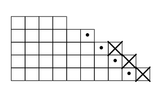


$\phi(\lambda)$ is still a strict partition and it has the property that the longest string of consecutive parts at the beginning of the partition is greater than or equal to $s$.

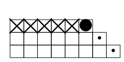
If $s \neq \ell(\lambda)$ and $r \leq s$ then we will let $\phi(\lambda)$ equal to the partition $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_r + 1, \lambda_{r+1}, \ldots, \lambda_{\ell(\lambda)})$. For example, if our diagram is similar to the one below with the cells marked with an $\times$ representing the row of size $r$ and those marked with the $\cdot$ represent the cells which correspond to the $s$ consecutive parts at the beginning of the partition.



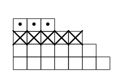
The partition corresponding to $\phi(\lambda)$ is then represented by the following picture.



Notice that it is also possible that $s = \ell(\lambda)$ and we consider this case separately because we need that $r$ is at least 2 more cells than $s$ does before we can move the $s$ cells to the top row. In this case if $r > s + 1$ then we will remove the $s$ cells along the diagonal and turn them into the shortest row so that $\phi(\lambda) = (\lambda_1 - 1, \lambda_2 - 1, \ldots, \lambda_s - 1, s)$. For example we have the picture on the left will be transformed to the one on the right.

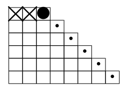


$\lambda$

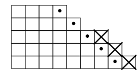


$\phi(\lambda)$

If $s = \ell(\lambda)$ and $r < s$ then it is still possible to move the shortest row of $\lambda$ to the first $r$ rows. We will set $\phi(\lambda) = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_r + 1, \lambda_r, \ldots, \lambda_{\ell(\lambda)-1})$, this corresponds to the case when we have a partition of the form of the one below.



$\lambda$



$\phi(\lambda)$

If we describe what is happening to the diagram the map $\phi$ does one of two things, either it removes the smallest row of $r = \lambda_{\ell(\lambda)}$ cells of the partition and places one cell more in each of the first $r$ rows (in the case that $r < s$ or $r = s$ and $s < \ell(\lambda)$) or it removes one cell from each of the first $s$ rows and adds a row of size $s$ to the top of the diagram (in the case that $r > s+1$ or $r = s+1$ and $s < \ell(\lambda)$).

Observe that if the weight of $\lambda$ is $(-1)^{\ell(\lambda)}$ then since $\phi(\lambda)$ has the same number of cells and either one more or one less row than $\lambda$ then the weight of $\phi(\lambda)$ is the negative of the weight of $\lambda$.

Also observe for each of the 4 cases we have considered, $\phi(\phi(\lambda))$ is just $\lambda$. This implies we can say that in the expansion of the expression $\sum_{\lambda\ strict}(-1)^{\ell(\lambda)}q^{|\lambda|}$, the term corresponding to the partition $\lambda$ will cancel with the term corresponding to the partition $\phi(\lambda)$ because the then $\phi(\lambda)$ will also cancel with $\phi(\phi(\lambda)) = \lambda$.

There are two cases that we have not considered. These terms do not cancel. One is that $r = s$ and $s = \ell(\lambda)$ and so we have a partition of the form $(2m-1, 2m-2, \ldots, m)$ and the other is that $r = s+1$ and $s = \ell(\lambda)$ and this is a partition of the form $(2m, 2m-1, \ldots, m+1)$. $\square$

We encourage the reader to take a pencil and draw an arrow between the diagrams of the strict partitions given in the example above to show that the involution works as expected.

What this implies is that we can derive a recurrence which will allow us to calculate the number of partitions of $n$. Notice if we multiply the formula of Euler and the generating function for the number of partitions together we get 1.

$$1 = \prod_{i\geq 1}\frac{1}{1-q^i}\prod_{i\geq 1}(1-q^i) = \left(\sum_{n\geq 0}p(n)q^n\right)\left(1 + \sum_{m\geq 1}(-1)^m\left(q^{\frac{m(3m-1)}{2}} + q^{\frac{m(3m+1)}{2}}\right)\right) \tag{2.5}$$

Now take the coefficient of $q^n$ in both sides of this equation. For $n > 0$ we see that

$$0 = p(n) + \sum_{m\geq 1}(-1)^m\left(p\left(n - m(3m-1)/2\right) + p\left(n - m(3m+1)/2\right)\right) \tag{2.6}$$

where we are assuming here that $p(k) = 0$ if $k$ is a negative number. By isolating $p(n)$ by itself we have the following recurrence.

Corollary 2.2. *$p(n) = 0$ for $n < 0$, $p(0) = 1$ and for $n > 0$ we have*

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) + \cdots$$

EXAMPLE 8.

$$\begin{aligned}
p(1) &= p(0) = 1\\
p(2) &= p(1) + p(0) = 2\\
p(3) &= p(2) + p(1) = 3\\
p(4) &= p(3) + p(2) = 5\\
p(5) &= p(4) + p(3) - p(0) = 7\\
p(6) &= p(5) + p(4) - p(1) = 11\\
p(7) &= p(6) + p(5) - p(2) - p(0) = 15\\
p(8) &= p(7) + p(6) - p(3) - p(1) = 22\\
p(9) &= p(8) + p(7) - p(4) - p(2) = 30\\
p(10) &= p(9) + p(8) - p(5) - p(3) = 42\\
p(11) &= p(10) + p(9) - p(6) - p(4) = 56\\
p(12) &= p(11) + p(10) - p(7) - p(5) + p(0) = 77
\end{aligned}$$

and this agrees with the generating function formula which we computed in an earlier example. This formulation is easier for implementing in an algorithm than the generating function formula.

It will be useful to consider the generating function for all partitions which fit inside of an $n \times k$ rectangle. This generating function $C_q(n,k) = \sum_{\lambda \subseteq (n^k)} q^{|\lambda|}$ must be finite and hence is a polynomial in $q$.

Every partition which fits inside of this $n \times k$ rectangle will have the property that either $\lambda$ has a part of size $n$ or it does not. If $\lambda$ has a part of size $n$ then $\lambda - (n)$ is a partition which fits inside of an $n \times (k-1)$ rectangle. In terms of the generating functions, this translates to the following recursion:

$$C_q(n,k) = \sum_{\substack{\lambda \subseteq (n^k)\\ \lambda_1 = n}} q^{|\lambda|} + \sum_{\substack{\lambda \subseteq (n^k)\\ \lambda_1 \neq n}} q^{|\lambda|} = q^n C_q(n,k-1) + C_q(n-1,k). \tag{2.7}$$

Since the generating function for the partitions which fit inside of an $n \times k$ rectangle is the same as the generating function which fit inside of a $k \times n$ rectangle, it follows that $C_q(n,k) = C_q(k,n)$ and hence using equation (2.7) we also have

$$C_q(n,k) = q^k C_q(n-1,k) + C_q(n,k-1). \tag{2.8}$$

Now set $[n]_q = \frac{1-q^n}{1-q} = \sum_{i=1}^n q^{i-1}$ and $[n]_q! = [n]_q[n-1]_q \cdots [1]_q = \frac{1}{(1-q)^n} \prod_{i=1}^n (1-q^i)$, and

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!} = \frac{\prod_{i=1}^n (1-q^i)}{\prod_{i=1}^k (1-q^i) \prod_{j=1}^{n-k} (1-q^j)}. \tag{2.9}$$

The symbol $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is called the $q$-binomial coefficient or Gaussian polynomial. It is not obvious from this definition that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is even an integer, in fact it seems to be no more than a rational function in $q$, but like the binomial coefficient the denominator always cancels with the numerator. This fact is not transparent until we make the following identification with the generating functions $C_q(n,k)$ which are clearly polynomials in $q$. What is clear however is that if $q = 1$, then $\begin{bmatrix} n \\ k \end{bmatrix}_1 = \binom{n}{k}$ because we can easily check $[n]_1 = n$ and $[n]_1! = n!$. The following statement makes it clear however that $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is a polynomial in $q$ with non-negative integer coefficients.

Exercise 6. (credit D. Stanton) If $n$ and $k$ are relatively prime then

$$\frac{1}{[n]_q} \begin{bmatrix} n \\ k \end{bmatrix}_q$$

is a polynomial in $q$ with positive integer coefficients. Is there a set of combinatorial objects for which this is a generating function?

Proposition 2.3. *For $n, k \geq 0$,*

$$C_q(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$$

Proof. We will establish that the $q$-binomial coefficients satisfy the same recurrence as the polynomials $C_q(n,k)$ and that they agree for the obvious bases cases and hence are the equal. For $n, k \geq 0$,

$$\begin{aligned} \begin{bmatrix} n+k \\ k \end{bmatrix}_q &= \frac{\prod_{i=1}^{n+k}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)\prod_{j=1}^{n}(1-q^j)} \\ &= \frac{(1-q^n+q^n-q^{n+k})\prod_{i=1}^{n+k-1}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)\prod_{j=1}^{n}(1-q^j)} \\ &= \frac{\prod_{i=1}^{n+k-1}(1-q^i)}{\prod_{i=1}^{k}(1-q^i)\prod_{j=1}^{n-1}(1-q^j)} + q^n \frac{\prod_{i=1}^{n+k-1}(1-q^i)}{\prod_{i=1}^{k-1}(1-q^i)\prod_{j=1}^{n}(1-q^j)} \\ &= \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q + q^n \begin{bmatrix} n+k-1 \\ k-1 \end{bmatrix}_q \end{aligned}$$

Now since the generating function $C_q(n,k)$ also satisfies $C_q(n,k) = C_q(n-1,k) + q^n C_q(n,k-1)$ and $C_q(n,0) = C_q(0,k) = \begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} k \\ k \end{bmatrix}_q = 1$ then we have that $C_q(n,k) = \begin{bmatrix} n+k \\ k \end{bmatrix}_q$ for all $n, k \geq 0$. $\square$

Example 9. Calculate the polynomial

$$\begin{aligned} \begin{bmatrix} 5 \\ 2 \end{bmatrix}_q &= \frac{(1-q)(1-q^2)(1-q^3)(1-q^4)(1-q^5)}{(1-q)(1-q^2)(1-q)(1-q^2)(1-q^3)} \\ &= (1+q^2)(1+q+q^2+q^3+q^4) \\ &= 1+q+2q^2+2q^3+2q^4+q^5+q^6 \end{aligned}$$

This polynomial is counting the number of partitions which fit inside of a $2 \times 3$ rectangle. We draw the $2 \times 3$ rectangle below 10 times and fill it with each of the partitions which fit in this rectangle.

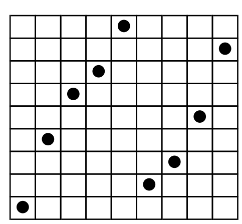
**2.0.1. Grassmannian permutations and partitions.** Recall that a permutation $\sigma \in Sym_n$ is called Grassmannian if it has at exactly one descent. If the descent is a position $k$ then the sequence $\sigma_1, \sigma_2, \ldots, \sigma_k$ is an increasing sequence of integers. In fact, this sequence completely determines the permutation since $\sigma_{k+1}, \sigma_{k+2}, \ldots, \sigma_n$ is also an increasing sequence and consists of the remaining integers. If $\{\sigma_1, \sigma_2, \ldots, \sigma_k\} = \{1, 2, \ldots k\}$ then there cannot be a descent at position $k$ because the next smallest value that can appear is $k+1$. If the first $k$ values are not just the values 1 through $k$ and they are in increasing order and the last $n-k$ values are in increasing order then $\sigma$ is a Grassmannian permutation.

So every subset of size $k$ of the integers $\{1, 2, \ldots, n\}$ except for the set $\{1, 2, \ldots, k\}$ determines a Grassmannian permutation with a descent at position $k$ therefore there are $\binom{n}{k} - 1$ Grassmannian permutations in $Sym_n$ with a descent at position $k$ and

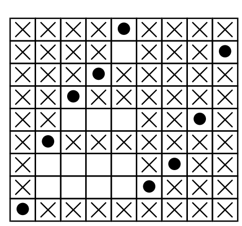
$$\sum_{k=1}^{n-1} \left( \binom{n}{k} - 1 \right) = \left( \sum_{k=0}^{n} \binom{n}{k} \right) - n = 2^{n-1} - n \tag{2.10}$$

Grassmannian permutations in $Sym_n$ with a descent anywhere.

To every $\sigma$ in $Sym_n$ we can associate an $n \times n$ subset of cells with a mark in the $(i, \sigma_i)$ cell for each $1 \leq i \leq n$ (the bottom left hand corner of this diagram will have the coordinate $(1, 1)$). For example, consider the Grassmannian permutation 146792358 which will be associated to the diagram



Now for every point in this grid, place an $\times$ in every cell that lies directly north or directly east of a $\bullet$.



Notice that the cells which do not have either a $\bullet$ or a $\times$ appear as empty rows and these will be decreasing in size. If the permutation is Grassmannian then the sequence of rows which are not filled can be viewed in a natural way as a partition. If the descent in the permutation is at position $k$ then this partition will have width less than or equal to $k$ and height less than or equal to $n-k$ since the rows $\sigma_1, \sigma_2, \ldots, \sigma_k$ will all be filled.

## 2.1. Tableaux

A tableau in the most general sense we will consider will be a function from a subset $S$ of points in $\mathbb{Z} \times \mathbb{Z}$ to a set of labels $M$. If $S = D(\lambda)$ (or $D(\lambda/\mu)$) then $\lambda$ (respectively $\lambda/\mu$) is referred to as the shape of the tableau $T$. $\lambda(T)$ will sometimes be used to denote the domain of the tableau $T$ and in particular when the domain of this function is a partition $D(\lambda)$ or $D(\lambda/\mu)$ it will represent $\lambda$ or $\lambda/\mu$.

A tableau can then be represented with a diagram where $\lambda(T)$ is drawn as the subset of cells on the $\mathbb{Z} \times \mathbb{Z}$ grid just as we drew the diagram for a partition in the previous section, and then the value of $T_s$ is recorded in the cell $s \in \lambda(T)$.

Example 10. Let $T$ be a tableau with $\lambda(T)$ equal to $D(\lambda)$ where $\lambda = (2, 1)$. In addition, say that $T_{(0,0)} = 1$, $T_{(1,0)} = 3$ and $T_{(0,1)} = 2$. The diagram for tableau $T$ will be

$$\begin{array}{|c|c}\hline 2 & \multicolumn{1}{|c}{} \\ \hline 1 & \multicolumn{1}{c|}{3} \\ \hline \end{array}\,.$$

Tableaux are a convenient means of recording information about partitions and it will be important for us to examine certain classes of tableaux. For the moment we will concentrate on tableaux with $\lambda(T)$ is the diagram for a partition or a skew-partition with labels of the tableaux (the range of the tableaux) limited to positive integers.

We will define the content of a tableau to be the sequence which records the number of cells labeled with each value $i$. Let $n_i(T) = \#\{s : T_s = i\}$ and define $\mu(T) = (n_1(T), n_2(T), n_3(T) \ldots)$. As long as $T$ is a finite tableau (we will never consider the infinite case) the sequence $\mu(T)$ will have all but a finite number of entries non-zero and largest label which appears in the tableau $T$ will be denoted $\ell(\mu(T))$.

We will call a tableau *column strict* if $T_{(i,j)} \leq T_{(i+1,j)}$ for all pairs $(i, j), (i + 1, j) \in \lambda(T)$ (the entries are weakly increasing in the rows of the partition) and $T_{(i,j)} < T_{(i,j+1)}$ whenever $(i, j), (i, j + 1) \in \lambda(T)$ (the entries are strictly increasing in the columns of the partition).

A column strict tableau can be seen as a sequence of partitions, if we restrict our attention to the set of cells $\{s : T_s \leq k\}$ (we shall denote $T$ restricted to this domain by $T\big|_{1\ldots k}$) then the shape will always be a partition. Since $T\big|_{1\ldots k-1}$ and $T\big|_{1\ldots k}$ have the shape of partitions and there can be at most one cell labeled by a $k$ in each column from our restrictions on the labeling of cells, then $\lambda(T\big|_{1\ldots k})/\lambda(T\big|_{1\ldots k-1}) \in \mathcal{H}$.

Stated in another way, a column strict tableau $T$ can be thought of as a sequence of partitions. Set $\lambda^{(k)} := \lambda(T\big|_{1\ldots k})$, then

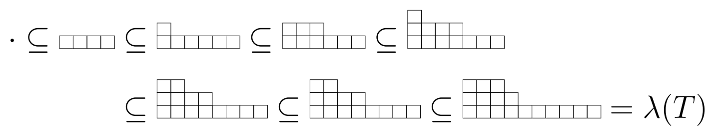
$$\lambda^{(1)} \subseteq \lambda^{(2)} \subseteq \cdots \subseteq \lambda^{(n)} \subseteq \lambda(T) \tag{2.11}$$

with the property that $\lambda^{(k)}/\lambda^{(k-1)} \in \mathcal{H}$.

EXAMPLE 11. Let

$$T = \begin{array}{|c|c|c|c|c|c|c|c|c|c|}\cline{1-3} 4&5&7 \\ \cline{1-4} 2&3&3&4 \\ \hline 1&1&1&1&2&2&4&5&7&7 \\ \hline\end{array}\,.$$

The content of the tableau $T$ is $\mu(T) = (4,3,2,3,2,0,3)$. Notice that the tableau $T$ satisfies the condition of being column strict. One can think of $T$ as an object which conveniently records a sequence of partitions and the condition of being column strict records the fact that each partition differs from the next by a horizontal strip.



Reading word of $T$. Content of $T$. $n_i(T)$ number of occurrences of the

**2.1.1. Crystal Operators.** If the number of cells labeled $i+1$ is greater than the number of cells labeled with an $i$ then an operation that we would like to be able to perform is to change one of the labels of $i+1$ to an $i$ in such a way that the operation is reversible. We will define these operations as crystal operators because they appear in crystal basis theory although they are also useful in the theory of symmetric functions.

Fix an integer $i$ and let $w = R(T)$ be the reading word of the tableau $T$. Consider the sequence of symbols ')' and '(' formed by reading $w$ from left to right and recording a close parenthesis ')' for each $i$ that occurs in $w$ and an open parenthesis for each $i+1$ that occurs in $w$.

We will let $\varepsilon_i(T)$ be the number of open parentheses that are not matched with a close parenthesis that lies to the right and let $\phi_i(T)$ be the number of close parentheses that do not match with an open parenthesis to the left.

Restated in another way, $\phi_i(T) = max\{n_i(w_{1\ldots k}) - n_{i+1}(w_{1\ldots k}) : 1 \le k \le \ell(w)\}$ and $\varepsilon_i(T) = max\{n_{i+1}(w_{k\ldots\ell(w)}) - n_i(w_{k\ldots\ell(w)}) : 1 \le k \le \ell(w)\}$ where $n_i(w)$ is the number of occurrences of $i$ in the word $w = w_1 w_2 \cdots w_{\ell(w)}$ and $w_{a\ldots b}$ is the subword of $w$ equal to $w_a w_{a+1} \cdots w_b$.

EXAMPLE 12.

$$T = \begin{array}{|c|c|c|c|c|c|c|c|}\cline{1-3} 5&6&6 \\ \cline{1-6} 3&4&5&7&7&8 \\ \hline 2&2&3&4&4&5&5&9 \\ \hline 1&1&1&1&2&2&4&8 \\ \hline\end{array}$$

For this tableaux $R(T) = w = 566345778223445591111 2248$. We list below the value of $i$ and the sequence of parentheses that is relevant to the calculation of $\phi_i(T)$ and $\varepsilon_i(T)$.

| | | | |
|---|---|---|---|
| $i = 1$ | (()))(( | $\phi_1(T) = 2$ | $\varepsilon_1(T) = 2$ |
| $i = 2$ | ())()) | $\phi_2(T) = 2$ | $\varepsilon_1(T) = 0$ |
| $i = 3$ | )()((( | $\phi_3(T) = 1$ | $\varepsilon_1(T) = 3$ |
| $i = 4$ | ()()(() | $\phi_4(T) = 0$ | $\varepsilon_1(T) = 1$ |
| $i = 5$ | )(())) | $\phi_5(T) = 2$ | $\varepsilon_1(T) = 0$ |
| $i = 6$ | ))(( | $\phi_6(T) = 2$ | $\varepsilon_1(T) = 2$ |
| $i = 7$ | ))(( | $\phi_7(T) = 2$ | $\varepsilon_1(T) = 2$ |
| $i = 8$ | )() | $\phi_8(T) = 1$ | $\varepsilon_1(T) = 0$ |

We will define the crystal operator $f_i$ to act on tableaux such that $\phi_i(T) > 0$ (and if $\phi_i(T) \le 0$ then $f_iT$ will be undefined or as it is sometimes stated, $f_iT = 0$). Let $s \in \lambda(T)$ be the cell such that $T_s = i$ and $s$ corresponds to the rightmost close parenthesis ')' which is unmatched in the parenthesis sequence. $f_iT$ is the tableau such that $(f_iT)_s = i + 1$ and $(f_iT)_c = T_c$ for $c \in \lambda(T)$ and $c \neq s$.

The crystal operator $e_i$ is the inverse of this operation. It is defined on tableau such that $\varepsilon_i(T) > 0$. If $s \in \lambda(T)$ is the cell of $T$ such that $T_s = i+1$ and it corresponds to the leftmost open parenthesis '(' which is unmatched in the parenthesis sequence then $(e_iT)_s = i$ and $e_iT$ has all of the other entries exactly the same as in $T$ (that is, $(e_iT)_c = T_c$ for $c \in \lambda(T)$ and $c \neq s$).

Example 13. Let

$$T = \begin{array}{|c|c|c|c|} \cline{1-3} 3 & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 3 & 4 \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array}$$

then we have

$$f_1T = \begin{array}{|c|c|c|c|} \cline{1-3} 3 & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 3 & 4 \\ \hline 1 & 1 & 2 & 3 \\ \hline \end{array} \qquad e_2T = \begin{array}{|c|c|c|c|} \cline{1-3} 3 & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 2 & 3 & 4 \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array}$$

$$f_3T = \begin{array}{|c|c|c|c|} \cline{1-3} 3 & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 4 & 4 \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array} \qquad e_4T = \begin{array}{|c|c|c|c|} \cline{1-3} 3 & 4 & 4 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 3 & 4 \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array}$$

$$f_4T = \begin{array}{|c|c|c|c|} \cline{1-3} 3 & 4 & 5 & \multicolumn{1}{c}{} \\ \hline 2 & 3 & 3 & 5 \\ \hline 1 & 1 & 1 & 3 \\ \hline \end{array}$$

while each of $f_2T$, $e_1T$ and $e_3T$ are undefined.

From these definitions we observe the following properties of these operators:

Proposition 2.4. *If $\phi_i(T) > 0$ then $\phi_i(f_iT) = \phi_i(T) - 1$ and $\varepsilon_i(f_iT) = \varepsilon_i(T) + 1$, and in addition*

$$e_if_iT = T. \tag{2.12}$$

*Similarly, if* $\varepsilon_i(T) > 0$ *then* $\phi_i(e_iT) = \phi_i(T) + 1$ *and* $\varepsilon_i(e_iT) = \varepsilon_i(T) - 1$*, and in addition*

$$f_ie_iT = T. \tag{2.13}$$

Proof. Notice that the operation $f_i$ has the effect of changing the corresponding parenthesis sequence of $T$ by changing an unmatched close parenthesis to an unmatched open parenthesis so that $\phi_i(f_iT)$ will be one smaller than $\phi(T)$ and $\varepsilon_i(f_iT)$ will be one larger than $\phi(T)$. In fact, since the rightmost unmatched close parenthesis is changed to the leftmost unmatched open parenthesis, $e_i$ will will have the inverse effect when it acts on $f_iT$.

Similar statements about the operator $e_i$ justify the second part of this proposition. □

The crystal operators can be used to define a symmetric group action on the content of tableaux. Let $s_iT = f_i^{\phi_i(T)-\varepsilon(T)}T$ where we have set $f_i^{-k}T = e_i^kT$ for $k > 0$.

The operators $s_i$ are the generators for a symmetric group action on the content of the tableau $T$, since if $\mu(T) = (a_1, a_2, \ldots, a_n)$ then $\mu(s_iT) = (a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_n)$. We must justify however that the operators $s_i$ which satisfy the Coxeter relations and therefore define a symmetric group action.

Proposition 2.5. *For* $1 \leq i \leq max\{T_s : s \in \lambda(T)\}$,

$$s_i^2T = T,$$
$$s_is_jT = s_js_iT \quad for \quad |i-j| > 1$$
$$s_is_{i+1}s_iT = s_{i+1}s_is_{i+1}T$$

Proof. From the previous proposition we see that $\phi_i(s_iT) = \phi_i(f_i^{\phi_i(T)-\varepsilon_i(T)}T) = \phi_i(T) - (\phi_i(T) - \varepsilon_i(T)) = \varepsilon_i(T)$ and similarly, $\varepsilon(s_iT) = \varepsilon_i(T) + \phi_i(T) - \varepsilon_i(T) = \phi_i(T)$. Therefore to compute

$$\begin{aligned} s_i^2T &= s_i(s_iT) = f_i^{\phi_i(s_iT)-\varepsilon_i(s_iT)}s_iT \\ &= f_i^{\varepsilon_i(T)-\phi_i(T)}f_i^{\phi_i(T)-\varepsilon_i(T)}T = T \end{aligned}$$

There are two cases to show why the last expression is $T$. If $\phi_i(T) - \varepsilon_i(T) > 0$, then $f_i^{\varepsilon_i(T)-\phi_i(T)}f_i^{\phi_i(T)-\varepsilon_i(T)}T = e_i^{\phi_i(T)-\varepsilon_i(T)}f_i^{\phi_i(T)-\varepsilon_i(T)}T = T$. If we have $\phi_i(T) - \varepsilon_i(T) < 0$, then $f_i^{\varepsilon_i(T)-\phi_i(T)}f_i^{\phi_i(T)-\varepsilon_i(T)}T = f_i^{\phi_i(T)-\varepsilon_i(T)}e_i^{\phi_i(T)-\varepsilon_i(T)}T = T$.

Now assume that $|i-j| > 1$. The action of $s_i$ on $T$ changes some of the labels with $i$ to $i+1$ (or the reverse). Since the entries labeled with an $i$ and $i+1$ are the same in both $T$ and $s_jT$, $s_i$ has the same effect on both of these tableaux. Similarly, the entries of $j$ and $j+1$ are the same in both $T$ and $s_iT$ and so $s_j$ has the effect on both these tableaux. For this reason $s_is_jT = s_js_iT$. □

A column strict tableau $T$ will be called *standard* if it is a bijection from the the set $\lambda(T)$ to the labels $\{1, 2, \ldots, |\lambda(T)|\}$.

THEOREM 2.6. *The number of standard tableaux of shape $\lambda$ with $\lambda \vdash n$ is equal to*

$$\frac{n!}{\prod_{s\in\lambda} h_\lambda(s)} \tag{2.14}$$

*where $h_\lambda(s)$ is hook length of the cell $s$ in the partition $\lambda$ (that is, $h_\lambda(i,j) = \lambda_i - i + 1 + \lambda'_j - j$).*

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