Introduction to Symmetric Functions
Chapter 3

Mike Zabrocki

Abstract. A development of the symmetric functions using the plethystic notation.

CHAPTER 2

Symmetric polynomials

Our presentation of the ring of symmetric functions has so far been non-standard and revisionist in the sense that the motivation for defining the ring Λ was historically to study the ring of polynomials which are invariant under the permutation of the variables. In this chapter we consider the relationship between Λ and this ring.

In this section we wish to consider polynomials $f(x_1, x_2, \ldots, x_n) \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ such that $f(x_{\sigma_1}, x_{\sigma_2}, \cdots, x_{\sigma_n}) = f(x_1, x_2, \ldots, x_n)$ for all $\sigma \in Sym_n$. These polynomials form a ring since clearly they are closed under multiplication and contain the element 1 as a unit.

We will denote this ring

$$\Lambda^{X_n} = \{f \in \mathbb{Q}[x_1, x_2, \ldots, x_n] : f(x_1, x_2, \ldots, x_n) = f(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) \text{ for all } \sigma \in Sym_n\} \tag{2.1}$$

Now there is a relationship between Λ and Λ^{X_n} by setting $p_k[X_n] := \sum_{k=1}^{n} x_i^k$ and define a map $\Lambda \longrightarrow \Lambda^{X_n}$ by the linear homomorphism

$$p_\lambda \mapsto p_{\lambda_1}[X_n] p_{\lambda_1}[X_n] \cdots p_{\lambda_{\ell(\lambda)}}[X_n] \tag{2.2}$$

with the natural extension to linear combinations of the p_λ.

In a more general setting we will take the elements Λ to be a set of functors on polynomials $p_k[x_i] = x_i^k$ and $p_k[cE + dF] = cp_k[E] + dp_k[F]$ for $E, F \in \mathbb{Q}[x_1, x_2, \ldots, x_n]$ and coefficients $c, d \in \mathbb{Q}$ then $p_\lambda[E] := p_{\lambda_1}[E] p_{\lambda_2}[E] \cdots p_{\lambda_{\ell(\lambda)}}[E]$. This means that $f \in \Lambda$ will also be a function from $\mathbb{Q}[x_1, x_2, \ldots, x_n]$ to itself with the additional property that if $E \in \Lambda^{X_n} \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$ then $f[E] \in \Lambda^{X_n}$ since if $\sigma E = E$ for a $\sigma \in Sym_n$ then we will also have $\sigma p_k[E] = p_k[\sigma E] = p_k[E]$ (similarly, $p_\lambda[E]$ and $f[E]$ will be invariant under σ).

Example 5. As a sample computation we determine $p_2[X_3], e_2[X_3]$ and $h_2[X_3]$.

$$p_2[x_1 + x_2 + x_3] = x_1^2 + x_2^2 + x_3^2$$

$$\begin{aligned} e_2[x_1 + x_2 + x_3] &= p_{(1,1)}[x_1 + x_2 + x_3]/2 - p_{(2)}[x_1 + x_2 + x_3]/2 \\ &= (x_1 + x_2 + x_3)^2/2 - (x_1^2 + x_2^2 + x_3^2)/2 = x_1x_2 + x_1x_3 + x_2x_3 \end{aligned}$$

$$\begin{aligned} h_2[x_1 + x_2 + x_3] &= p_{(1,1)}[x_1 + x_2 + x_3]/2 + p_{(2)}[x_1 + x_2 + x_3]/2 \\ &= (x_1 + x_2 + x_3)^2/2 + (x_1^2 + x_2^2 + x_3^2)/2 \\ &= x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_1x_3 + x_2x_3 \end{aligned}$$

Example 6. Calculate $e_4[X_3]$.

$$\begin{aligned}
e_4[X_3] &= \frac{p_{(1^4)}[X_3]}{24} - \frac{p_{(211)}[X_3]}{4} + \frac{p_{(22)}[X_3]}{8} + \frac{p_{(31)}[X_3]}{3} - \frac{p_{(4)}[X_3]}{4} \\
&= \frac{(x_1+x_2+x_3)^4}{24} - \frac{(x_1^2+x_2^2+x_3^2)(x_1+x_2+x_3)^2}{4} + \frac{(x_1^2+x_2^2+x_3^2)^2}{8} \\
&\quad + \frac{(x_1^3+x_2^3+x_3^3)(x_1+x_2+x_3)}{3} - \frac{x_1^4+x_2^4+x_3^4}{4} \\
&= 0
\end{aligned}$$

Remark 2. p_k is linear homomorphism because as we stated above $p_k[cE+dF] = cp_k[E] + dp_k[F]$ for $E, F \in \mathbb{Q}[x_1, x_2, \ldots x_n]$ and coefficients $c, d \in \mathbb{Q}$, but this is not true for p_λ in general (e.g. $p_{(2,1)}[x_1+x_2] = (x_1^2+x_2^2)(x_1+x_2) \neq p_{(2,1)}[x_1] + p_{(2,1)}[x_2] = x_1^3 + x_2^3$).

Remark 3. This notation is an extension of the linear homomorphism defined in equation (1.2) where we set $X_n := x_1 + x_2 + \cdots + x_n$.

In addition we will also consider Λ as functors on formal power series. Let $R = \mathbb{Q}[x_1, x_2, x_3, \ldots]$ and $R^{(k)}$ as the subspace of elements in R of degree k. $\widehat{R^{(k)}}$ will denote the completion of this subspace consisting of polynomials and formal series of monomials of degree k. Next define the ring

$$\begin{aligned}
\Lambda^X = \{ & f(x_1, x_2, \ldots) \in \widehat{R^{(k)}} : f(x_1, x_2, \ldots) = \\
& f(x_{\sigma_1}, x_{\sigma_2}, \ldots) \text{ for any permutation } \sigma, k \geq 0 \}.
\end{aligned} \tag{2.3}$$

Just as we had for $\Lambda^{X_n} \subseteq \mathbb{Q}[x_1, x_2, \ldots, x_n]$, $f \in \Lambda$ acts on $E \in \Lambda^X \subseteq \bigoplus_{k\geq 0} \widehat{R^{(k)}}$. Denote $X = x_1 + x_2 + x_3 + \cdots \in \Lambda^X$ so that $p_k[X] = \sum_{i\geq 1} x_i^k$. The operation of setting $x_{n+1} = x_{n+2} = \cdots = 0$ maps X to X_n and Λ^X to Λ^{X_n} such that the following diagram commutes.

$$\begin{tikzcd}
\Lambda \arrow[r, "f \mapsto f[X]"] \arrow[dr, "f \mapsto f[X_n]"'] & \Lambda^X \arrow[d, "x_1 = x_2 = \cdots = 0"] \\
& \Lambda^{X_n}
\end{tikzcd}$$

Remark 4. This notation we have just introduced is quite useful, though there is one pitfall with which the reader should be aware.

Constants and variables are very different.

What we mean by this comment is that in polynomial notation where if $f(x) = x^k$ then $f(2) = 2^k$ and $f(-1) = (-1)^k$ (here constants have the same properties that variables do). In our notation, $p_k[2] = 2$ and $p_k[-1] = -1$ because $p_k[x_i] = x_i^k$, while p_k does nothing when it acts on constants. The reader should spend a few minutes to try to figure out a 'meaning' of $p_k[c]$ or $p_k[cX_n]$ because these **do not** represent $p_k(c)$ and $p_k(cx_1, cx_2, \ldots, cx_n)$ and it is important in doing calculations to be aware of this difference.

One interpretation of the expression $f[n]$ for a positive integer n should be thought of as $f[1+1+\cdots+1]$ and represents the symmetric function $f \in \Lambda$ with each p_k replaced by $x_1^k + x_2^k + \cdots + x_n^k$ and with each of these variables set to 1. We will derive more formulas for our symmetric functions below and using (1.5) we see that $e_k[n] = 0$ for $n = 0, 1, 2, \ldots, k-1$ and $e_k[k] = 1$. $f[c]$ when c is not a non negative integer does not have such a concrete realization and is instead a polynomial interpolation of $f[n]$.

Similarly, we have that $kX_n = X_n + X_n + \cdots + X_n$ (k-times) and hence $f[kX_n]$ represents the symmetric function f evaluated at a set of n variables repeated k times.

In some cases we will use a parameter q in some of our formulas. This parameter will act as a variable and has the property that $p_k[qX] = q^k p_k[X]$. The contrast between variables (q has the same properties as a variable) and constants can be seen here since $p_\lambda[qX] = q^{|\lambda|} p_\lambda[X]$ while for a constant c, $p_\lambda[cX] = c^{\ell(\lambda)} p_\lambda[X]$.

From now on we will denote $X_n := x_1 + x_2 + \cdots + x_n$ so that we have $p_k[X_n] = x_1^k + x_2^k + \cdots + x_n^k$. We are now ready to state and prove the fundamental theorem of symmetric functions which relates the algebra of symmetric functions and the algebra of symmetric polynomials.

We have defined $p_k[X]$ to be $\sum_i x_i^k$ and $p_k[X_n] = \sum_{i=1}^n x_i^k$. Therefore we realize $p_\lambda[X]$ and $p_\lambda[X_n]$ as just products of these elements. This does give us an explicit formula for $h_n[X]$ and $e_n[X]$ because they are just defined to be $h_n[X] = \sum_{\lambda \vdash n} p_\lambda[X]/z_\lambda$ and $e_n[X] = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} p_\lambda[X]/z_\lambda$. This formula is not at all adequate because we need only compute a few of these elements by hand or by computer to realize that the coefficient of any monomial of degree n in $h_n[X]$ is always 1. It is not immediately clear from the definition that the coefficients should even be integers.

Proposition 2.1. *For $n \geq 1$,*

$$h_n[X] = \sum_{i_1 \leq i_2 \leq \cdots \leq i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{2.4}$$

$$e_n[X] = \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{2.5}$$

These results are implied by the following expressions for the generating functions of $h_n[X]$ and $e_n[X]$

$$H(t)[X] = \sum_{n \geq 0} h_n[X] t^n = \prod_i \frac{1}{1 - t x_i} \tag{2.6}$$

$$E(t)[X] = \sum_{n \geq 0} (-1)^n e_n[X] t^n = \prod_i (1 - t x_i) \tag{2.7}$$

PROOF. Consider the generating function $P(t)[X] = \sum_{r\geq 1} p_r[X]/rt^r$. This may be rewritten as

$$
\begin{aligned}
P(t)[X] &= \sum_{r\geq 1} \frac{p_r[X]}{r} t^r = \sum_{r\geq 1}\sum_i \frac{x_i^r}{r} t^r \\
&= \sum_i \sum_{r\geq 1} \frac{x_i^r}{r} t^r = -\sum_i log(1-tx_i) \qquad (2.8)\\
&= log\left(\prod_i \frac{1}{1-tx_i}\right)
\end{aligned}
$$

However we have already seen in equation (1.9) that $H(t)[X] = exp(P(t))[X] = exp(P(t)[X])$ and a similar calculation yields $E(t)[X] = exp(-P(t)[X])$. This demonstrates equations (1.6) and (1.7).

The equation for $h_n[X]$ follows from taking the coefficient of t^n in (1.6). In each monomial there are n variables and each x_i can appear with repetition because $\frac{1}{1-tx_i} = 1+tx_i+(tx_i)^2+(tx_i)^3+\cdots$.

The equation for $e_n[X]$ can be arrived at by taking the coefficient of t^n in (1.7). In each monomial each x_i can appear at most once and each variable that appears contributes a factor of -1 and exactly n variables will appear in each monomial. $\square$

We are now prepared to explicitly state the relationship between Λ and Λ^{X_n}. These spaces are not isomorphic, however the degree k components of each of these spaces is isomorphic as long as $k \leq n$.

PROPOSITION 2.2. *Λ^{X_n} is algebraically generated by the elements $e_1[X_n]$, $e_2[X_n]$, ..., $e_n[X_n]$ and every $f(X_n) \in \Lambda^{X_n}$ is uniquely expressible as a linear combination of the elements $e_\lambda[X_n]$ for λ partitions with parts smaller or equal to n. In particular, the subspace of Λ^{X_n} of degree k is isomorphic to the subspace of degree k elements of Λ under the map which sends $f \mapsto f[X_n]$.*

PROOF. For any $f(x_1, x_2, \cdots, x_n) \in \Lambda^{X_n}$ we note that the coefficient of $x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n}$ and $x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_n^{\lambda_n}$ must be the same if $\alpha = \sigma\lambda$ for some $\sigma \in Sym_n$. Therefore a linear basis of this space is given by the functions

$$
\hat{m}_\lambda^{X_n} = \sum_{\alpha\sim\lambda} x_1^{\alpha_1}x_2^{\alpha_2}\cdots x_n^{\alpha_n} \qquad (2.9)
$$

where λ is a partition with no more than n parts and the sum is over compositions α such that when the entries sorted in decreasing order the resulting partition is equal to λ. The dimension of Λ^{X_n} at degree m is the number of partitions of m with no more than n parts.

Now consider the elements $e_\lambda[X_n] \in \Lambda^{X_n}$ which are symmetric and hence can be expressed in the $\hat{m}_\mu^{X_n}$ basis. Notice that if $\lambda_1 \leq n$ then

$$
e_\lambda[X_n] = \hat{m}_{\lambda'}^{X_n} + \text{ terms containing } \hat{m}_\mu^{X_n} \text{ with } \mu \text{ finer than } \lambda',
$$

otherwise $e_\lambda[X_n] = 0$. Therefore $\{e_\lambda[X_n]\}_{\substack{\lambda\vdash n\\ \lambda_1\leq n}}$ is a basis for Λ^{X_n} and hence Λ^{X_n} is algebraically generated by the elements $e_1[X_n]$, $e_2[X_n]$, …, $e_n[X_n]$. $\square$

Since e_n is a linear combination of the h_λ with $\lambda \vdash n$ then we also have the following corollary.

Corollary 2.3. *Λ^{X_n} is algebraically generated by the elements $h_1[X_n]$, $h_2[X_n]$, …, $h_n[X_n]$ and every $f(X_n) \in \Lambda^{X_n}$ is uniquely expressible as a linear combination of the elements $h_\lambda[X_n]$ for λ partitions with parts smaller or equal to n.*

Similarly, p_n can is a linear combination of the e_λ with $\lambda \vdash n$ and we can also state the previous corollary with $p_i[X_n]$ in place of $h_i[X_n]$. There is however a difference between the $p_i[X_n]$ and the $e_i[X_n]$ or $h_i[X_n]$ since if $f(X_n)$ is a symmetric polynomial with integer coefficients then when it is an expressed as a polynomial in either the $\{e_\lambda[X_n]\}_{\lambda_i\leq n}$ basis or the $\{h_\lambda[X_n]\}_{\lambda_i\leq n}$ basis it will have integer coefficients. This is not true in general of the $\{p_\lambda[X_n]\}_{\lambda_i\leq n}$ basis (see exercise 2.8).

This leads us to what we will refer to as the fundamental theorem of symmetric functions. It says essentially that Λ and Λ^X are isomorphic and as long as the degree of the symmetric functions you are working with is smaller than n then Λ, Λ^{X_n} and Λ^X are all the same.

Theorem 2.4. *For $f, g \in \Lambda$ with deg$(f) \leq n$ and deg$(g) \leq n$, the following are equivalent:*

(1) $f = g$
(2) $f[E] = g[E]$ *for every expression* $E \in \bigoplus_{k\geq 0} R^{(k)}$
(3) $f[X] = g[X]$ *where* $X = x_1 + x_2 + x_3 + \cdots$
(4) $f[X_n] = g[X_n]$ *where* $X_n = x_1 + x_2 + \cdots + x_n$

Proof. The implications (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (4). If $f[X] = g[X]$, then this expression holds independent of the values of x_i. In particular, if we set $x_{n+1} = x_{n+2} = \cdots = 0$, then we see that it must also hold that $f[X_n] = g[X_n]$.

(4) $\Rightarrow$ (1). Assume that $f \neq g$ then $f - g \in \Lambda$ can be expressed in the e_λ basis with at least one coefficient not equal to 0. As we showed in the last proposition, $e_\lambda[X_n]$ is a basis of Λ^{X_n} and hence $f[X_n] - g[X_n]$ is not equal to 0. $\square$

The formulas for $h_k[X]$ and $e_k[X]$ are interesting because it also gives us recurrences in terms of variables. Because any single variable appears in any monomial in $h_k[X]$ with exponent $0, 1, 2, \ldots, k$ and in $e_k[X]$ with exponent either 0 or 1, then we can grade $h_k[X+z]$ or $e_k[X+z]$ depending on the coefficient of z.

$$h_k[X + z] = \sum_{i=0}^{k} z^i h_{k-i}[X] \tag{2.10}$$

$$e_k[X+z] = e_k[X] + ze_{k-1}[X] \tag{2.11}$$

This is useful because it can be used to derive a formula for the homogeneous and elementary symmetric functions at $\frac{1-q^n}{1-q} = 1 + q + q^2 + \cdots + q^{n-1}$

EXAMPLE 7.

$$p_2\left[\frac{1-q^3}{1-q}\right] = p_2[1+q+q^2] = 1 + q^2 + q^4 = \frac{1-q^6}{1-q^2}$$

$$\begin{aligned} e_2\left[\frac{1-q^3}{1-q}\right] &= p_{(1,1)}\left[\frac{1-q^3}{1-q}\right]/2 - p_{(2)}\left[\frac{1-q^3}{1-q}\right]/2 \\ &= \frac{(1-q^3)^2}{2(1-q)^2} - \frac{1-q^6}{2(1-q^2)} \\ &= \frac{(1-2q^3+q^6)(1+q) - (1-q^6)(1-q)}{2(1-q)(1-q^2)} = q\frac{1-q^3}{1-q} \end{aligned}$$

$$\begin{aligned} h_2\left[\frac{1-q^3}{1-q}\right] &= p_{(1,1)}\left[\frac{1-q^3}{1-q}\right]/2 - p_{(2)}\left[\frac{1-q^3}{1-q}\right]/2 \\ &= \frac{(1-q^3)^2}{2(1-q)^2} + \frac{1-q^6}{2(1-q^2)} \\ &= \frac{(1-2q^3+q^6)(1+q) + (1-q^6)(1-q)}{2(1-q)(1-q^2)} = \frac{(1-q^3)(1-q^4)}{(1-q)(1-q^2)} \end{aligned}$$

The use of $H(t)[X]$ as a generating function in which we can take coefficients a useful technique for deriving results in the theory of symmetric functions. To this end we define the special element $\Omega = \sum_{n\geq 0} h_n$ which lies in the completion of Λ. In some sense, Ω is still a generating function for the homogeneous generators like $H(t)$ from the previous section and we have $\Omega = H(1)$, $H(t)[X] = \Omega[tX]$, $E(t)[X] = \Omega[-X]$. This special element has some remarkable properties and we call it the Cauchy element.

PROPOSITION 2.5.

$$\Omega[X+Y] = \Omega[X]\Omega[Y] \tag{2.12}$$

and consequently

$$\Omega[-X] = \Omega[X]^{-1} \tag{2.13}$$

PROOF. Note that since $\Omega = H(1)$ so we have from equation (1.6) that

$$\Omega[X] = \prod_i \frac{1}{1-x_i}. \tag{2.14}$$

Therefore we also have

$$\Omega[X+Y] = \prod_i \frac{1}{1-x_i} \prod_i \frac{1}{1-y_i} = \Omega[X]\Omega[Y] \tag{2.15}$$

Notice that $\Omega[X - X] = 1$ since for $k > 0$, $p_k[X - X] = p_k[X] - p_k[X] = 0$. This implies that the operation of sending f to $f[X - X]$ gives the constant term of f and for Ω this is just 1. Therefore

$$\Omega[X - X] = \Omega[X]\Omega[-X] = 1 \tag{2.16}$$

and so $\Omega[-X] = \Omega[X]^{-1} = \prod_i 1 - x_i$. □

Algebra with infinite series sometimes has unusual consequences and there is one relation involving the element Ω which we shall exploit as often as possible.

PROPOSITION 2.6. *(The phantom relation) Let $\phi(z, u) = \sum_{k\in\mathbb{Z}} z^k u^{-k}$, then for any alphabet X,*

$$\phi(z, u)\Omega[zX]\Omega[-uX] = \phi(z, u) \tag{2.17}$$

What this means is that $\phi(z, u) (1 - \Omega[zX]\Omega[-uX]) = 0$ which is slightly unexpected since elements of our polynomial algebra are not zero divisors, however playing with these infinite series we can arrive at these unusual relations.

PROOF. Take the coefficient of $z^m u^n$ in $\phi(z, u)\Omega[zX]\Omega[-uX]$. This will be equal to

$$\cdots + (-1)^{n-1}h_{m+1}[X]e_{n-1}[X] + (-1)^n h_m[X]e_n[X]$$
$$+ (-1)^{n+1}h_{m-1}[X]e_{n+1}[X] + (-1)^{n+2}h_{m-2}[X]e_{n+2}[X] + \cdots$$

Because $h_n[X] = e_n[X] = 0$ for $n < 0$, this sum is not infinite for each m and n, instead we have that for $m + n > 0$ the sum is equal to

$$\sum_{i=0}^{m+n} (-1)^i h_{m+n-i}[X]e_i[X] = 0. \tag{2.18}$$

If $m + n < 0$, then the sum is 0 simply because all terms are equal to 0, and if $m + n = 0$ then exactly one term is non-zero and it is equal to 1. This means that the coefficient of $z^m u^n$ in $\phi(z, u)\Omega[zX]\Omega[-uX]$ is equal to 1 if $m = -n$ and 0 otherwise and hence the series is equal to $\phi(z, u)$. □

Another remarkable property of the element Ω is that it plays the role of the identity element with respect to the Kronecker product. This means that for any symmetric function $\Omega * f = f * \Omega = f$. The bialgebra structure with product $*$ and coproduct Δ' has an identity element but that element does not lie in the algebra Λ, instead it is in the completion of Λ.

2.1. The monomial symmetric functions

For any given basis $\{a_\lambda\}_\lambda$ of Λ (so far we are essentially working with just the power, homogeneous and elementary) we can ask "what is the set of elements of Λ, $\{b_\lambda\}_\lambda$, such that $\langle a_\lambda, b_\mu\rangle = \delta_{\lambda\mu}$?"

It is a basic fact of linear algebra that the $\{b_\lambda\}_\lambda$ must also be a basis since if there is some linear dependence $\sum_\mu c_\mu b_\mu = 0$ with at least one $c_\lambda \neq 0$, then $\sum_\mu c_\mu b_\mu$ cannot be 0, because then $c_\lambda = \left\langle \sum_\mu c_\mu b_\mu, a_\lambda \right\rangle = \langle 0, a_\lambda \rangle = 0$. Therefore because the set $\{b_\lambda\}_\lambda$ has exactly the number of partitions of n elements at each degree and this set is linearly independent and therefore it spans and is a basis. We will call $\{b_\lambda\}_\lambda$ the basis dual to $\{a_\lambda\}_\lambda$. Notice also that this property is reflexive and $\{a_\lambda\}_\lambda$ is the dual basis to $\{b_\lambda\}_\lambda$ as well.

The bases $\{p_\lambda\}_\lambda$ and $\{p_\lambda/z_\lambda\}_\lambda$ are a pair of dual bases. As we have only just developed two other bases $\{h_\lambda\}_\lambda$ and $\{e_\lambda\}_\lambda$, we should ask what their dual bases are. For this reason we develop the following amazing property of the element Ω. In the expression below, XY is the product of $X = \sum_i x_i$ and $Y = \sum_j y_j$ and hence $XY = \sum_{i,j} x_i y_j$. Therefore by definition, $\Omega[XY] = \prod_{i,j} \frac{1}{1-x_i y_j}$.

Proposition 2.7. *Let $\{a_\lambda\}_\lambda$ be a basis for the symmetric functions then $\{b_\lambda\}_\lambda$ is the dual basis if and only if*

$$\Omega[XY] = \sum_\lambda a_\lambda[X] b_\lambda[Y] \tag{2.19}$$

It follows then that

$$\langle f[X], \Omega[XY] \rangle_X = f[Y] \tag{2.20}$$

Proof. Since $\Omega = \sum_{n\geq 0} \sum_{\lambda \vdash n} p_\lambda/z_\lambda$, then we see that

$$\begin{aligned}
\Omega[XY] &= \sum_{n\geq 0} \sum_{\lambda\vdash n} p_\lambda[X] p_\lambda[Y]/z_\lambda \\
&= \sum_{n\geq 0} \sum_{\lambda\vdash n} \sum_{\mu\vdash n} a_\mu[X] \langle p_\lambda[X], b_\mu[X] \rangle_X p_\lambda[Y]/z_\lambda \\
&= \sum_{n\geq 0} \sum_{\mu\vdash n} a_\mu[X] \sum_{\lambda\vdash n} \langle p_\lambda[X], b_\mu[X] \rangle_X p_\lambda[Y]/z_\lambda \\
&= \sum_{n\geq 0} \sum_{\mu\vdash n} a_\mu[X] b_\mu[Y]
\end{aligned} \tag{2.21}$$

The reverse implication can be seen from the same calculation since

$$\sum_{n\geq 0} \sum_{\lambda\vdash n} p_\lambda[X] p_\lambda[Y]/z_\lambda = \sum_{n\geq 0} \sum_{\lambda\vdash n} \sum_{\mu\vdash n} a_\mu[X] \langle p_\lambda[X], b_\mu[X] \rangle_X p_\lambda[Y]/z_\lambda$$

so we can conclude by taking the coefficient of $p_\lambda[Y]/z_\lambda$ that $p_\lambda = \sum_{\mu\vdash n} a_\mu \langle p_\lambda, b_\mu \rangle$. This means that if $a_\gamma = \sum_\lambda c_{\gamma\lambda} p_\lambda$,

$$a_\gamma = \sum_\lambda c_{\gamma\lambda} p_\lambda = \sum_\lambda c_{\gamma\lambda} \sum_{\mu\vdash n} a_\mu \langle p_\lambda, b_\mu \rangle = \sum_{\mu\vdash n} \langle a_\gamma, b_\mu \rangle a_\mu$$

Since $\{a_\lambda\}_\lambda$ is a basis, we can take the coefficient of a_λ on both sides of this equation and conclude that $\langle a_\gamma, b_\lambda \rangle = \delta_{\lambda\gamma}$ and hence $\{b_\mu\}_\mu$ is the dual basis to $\{a_\lambda\}_\lambda$.

To show the last result, we take for $f \in \Lambda$ and expand it in the power symmetric function basis using some coefficients c_λ, $f = \sum_\lambda c_\lambda p_\lambda$.

$$\begin{aligned}\langle f[X], \Omega[XY]\rangle &= \sum_\lambda c_\lambda \sum_\mu \langle p_\lambda[X], p_\mu[X]/z_\mu\rangle_X \, p_\mu[Y] \\ &= \sum_\lambda c_\lambda p_\lambda[Y] = f[Y]. \end{aligned} \tag{2.22}$$

□

We will define the basis dual to $\{h_\lambda\}_\lambda$ to be the monomial basis $\{m_\lambda\}_\lambda$ and the basis dual to the elementary symmetric functions $\{e_\lambda\}_\lambda$ are usually referred to as the forgotten symmetric functions. The last proposition can be used to find a direct formula for the monomial symmetric functions.

PROPOSITION 2.8. *Let $\lambda \vdash n$,*

$$m_\lambda[X] = \sum_{\alpha\sim\lambda} \prod_i x_i^{\alpha_i} \tag{2.23}$$

where the sum is over all sequences $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ and we have taken $\alpha \sim \lambda$ to mean the number of non-zero entries in α is $\ell(\lambda)$ and if they are sorted in decreasing order the sequence is equal to λ.

PROOF. From the last propostion we know that $\Omega[XY] = \prod_i \frac{1}{1-x_iy_j} = \sum_\lambda h_\lambda[X]m_\lambda[Y]$. Now consider the coefficient of $y_{i_1}^{\alpha_1} y_{i_2}^{\alpha_2} \cdots y_{i_k}^{\alpha_k}$ in $\Omega[XY]$ is

$$\prod_j \frac{1}{1-x_jy_{i_1}}\Big|_{y_{i_1}^{\alpha_1}} \prod_j \frac{1}{1-x_jy_{i_2}}\Big|_{y_{i_2}^{\alpha_2}} \cdots \prod_j \frac{1}{1-x_jy_{i_k}}\Big|_{y_{i_k}^{\alpha_k}} \tag{2.24}$$

which is equal to $h_{\alpha_1}[X]h_{\alpha_2}[X]\cdots h_{\alpha_k}[X]$. Therefore we may realize $\Omega[XY]$ as a sum over all sequences $\alpha = (\alpha_1, \alpha_2, \alpha_3, \ldots)$ with $\alpha_i \geq 0$ and a finite number of non-zero entries we find that

$$\Omega[XY] = \sum_\alpha h_\alpha[X] y^\alpha \tag{2.25}$$

where $h_\alpha = h_\lambda$ if $\alpha \sim \lambda$. This means that $m_\lambda[Y]$ is equal to the coefficient of $h_\lambda[X]$ in the expression above and hence it is equal to $\sum_{\alpha\sim\lambda} y^\alpha$. □

We have defined the monomial symmetric functions $\{m_\lambda\}_\lambda$ as the basis which is dual to the homogeneous basis $\{h_\lambda\}_\lambda$, but now knowing an explicit formula for $m_\lambda[X]$ allows us to easily deduce relations between these bases that are difficult to show otherwise. For instance, we immediately see that $m_{(k)}[X] = p_k[X]$ and $m_{(1^k)}[X] = e_k[X]$ and therefore $m_{(k)} = p_k$ and $m_{(1^k)} = e_k$ and so unlike our other bases m_λ is not generated as a product of elements. We can also see that $h_k = \sum_{\lambda\vdash k} m_\lambda$ either from Proposition 1.1 or by recalling that we have calculated $\langle h_k, h_\lambda\rangle = 1$ as an exercise. The formula for $m_\lambda[X]$ can also be used to derive a combinatorial rule for multiplying two monomial symmetric functions together.

PROPOSITION 2.9. *Let $\lambda \vdash n$ and $\mu \vdash k$*

$$m_\lambda \cdot m_\mu = \sum_{\nu \vdash n+k} r^\nu_{\lambda\mu} m_\nu \tag{2.26}$$

where $r^\nu_{\lambda\mu}$ is the number of pairs of sequences (α, β) with $\alpha_i, \beta_i \geq 0$ where $\alpha \sim \lambda$ and $\beta \sim \mu$ such that $\alpha + \beta = \nu$.

PROOF. This is easily seen in the expansion of $m_\lambda[X]m_\mu[X]$, we need only take the coefficient of x^ν in this expression. There is a contribution of weight 1 to each monomial of type x^ν in the product for each $\alpha \sim \lambda$ and $\beta \sim \mu$ such that $x^\alpha x^\beta = x^\nu$. This is equivalent to the condition that $\alpha + \beta = \nu$. □

EXAMPLE 8. We ask what the coefficient of $m_{(4,3,3)}$ is in $m^2_{(2,2,1)}$. This must be 2 because the only pairs $(\alpha, \beta) \sim ((2,2,1),(2,2,1))$ such that $\alpha + \beta = (4,3,3)$ are $((2,1,2),(2,2,1))$ and $((2,2,1),(2,1,2))$. As a more pictorial way of expressing this result, we may ask how many ways are there of coloring the Young diagram of the partition $(4,3,3)$ with two colors (the first color always lies to the left of the second) such that the horizontal pieces of the first color are of size $(2,2,1)$ and of the second color are of size $(2,2,1)$. The two diagrams are expressed as

Perhaps this combinatorial rule looks familiar since the coefficient of m_ν in $m_\lambda m_\mu$ will be the same as the coefficient of $h_\lambda \otimes h_\mu$ in the expression $\Delta(h_\nu)$ (a fact which we leave to the reader as an exercise). We used the same picture as appeared in chapter 1 to demonstrate exactly that connection.

From this we can arrive at a combinatorial method for computing the scalar product of $\langle h_\lambda, h_\mu \rangle$, $\langle e_\lambda, h_\mu \rangle$ or $\langle p_\lambda, h_\mu \rangle$. The scalar product of $\langle h_n, h_\lambda \rangle$ appears in the exercises of the last section, but the solution relied on the use of the $h_k^\perp$ operators on the h_μ basis. This time we give a proof that relies on a simple observation about symmetric functions given in terms of their variables. Since the homogeneous basis is dual to the monomial basis, we know that $\langle f, h_\mu \rangle$ is the coefficient of $m_\mu[X]$ in $f[X]$.

PROPOSITION 2.10. *For μ a partition of n, $\langle h_\lambda, h_\mu \rangle = A_{\lambda\mu}$ or*

$$h_\mu = \sum_\lambda A_{\lambda\mu} m_\lambda \tag{2.27}$$

where $A_{\lambda\mu}$ is the number of matricies with entries in $\mathbb{N}$ whose column sum is μ and row sum is equal to λ. $\langle h_\lambda, e_\mu \rangle = B_{\lambda\mu}$ or

$$e_\mu = \sum_\lambda B_{\lambda\mu} m_\lambda \tag{2.28}$$

where $B_{\lambda\mu}$ is the number of matricies with entries in $\{0,1\}$ whose column sum is μ and row sum is equal to λ. $\langle h_\lambda, p_\mu\rangle = C_{\lambda\mu}$ or

$$p_\mu = \sum_\lambda C_{\lambda\mu} m_\lambda \tag{2.29}$$

where $C_{\lambda\mu}$ is the number of matricies with entries in $\mathbb{N}$ whose column sum is μ and row sum is equal to λ and there is at most one non-zero entry for each column.

Proof. The coefficient of m_λ in h_μ will be equal to the coefficient of x^λ in $h_\mu[X]$ so we need only count the number of ways the coefficient x^λ may arise in $h_\mu[X]$. Since $h_n[X] = \sum_{|\alpha|=n} x^\alpha$ where the sum is over all compositions α whose entries sum to n, we see that the coefficient of x^λ will be the number of sequences $(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\ell(\mu))})$ such that $x^{\alpha^{(1)}} x^{\alpha^{(2)}} \cdots x^{\alpha^{(\ell(\mu))}} = x^\lambda$ where $\alpha^{(i)}$ is a composition such that $|\alpha^{(i)}| = \mu_i$. We may think of $\alpha^{(i)}$ as a column vector of length $\ell(\mu)$ since the last non-zero entry occurs before $\ell(\mu)$ and the sum of the entries in that column are of course μ_i. The sum of the rows of the matrix $(\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(\ell(\mu))})$ are exactly λ. Since there is a contribution of 1 to the coefficient of x^λ in $h_\mu[X]$ for every such matrix $A_{\lambda\mu}$ is exactly the number of such matricies.

The interpretation for $B_{\lambda\mu}$ and $C_{\lambda\mu}$ are very similar. Since the coefficient of m_λ in e_μ is equal to the coefficient of x^λ in $e_\mu[X]$, we are counting the number of ways that x^λ arises in $e_\mu[X]$. Since $e_n[X] = \sum_{|\alpha|=n} x^\alpha$ with the sum running over all compositions α with entries in $\{0,1\}$. This means that the coefficient of x^λ is again counting the number of matricies whose column sums are μ_i and whose row sums are λ_j, but with the additional restriction that the entries in these matricies are either 0 or 1.

Similarly, the interpretation for $C_{\lambda\mu}$ arises because $p_n[X] = \sum_i x_i^n = \sum_{|\alpha|=n} x^\alpha$, where the sum is over all compositions α with exactly one entry equal to n and the other enetries 0. This implies that the coefficient of x^λ in $p_\mu[X]$ is the number of matricies whose row sum λ_j and whose column sum is μ but at most one entry in the each column is allowed to be non-zero. □

Example 9. It is useful to see this proposition work in an example. We have established the coefficient of $m_{(2,2,2)}$ in $e_{(3,2,1)}$ is $B_{(2,2,2),(3,2,1)}$ and recall that this will also be the scalar product $\left\langle h_{(2,2,2)}, e_{(3,2,1)}\right\rangle$. One method for computing this scalar product could be to compute this directly by expanding both expressions in the power basis and using the definition of the scalar product, but we can also construct each of the $\{0,1\}$ matricies with row sums equal to $(2,2,2)$ and column sums equal to $(3,2,1)$. By exhaustively writing them out, we see there are exactly 3. They are

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}. \tag{2.30}$$

This is not however the only combinatorial interpretation possible for these coefficients. We can provide another set of objects with the same number of elements as an interpretation that is perhaps easier to visualize. These interpretations are not significantly different however since there is a direct bijection between the elements in one set and the other.

Corollary 2.11. *Alternatively, $A_{\lambda\mu}$ is the number of ways of filling the the Young diagram for the partition λ with μ_1 1s, μ_2 2s, etc. that are weakly increasing in the rows and there is no restriction on the relationship between the values in the columns.*

$B_{\lambda\mu}$ is the number of ways of filling the the Young diagram for the partition λ with μ_1 1s, μ_2 2s, etc. that are strictly increasing in the rows and there is no restriction on the relationship between the values in the columns.

$C_{\lambda\mu}$ is the number of ways of filling the the Young diagram for the partition λ with μ_1 1s, μ_2 2s, etc. that are weakly increasing in the rows and we require that all cells with label i must lie in the same row.

Example 10. We again compute the same coefficient $\langle h_{(2,2,2)}, e_{(3,2,1)} \rangle$ by giving the possible fillings of the Young diagram of shape $(2,2,2)$ with 3 1s, 2 2s and 1 3.

$$\begin{array}{|c|c|}\hline 1&3\\\hline 1&2\\\hline 1&2\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 1&3\\\hline 1&2\\\hline\end{array}\quad\begin{array}{|c|c|}\hline 1&2\\\hline 1&2\\\hline 1&3\\\hline\end{array}\tag{2.31}$$

Notice the relationship between these tabloid and to the matricies listed in the previous example. A bijection between the two sets of objects should be clear.

Example 11. To expand $h_{(3,2)}$ in terms of the monomial symmetric functions we examine all possible ways of filling the Young diagrams for the partitions of size 5 with 3 1s and 2 2s such that the entries are weakly increasing in the rows. We draw all of the possible tabloid as follows:

$$\begin{array}{ccccc}1&1&1&2&2\end{array}\quad\begin{array}{cccc}2&&&\\1&1&1&2\end{array}\quad\begin{array}{cccc}1&&&\\1&1&2&2\end{array}\quad\begin{array}{ccc}2&2&\\1&1&1\end{array}\quad\begin{array}{ccc}1&2&\\1&1&2\end{array}\quad\begin{array}{ccc}1&1&\\1&2&2\end{array}$$

$$\begin{array}{ccc}2&&\\2&&\\1&1&1\end{array}\quad\begin{array}{ccc}2&&\\1&&\\1&1&2\end{array}\quad\begin{array}{ccc}1&&\\2&&\\1&1&2\end{array}\quad\begin{array}{ccc}1&&\\1&&\\1&2&2\end{array}\quad\begin{array}{cc}2&\\2&\\1&\\1&1\end{array}\quad\begin{array}{cc}2&\\1&\\2&\\1&1\end{array}$$

$$\begin{array}{cc}2&\\1&2\\1&1\end{array}\quad\begin{array}{cc}2&\\1&1\\1&2\end{array}\quad\begin{array}{cc}1&\\1&1\\2&2\end{array}\quad\begin{array}{cc}1&\\2&2\\1&1\end{array}\quad\begin{array}{cc}1&\\1&2\\1&2\end{array}$$

$$\begin{array}{cc}1&\\2&\\2&\\1&1\end{array}\quad\begin{array}{cc}2&\\1&\\1&\\1&2\end{array}\quad\begin{array}{cc}1&\\2&\\1&\\1&2\end{array}\quad\begin{array}{cc}1&\\1&\\2&\\1&2\end{array}\quad\begin{array}{cc}1&\\1&\\1&\\2&2\end{array}$$

There are also $\binom{5}{3} = 10$ ways of filling the Young diagram of shape (11111) in this manner. This implies that

$$h_{(32)} = m_{(5)} + 2m_{(41)} + 3m_{(32)} + 4m_{(311)} + 5m_{(221)} + 7m_{(2111)} + 10m_{(11111)}.$$

Example 12. To express $p_{(321)}$ in the monomial basis we need only examine partitions of size 6 such that the partition are sums of the parts of $(3,2,1)$. We list all of the possible tabloid for the partitions $(3,2,1), (3,3), (4,2), (5,1), (6)$.

$$\begin{array}{ccc}3&&\\2&2&\\1&1&1\end{array}\quad\begin{array}{ccc}2&2&3\\1&1&1\end{array}\quad\begin{array}{ccc}1&1&1\\2&2&3\end{array}\quad\begin{array}{cccc}2&2&&\\1&1&1&3\end{array}\quad\begin{array}{ccccc}3&&&&\\1&1&1&2&2\end{array}\quad\begin{array}{cccccc}1&1&1&2&2&3\end{array}$$

This implies that $p_{(3,2,1)}$ has the expansion

$$p_{(321)} = m_{(321)} + 2m_{(33)} + m_{(42)} + m_{(51)} + m_{(6)}.$$

The forgotten symmetric functions are the basis which is dual to the elementary symmetric functions. Because we have that $\langle \omega f, \omega g \rangle = \langle f, g \rangle$ where f and g are elements of Λ, we have that $\langle e_\mu, \omega(m_\lambda) \rangle = \langle h_\mu, m_\lambda \rangle = \delta_{\lambda\mu}$ and so $\{\omega(m_\lambda)\}_\lambda$ is the basis which is dual to the $\{e_\lambda\}_\lambda$. We name this basis $f_\lambda := \omega(m_\lambda)$, the forgotten symmetric functions.

Just by the definition, we have the following formulas:

$$\Omega[XY] = \sum_\lambda e_\lambda[X] f_\lambda[Y] \tag{2.32}$$

For a partition μ of n,

$$\begin{aligned} e_\mu &= \sum_\lambda A_{\lambda\mu} f_\lambda \\ h_\mu &= \sum_\lambda B_{\lambda\mu} f_\lambda \\ p_\mu &= (-1)^{|\mu| - \ell(\mu)} \sum_\lambda C_{\lambda\mu} f_\lambda \end{aligned} \tag{2.33}$$

where the coefficients $A_{\lambda\mu}, B_{\lambda\mu}$ and $C_{\lambda,\mu}$ are given in Proposition 1.10.

$$f_\lambda \cdot f_\mu = \sum_{\nu \vdash |\lambda| + |\mu|} r^\nu_{\lambda\mu} f_\nu \tag{2.34}$$

where the coefficients $r^\nu_{\lambda\mu}$ are given in Proposition 1.9.

To expand the monomial symmetric functions in terms of the forgotten basis we have the usual expansion

$$m_\lambda = \sum_{\mu \vdash |\lambda|} \langle m_\lambda, e_\mu \rangle f_\mu \tag{2.35}$$

Notice also that if we expand the elementary basis in terms of the homogeneous basis we see the same coefficients

$$e_\mu = \sum_{\lambda \vdash |\mu|} \langle m_\lambda, e_\mu \rangle h_\lambda \tag{2.36}$$

That is if we define the coefficient $D_{\lambda\mu} := \langle m_\lambda, e_\mu \rangle$, then we have the symmetry $m_\lambda = \sum_\mu D_{\lambda\mu} f_\mu$ and $e_\lambda = \sum_\mu D_{\mu\lambda} h_\mu$.

We have not exploited completely the formulas that we have derived for the coefficient of e_n in the h_λ basis. We can use this formula to give a rough combinatorial formula for the coefficient of h_λ in the expansion of e_μ. By the solution to exercise 1.9, we know that

$$e_n = \sum_{\lambda \vdash n} (-1)^{n-\ell(\lambda)} \frac{\ell(\lambda)!}{\prod_i m_i(\lambda)!} h_\lambda. \tag{2.37}$$

This implies that

$$e_\mu = \prod_{i=1}^{\ell(\mu)} \sum_{\nu \vdash \mu_i} (-1)^{\mu_i - \ell(\nu)} \frac{\ell(\nu)!}{\prod_i m_i(\nu)!} h_\nu. \tag{2.38}$$

Now in order to take the coefficient of h_λ in this equation we say that there will be a contribution to the coefficient of h_λ for every sequence of partitions $(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))})$ such that $\nu^{(1)} \uplus \nu^{(2)} \uplus \ldots \uplus \nu^{(\ell(\mu))} = \lambda$ and $\nu^{(i)} \vdash \mu_i$. We can see immediately that the sign of $e_\mu\Big|_{h_\lambda}$ is simply $(-1)^{n-\ell(\lambda)}$ because the sign of each contribution to the coefficient of h_λ in the product is always $\prod_{i=1}^{\ell(\mu)} (-1)^{\mu_i - \ell(\nu^{(i)})} = (-1)^{|\mu| - \ell(\lambda)}$. The contribution for each sequence $(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))})$ which satisfies these conditions is

$$\prod_{i=1}^{\ell(\mu)} \frac{\ell(\nu^{(i)})!}{\prod_{j=1}^{\nu_1^{(i)}} m_j(\nu^{(i)})!} \tag{2.39}$$

This implies the following proposition.

PROPOSITION 2.12.

$$\begin{aligned} e_\mu &= \textstyle\sum_{\lambda \vdash |\mu|} D_{\lambda\mu} h_\lambda & \qquad h_\mu &= \textstyle\sum_{\lambda \vdash |\mu|} D_{\lambda\mu} e_\lambda \\ m_\mu &= \textstyle\sum_{\lambda \vdash |\mu|} D_{\mu\lambda} f_\lambda & \qquad f_\mu &= \textstyle\sum_{\lambda \vdash |\mu|} D_{\mu\lambda} m_\lambda \end{aligned} \tag{2.40}$$

where

$$D_{\lambda\mu} = (-1)^{|\mu| - \ell(\lambda)} \sum_{(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))})} \prod_{i=1}^{\ell(\mu)} \frac{\ell(\nu^{(i)})!}{\prod_{j=1}^{\nu_1^{(i)}} m_j(\nu^{(i)})!} \tag{2.41}$$

is the sum over all sequences of partitions such that $\nu^{(1)} \uplus \nu^{(2)} \uplus \ldots \uplus \nu^{(\ell(\mu))} = \lambda$ *and* $\nu^{(i)} \vdash \mu_i$.

The formula for $D_{\lambda\mu}$ is very similar to that for the coefficient of h_λ in p_μ, an explicit formula for these coefficients was calculated in exercise (1.11). These coefficients also appear in the expansion of the the monomial and forgotten bases in terms of the power basis.

PROPOSITION 2.13.

$$p_\mu = \sum_{\lambda \vdash |\mu|} E_{\lambda\mu} h_\lambda \qquad\qquad p_\mu = (-1)^{|\mu| - \ell(\mu)} \sum_{\lambda \vdash |\mu|} E_{\lambda\mu} e_\lambda$$

$$m_\mu = \sum_{\lambda \vdash |\mu|} E_{\mu\lambda} p_\lambda / z_\lambda \qquad\qquad f_\mu = \sum_{\lambda \vdash |\mu|} (-1)^{|\lambda| - \ell(\lambda)} E_{\mu\lambda} p_\lambda / z_\lambda$$

with

$$E_{\lambda\mu} = (-1)^{\ell(\lambda) - \ell(\mu)} \sum_{(\nu^{(1)}, \nu^{(2)}, \ldots, \nu^{(\ell(\mu))})} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i (\ell(\nu^{(i)}) - 1)!}{\prod_{j \geq 1} m_j(\nu^{(i)})!}$$

where the sum is over all sequences of partitions with $\nu^{(i)} \vdash \mu_i$ *and* $\nu^{(1)} \uplus \nu^{(2)} \uplus \cdots \uplus \nu^{(\ell(\mu))} = \lambda$.

PROOF. The justification of the expansion of p_μ in the homogeneous basis is exercise (1.10) and (1.11) from the previous chapter. An application of ω to this formula justifies the expansion of p_μ in the elementary basis.

Now to show the expansion of the monomial basis in the power basis, we recall that for $\mu \vdash n$

$$\begin{aligned} m_\mu &= \sum_{\nu \vdash n} \langle m_\mu, p_\lambda \rangle \, p_\lambda / z_\lambda \\ &= \sum_{\nu \vdash n} \left\langle m_\mu, \sum_{\nu \vdash n} E_{\nu\lambda} h_\nu \right\rangle p_\lambda / z_\lambda \\ &= \sum_{\nu \vdash n} E_{\mu\lambda} p_\lambda / z_\lambda \end{aligned}$$

The expansion of the forgotten basis in terms of the power basis also follows by an application of the involution ω on the previous formula. □

We should note that the only one of these formulas where the coefficients all have the same sign is the expansion of f_μ in the power sum basis. The coefficients of p_λ will be positive (or 0) if $|\mu| + \ell(\mu)$ is even and negative otherwise.

Example 13. We will give an example of a computation of the expansion of $e_{(421)}$ and $p_{(421)}$ in the homogeneous basis. The sum is over the same set of objects so it is easy to both of the computations at the same time.

Each of the following pictures represents how to divide the partition λ in to sub-partitions $(\nu^{(1)}, \nu^{(2)}, \nu^{(3)})$ such that $\nu^{(1)} \vdash 4$, $\nu^{(2)} \vdash 2$ and $\nu^{(3)} \vdash 1$. For each of these tableaux we will count each with a weight.

$$\ytableaushort{3,2,2,1,1,1,1} \quad \ytableaushort{3,2,2,1,1,11} \quad \ytableaushort{3,1,1,1,1,22} \quad \ytableaushort{3,2,2,1,111} \quad \ytableaushort{3,2,2,11,11} \quad \ytableaushort{3,1,1,22,11}$$

$$\ytableaushort{3,22,11,11} \quad \ytableaushort{3,1,22,111} \quad \ytableaushort{3,2,2,1111} \quad \ytableaushort{3,22,1111}$$

Now in order to expand $e_{(421)}$ in terms of h_λ we count each of these tableaux with the weight

$$(-1)^{7-\ell(\lambda)} \prod_{i=1}^{3} \frac{\ell(\nu^{(i)})!}{\prod_{j \geq 1} m_j(\nu^{(i)})!},$$

where $\nu^{(i)}$ is the partition whose rows are labeled with i. This implies that

$$\begin{aligned} e_{(421)} = h_{(421)} &- h_{(41^3)} - 2\ h_{(3211)} + 2\ h_{(31^4)} - h_{(2^31)} + 4\ h_{(221^3)} \\ &- 4\ h_{(21^5)} + h_{(1^7)} \end{aligned}$$

In order to expand $p_{(421)}$ in terms of h_λ, we weight each of the tableaux listed above with the coefficient

$$(-1)^{3-\ell(\lambda)} 4 \cdot 2 \cdot 1 \prod_{i=1}^{3} \frac{(\ell(\nu^{(i)}) - 1)!}{\prod_{j \geq 1} m_j(\nu^{(i)})!}.$$

$$\begin{aligned} p_{(421)} = 8\ h_{(421)} &- 4\ h_{(41^3)} - 8\ h_{(3211)} + 4\ h_{(31^4)} - 4\ h_{(2^31)} + 10\ h_{(221^3)} \\ &- 6\ h_{(21^5)} + h_{(1^7)} \end{aligned}$$

From the previous discussion we have determined a formula or combinatorial interpretation for the coefficient of every one of the 5 bases in every other one of the 5 bases. The coefficients $A_{\lambda\mu}, B_{\lambda\mu}$ and $C_{\lambda\mu}$ can be found in Proposition 1.10 and Corollary 1.11. A formula/combinatorial interpretation for the coefficients $D_{\lambda\mu}$ is in Proposition 1.12 and the preceding discussion and $E_{\lambda\mu}$ is in Proposition 1.13. From these definitions we have the following table for the coefficient of b_λ in a_μ where a_μ represents the entry down the left side of the table and b_λ represents the label across the top of the table.

	p_λ	h_λ	e_λ	m_λ	f_λ
p_μ	$\delta_{\lambda\mu}$	$E_{\lambda\mu}$	$(-1)^{\vert\mu\vert-\ell(\mu)}E_{\lambda\mu}$	$C_{\lambda\mu}$	$(-1)^{\vert\mu\vert-\ell(\mu)}C_{\lambda\mu}$
h_μ	$C_{\mu\lambda}/z_\lambda$	$\delta_{\lambda\mu}$	$D_{\lambda\mu}$	$A_{\lambda\mu}$	$B_{\lambda\mu}$
e_μ	$(-1)^{\vert\lambda\vert-\ell(\lambda)}C_{\mu\lambda}/z_\lambda$	$D_{\lambda\mu}$	$\delta_{\lambda\mu}$	$B_{\lambda\mu}$	$A_{\lambda\mu}$
m_μ	$E_{\mu\lambda}/z_\lambda$	$F_{\lambda\mu}$	$G_{\lambda\mu}$	$\delta_{\lambda\mu}$	$D_{\mu\lambda}$
f_μ	$(-1)^{\vert\lambda\vert-\ell(\lambda)}E_{\mu\lambda}/z_\lambda$	$G_{\lambda\mu}$	$F_{\lambda\mu}$	$D_{\mu\lambda}$	$\delta_{\lambda\mu}$

This leaves two coefficients that we have not yet determined, $F_{\lambda\mu}$ and $G_{\lambda\mu}$, we leave it as an exercise to determine some sort of formula for these coefficients. For a more detailed account of the combinatorial interpretation of change of basis coefficients see [**3**].

2.2. Algebra operations and sets of variables

The notation that we have introduced allows us to express the operations of our Hopf algebra and bialgebra that we have already discussed as addition, subtraction and multiplication of alphabets.

Notice that $p_k[X+Y] = p_k[X] + p_k[Y]$ while $\Delta(p_k) = p_k \otimes 1 + 1 \otimes p_k$. Because we have defined $p_\lambda[X+Y] = \prod_i p_{\lambda_i}[X+Y] = \prod_i (p_{\lambda_i}[X] + p_{\lambda_i}[Y])$. It then follows that the coefficient of $p_\mu[X]p_\nu[Y]$ in $p_\lambda[X+Y]$ is equal to the coefficient of $p_\mu \otimes p_\nu$ in $\Delta(p_\lambda)$. More generally it follows that that if $\Delta(f) = \sum_i f_i \otimes g_i$, then $f[X+Y] = \sum_i f_i[X]g_i[Y]$..

This means that there is a clear isomorphism between $\Lambda \otimes \Lambda$ and Λ^{X+Y}, that is, a basis element $p_\lambda[X]p_\mu[Y]$ of Λ^{X+Y} is isomorphic to the basis element $p_\lambda \otimes p_\mu \in \Lambda \otimes \Lambda$. More generally, an element of Λ^{X+Y} $\sum_i f_i[X]g_i[Y]$ is isomorphic to $\sum_i f_i \otimes g_i$. That means that addition of two sets of variables encodes the coproduct Δ which we express in the following proposition.

Proposition 2.14. *Given $f \in \Lambda$ such that $\Delta(f)$ is given by $\Delta(f) = \sum_i f_i \otimes g_i$, then*

$$f[X+Y] = \sum_i f_i[X]g_i[Y]. \tag{2.42}$$

Moreover, if $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$ are dual bases for Λ, then for all $f \in \Lambda$

$$f[X+Y] = \sum_{k\geq 0}\sum_{\lambda\vdash k} (a_\lambda^\perp f)[X]b_\lambda[Y] \tag{2.43}$$

PROOF. We know that for $f = p_\lambda$ we see that

$$
\begin{aligned}
p_\lambda[X+Y] &= \prod_{i=1}^{\ell(\lambda)} (p_{\lambda_i}[X] + p_{\lambda_i}[Y]) \\
&= \sum_{S\subseteq\{1,\ldots,\ell(\lambda)\}} \prod_{i\in S} p_{\lambda_i}[X] \prod_{i\notin S} p_{\lambda_i}[Y] \qquad (2.44)\\
&= \sum_{k\geq 0}\sum_{\mu\vdash k} \left(\frac{p_\mu^\perp}{z_\mu} p_\lambda\right)[X] p_\mu[Y]
\end{aligned}
$$

Now for $f = \sum_\lambda c_\lambda p_\lambda$ we have that

$$
\begin{aligned}
f[X+Y] &= \sum_\lambda c_\lambda p_\lambda[X+Y] \\
&= \sum_\lambda c_\lambda \sum_{k\geq 0}\sum_{\mu\vdash k} \left(\frac{p_\mu^\perp}{z_\mu} p_\lambda\right)[X] p_\mu[Y] \qquad (2.45)\\
&= \sum_{k\geq 0}\sum_{\mu\vdash k}\sum_\lambda c_\lambda \left(\frac{p_\mu^\perp}{z_\mu} p_\lambda\right)[X] p_\mu[Y] \\
&= \sum_{k\geq 0}\sum_{\mu\vdash k} \left(\frac{p_\mu^\perp}{z_\mu} f\right)[X] p_\mu[Y]
\end{aligned}
$$

Now that we know that (1.43) holds for $a_\lambda = p_\lambda/z_\lambda$ and $b_\lambda = p_\lambda$, we show more generally that

$$
\begin{aligned}
f[X+Y] &= \sum_{k\geq 0}\sum_{\mu\vdash k}\sum_{\nu\vdash k} \left(\frac{p_\mu^\perp}{z_\mu} f\right)[X] < p_\mu, a_\nu > b_\nu[Y] \\
&= \sum_{k\geq 0}\sum_{\nu\vdash k}\sum_{\mu\vdash k} \left(< p_\mu, a_\nu > \frac{p_\mu^\perp}{z_\mu} f\right)[X] b_\nu[Y] \qquad (2.46)\\
&= \sum_{k\geq 0}\sum_{\nu\vdash k} \left(a_\nu^\perp f\right)[X] b_\nu[Y].
\end{aligned}
$$

□

Subtraction of variables is equivalent to addition of a negative set of variables and a symmetric function evaluated at a negative set of variables is equal to an application of the antipode map.

PROPOSITION 2.15. *For $f \in \Lambda$ such that f is homogeneous of degree k*

$$
f[-X] = S(f)[X] = (-1)^k \omega(f)[X] \qquad (2.47)
$$

PROOF. Recall that $S(p_k) = -p_k = (-1)^k\omega(p_k)$ and $S(p_\lambda) = (-1)^{\ell(\lambda)} p_\lambda = (-1)^{|\lambda|-\ell(\lambda)} p_\lambda$. We also have $p_\lambda[-X] = (-1)^{\ell(\lambda)} p_\lambda[X] = S(p_\lambda)[X]$. This means that for $f = \sum_{\lambda\vdash k} c_\lambda p_\lambda$ for some coefficients c_λ,

$$f[-X] = \sum_{\lambda \vdash k} c_\lambda p_\lambda[-X] = \sum_{\lambda \vdash k} c_\lambda S(p_\lambda)[X] = S(f)[X].$$

□

REMARK 5. We can encode the antipode map S which is much like the involution ω, but not quite the same since it is off by -1 raised to the degree of the symmetric function it is acting on. It is possible to introduce notation which eliminates the sign. To this end one may introduce a variable q and at the end of the calculation set $q = -1$. This will be denoted ϵ. That is, for $f \in \Lambda$, f homogeneous of degree k

$$f[-\epsilon X] = (-1)^k \omega(f)[qX]\Big|_{q=-1} = (-1)^k q^k \omega(f)[X]\Big|_{q=-1} = \omega(f)[X].$$

Note that ϵ is very different from -1 because it is a variable. This notation is more useful when working with non-homogeneous symmetric functions since it allows us to encode the involution ω without referring to the degree of the symmetric function. We will not use this notation here.

Using this notation we can see multiplication as an operation that maps Λ^{X+Y} to Λ^X by setting the Y variables equal to the X variables. This can be seen since $\mu(f \otimes g) = f \cdot g$ while at the same time $f[X]g[Y]\Big|_{Y=X} = f[X]g[X] = (f \cdot g)[X]$. This means that we will be representing multiplication by the symbol $\Big|_{Y=X}$ which acts on the expression that lies to the left of this symbol by changing the Y variables to the X variables.

This notation implies that we have already computed such expressions as $h_m[X+Y]$ since we have already computed that $\Delta(h_m) = \sum_i h_i \otimes h_{m-i}$ in Proposition 1.15. Using the previous remark, this means $h_m[X+Y] = \sum_{i=0}^m h_i[X]h_{m-i}[Y]$, and similarly that $e_m[X+Y] = \sum_{i=0}^m e_i[X]e_{m-i}[Y]$.

Notice that for any $f, g \in \Lambda$,

$$\langle g[X+Y], f[Y]\rangle = (f^\perp g)[X]. \tag{2.48}$$

This is perhaps an unusual means for computing $f^\perp g$, but it is important interpretation of the operation $f[X+Y]$. We may also use this operation to compute specific operators $f^\perp$.

PROPOSITION 2.16. *For $k \in \mathbb{Z}$ and $f \in \Lambda$,*

$$h_k^\perp f[X] = f[X+z]\Big|_{z^k} \tag{2.49}$$

$$e_k^\perp f[X] = f[X-z]\Big|_{z^k}(-1)^k \tag{2.50}$$

PROOF. The first identity follows from equation (1.43),

$$f[X+z] = \sum_\lambda \left(h_\lambda^\perp f\right)[X] m_\lambda[z] \tag{2.51}$$

all terms of this sum are 0 unless λ has exactly one part. The coefficient of z^k in this equation will be $(h_k^\perp f)[X]$.

This same argument using the dual bases $\{e_\lambda\}_\lambda$ and $\{f_\lambda\}_\lambda$ and the relation $f_\lambda[-z] = (-1)^{|\lambda|} m_\lambda[z]$ shows that

$$f[X-z] = \sum_\lambda \left(e_\lambda^\perp f\right)[X] f_\lambda[-z] = \sum_\lambda (-1)^{|\lambda|} \left(e_\lambda^\perp f\right)[X] m_\lambda[z]. \tag{2.52}$$

The coefficient of z^k in this equation will be $(-1)^k(e_k^\perp f)[X]$. □

This last proposition implies that

$$f[X+z] = \sum_{k \geq 0} z^k (h_k^\perp f)[X] = \Omega[zX]^\perp f[X] \tag{2.53}$$

where by $\Omega[zX]^\perp$ is the operator which is dual to multiplication by the series $\Omega[zX]$ with respect to the scalar product in the X variables. As a manipulation, we know then for any two symmetric functions $f, g \in \Lambda$,

$$\langle f[X], g[X+z] \rangle = \langle \Omega[zX] f[X], g[X] \rangle\,. \tag{2.54}$$

The defining relation of the antipode, $\mu \circ (id \otimes S) \circ \Delta = u \circ \varepsilon$ can easily be seen in this notation.

$$\begin{aligned} \mu \circ (id \otimes S) \circ \Delta(f)[X] &= \mu \circ (id^X S^Y) f[X+Y] \\ &= \mu f[X-Y] = f[X-X] = f[0] \end{aligned} \tag{2.55}$$

This means that $u \circ \varepsilon(f)[X] = f[0]$, but this was something that we already knew since $u \circ \varepsilon(f) = f\Big|_{p_k=0}$.

Similarly, because we have that $p_k[XY] = p_k[X]p_k[Y]$ then just by the definition $p_\lambda[XY] = p_\lambda[X]p_\lambda[Y]$. Comparing this to $\Delta'(p_\lambda) = p_\lambda \otimes p_\lambda$, it follows that again the coefficient of $p_\mu \otimes p_\nu$ in $\Delta'(p_\lambda)$ is equal to the coefficient of $p_\mu[X]p_\nu[Y]$ in $p_\lambda[XY]$ (that is, they are both equal to $\delta_{\mu\lambda}\delta_{\nu\lambda}$). More generally this shows that if $\Delta'(f) = \sum_i f_i \otimes g_i$, then $f[XY] = \sum_i f_i[X] g_i[Y]$. This means that the coproduct Δ' is encoded in the multiplication of two sets of variables in the sense of the following propostion.

PROPOSITION 2.17. *For any symmetric function f inΛ, if $\Delta'(f) = \sum_i f_i \otimes g_i$, then*

$$f[XY] = \sum_i f_i[X] g_i[Y]. \tag{2.56}$$

Moreover for any dual bases $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$, we have that

$$f[XY] = \sum_{k \geq 0} \sum_{\lambda \vdash k} (a_\lambda * f)[X] b_\lambda[Y]. \tag{2.57}$$

PROOF. We have yet to show (1.57) which we know will hold for $f = p_\lambda$ $a_\mu = p_\mu/z_\mu$ and $b_\mu = p_\mu$ since

$$p_\lambda[XY] = p_\lambda[X]p_\lambda[Y] = \sum_\mu \left(\frac{p_\mu}{z_\mu} * p_\lambda\right)[X]p_\mu[Y] \tag{2.58}$$

since all but one term of this sum is equal to 0. More generally, if $f = \sum_\lambda c_\lambda p_\lambda$ and $\{a_\lambda\}_\lambda$ and $\{b_\lambda\}_\lambda$ are any pair of dual bases, then

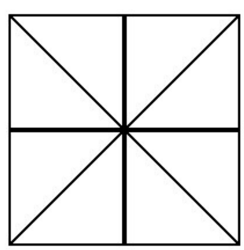
$$\begin{aligned} f[XY] &= \sum_\lambda \sum_\mu c_\lambda \left(\frac{p_\mu}{z_\mu} * p_\lambda\right)[X]p_\mu[Y] \\ &= \sum_\lambda \sum_\mu \sum_\nu c_\lambda \left(\frac{p_\mu}{z_\mu} * p_\lambda\right)[X]\langle p_\mu, a_\nu\rangle b_\nu[Y] \\ &= \sum_\lambda \sum_\nu \sum_\mu c_\lambda \left(\langle p_\mu, a_\nu\rangle \frac{p_\mu}{z_\mu} * p_\lambda\right)[X]b_\nu[Y] \\ &= \sum_\nu (a_\nu * f)[X]b_\nu[Y] \end{aligned} \tag{2.59}$$

□

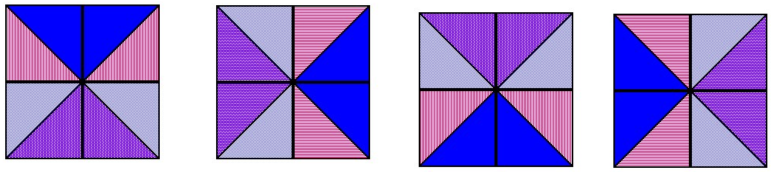
2.3. Application: Coloring enumeration

Starting with a figure like the one below, we can ask how many distinct ways there are of coloring the regions of the figure with k colors. If the figure is fixed in place and not allowed to move the answer is simply k^8 since there are 8 regions and each region can be colored independently with one of k different colors.

FIGURE 14. A square figure with 8 regions.

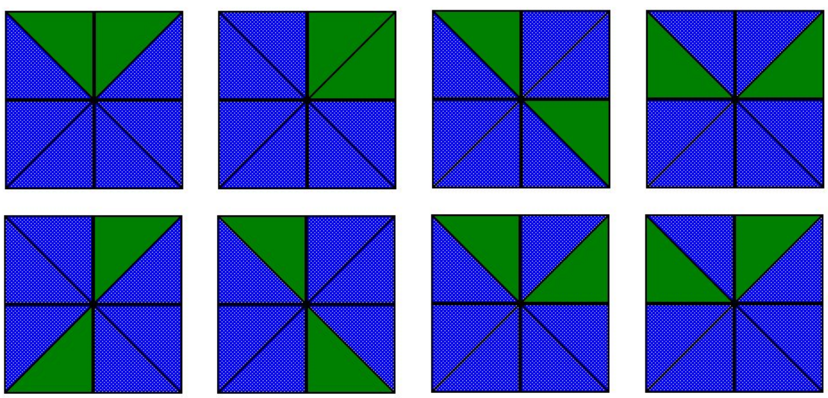


Now allow a group of isometries to act on the figure and say that two colorings are equal if there is some group element that transforms one to the other. With this condition there must be fewer than k^8 colorings because some of the k^8 colorings are now the equal. For instance, if we allow the group of four rotations to act on this figure then the following 4 colorings will be equivalent.



In general, to count the colorings we cannot simply divide by four since not all colorings will have the same order in their symmetry. In fact we will show that the number of colorings of figure 10 with k colors and this group of isometries acting on it is $\frac{1}{4}(k^8 + 2k^2 + k^4)$, something that is difficult to do with a simple counting argument.

A more general problem is to count the number of ways of coloring a figure like the one in figure 10 with a_1 regions blue, a_2 regions red, a_3 regions green, etc. such that $a_1+a_2+a_3+\cdots$ is equal to the number of regions. For example to color figure 10 with 6 blue regions and 2 green regions it is not difficult to determine that there are 8 distinct colorings like those given below. We wish to approach this problem in a general setting and give a formula for these enumerations.



It turns out that both of these types of problems can be solved using symmetric functions and the link between enumerating colorings and symmetric polynomials is a generating function for the number of colorings. Notice that if we assign a monomial weight $w(c)$ to each possible coloring c of a figure where $w(c)$ is x_1 raised to the number times the first color appears, x_2 to the number of times the second color appears, etc., then the sum over all possible colorings of $w(c)$ will be symmetric in the variables x_i and hence this expression will be a symmetric polynomial.

To begin we introduce some notation. A *group action* of a group G on a set R of regions to be colored satisfying the following properties:

(1) For the identity element $e \in G$, $e(r) = r$ for $r \in R$.
(2) For $g_1, g_2 \in G$, $g_1(g_2(r)) = (g_1 \cdot g_2)(r)$ for all $r \in R$.

Now the elements $g \in G$ act on the set R and permute the elements. The cycle type of G when it acts on R will be denoted by $\lambda_R(g)$ and is equal to the cycle type of the following permutation:

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ g(r_1) & g(r_2) & \cdots & g(r_n) \end{pmatrix}.$$

Define the cycle index symmetric function of a group G acting on a set R will be denoted

$$\mathcal{C}_R^G = \frac{1}{|G|} \sum_{g \in G} p_{\lambda_R(g)} \tag{2.60}$$

and it is a symmetric function of degree equal to the number of elements of R.

Example 15. Let $G = C_4$ be the cyclic group of permutations which rotate figure 10. That is, there are four elements of this group $C_4 = \{e, r, r^2, r^3\}$ where $r^4 = e$ and r acting on this figure is a rotation by 90 degrees. There are 8 regions of this figure to be colored and $\lambda_R(e) = (1^8)$, $\lambda_R(r) = \lambda_R(r^3) = (4,4)$, and $\lambda_R(r^2) = (2,2,2,2)$. Therefore

$$\mathcal{C}_R^{C_4} = \frac{1}{4}\left(p_1^8 + 2p_4^2 + p_2^4\right).$$

Example 16. Let $G = D_4$ be the group of rotations and reflections acting on figure 10. This time there are 8 elements in the group, $D_4 = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$ where $r^4 = e$ and $s^2 = e$ and $sr = r^3 s$. r acting on this figure is again a rotation by 90, s will be a flip across the horizontal passing through the center of the figure. We have already calculated the cycle type of the elements e, r, r^2, r^3 and we also have $\lambda_R(s) = \lambda_R(sr) = \lambda_R(sr^2) = \lambda_R(sr^3) = (2,2,2,2)$. Therefore

$$\mathcal{C}_R^{D_4} = \frac{1}{8}\left(p_1^8 + 2p_4^2 + 5p_2^4\right).$$

If we have a group action of G on a set X then the *orbit* of an element $x \in X$ is the set

$$Orbit(G; x) = \{g(x) : g \in G\}.$$

We also define the *stablilizer* of an element x to be the set

$$Stab(G; x) = \{g \in G : g(x) = x\}.$$

If $g \in Stab(G;x)$ then $g^{-1}(x) = g^{-1}(g(x)) = (g^{-1}g)(x) = x$ and hence $g^{-1} \in Stab(G;x)$. If $g, h \in Stab(G;x)$ then $(gh)(x) = g(h(x)) = x$ and so $gh \in Stab(G;x)$ and hence $Stab(G;x)$ is a subgroup of G. We also have that for $g \in G$, if $g(x) = y$ then $g^{-1}(y) = g^{-1}(g(x)) = (g^{-1}g)(x) = x$. The orbit and the stabilizer of an element x are related by the set $Orbit(G;x)$ is isomorphic to the set of left cosets of $Stab(G;x)$. We show this in the following lemma.

Proposition 2.18. *For $x \in X$, we have $Orbit(G;x)$ is isomorphic to the set of left cosets of $Stab(G;x)$ in G. This implies*

$$|Orbit(G;x)|\,|Stab(G;x)| = |G|.$$

PROOF. The correspondence between the orbit and the cosets of the stabilizer simply sends the element gx to the coset $g \cdot Stab(G;x)$. We show

$$\begin{aligned} g \cdot Stab(G;x) = h \cdot Stab(G;x) &\Leftrightarrow (h^{-1}g) \cdot Stab(G;x) = Stab(G;x) \\ &\Leftrightarrow h^{-1}g \in Stab(G;x) \\ &\Leftrightarrow (h^{-1}g)(x) = x \\ &\Leftrightarrow g(x) = h(x). \end{aligned}$$

This shows that the map is onto since for $g \in G$, $g(x) = g_i(x)$ for some representative element $g_i(x) \in Orbit(G;x)$ and $g \cdot Stab(G;x) = g_i \cdot Stab(G;x)$. This also implies that the map is one-to-one since $g(x) = h(x)$ implies $g \cdot Stab(G;x) = h \cdot Stab(G;x)$. Since the cosets of $Stab(G;x)$ partition the group G into equal parts and every element is in exactly one coset, we know that

$$|Orbit(G;x)| = \#\text{ cosets of } Stab(G;x) \text{ in } G = \frac{|G|}{|Stab(G;x)|}.$$

□

If we have a finite set X and G acts on X by permuting the elements, then there are a finite number of sets $Orbit(G;c)$ and every element of X will be in exactly one of the orbits (the set of orbits forms a partition of the set X). Let m be the number of elements in $\{Orbit(G;c) : c \in X\}$ and let $c_1, c_2, \ldots, c_m$ be representative elements of this set of orbits so that every $c \in X$ is in exactly one set $Orbit(G;c_i)$.

PROPOSITION 2.19. *If $c \in Orbit(G;d)$, then $Orbit(G;c) = Orbit(G;d)$ and consequently $|Stab(G;c)| = |Stab(G;d)|$.*

PROOF. If $c \in Orbit(G;d)$ then $c = g(d)$ for some $g \in G$. This mean that

$$Orbit(G;c) = \{h(c) : h \in G\} = \{h(g(d)) : h \in G\} = Orbit(G;d).$$

□

For a finite set R of regions, a *coloring* of R with k colors is a map $c : R \to \{1, 2, \ldots k\}$. If g acts on R then the definition of g on the coloring c is $g(c)(r) = c(g(r))$ for $r \in R$ (in other words, a group acts on a coloring of the regions by permuting the regions).

A coloring c will be invariant under the action of a group element g if every r in the same orbit of g is colored with the same value, that is, if $g(r) = r'$ then $c(r) = c(r')$. This implies that each cycle in the permutation

$$\begin{pmatrix} r_1 & r_2 & \cdots & r_n \\ g(r_1) & g(r_2) & \cdots & g(r_n) \end{pmatrix}$$

may be assigned a value independently. The number of colorings invariant under the action of the element $g \in G$ is equal to k raised to the number of cycles in this permutation or $k^{\ell(\lambda_R(g))} = p_{\lambda_R(g)}[k]$. That is, we have

$$p_{\lambda_R(g)}[k] = \#\{c : R \to \{1, 2, \ldots k\} | g(c) = c\}.$$

This leads us to our first enumeration formula:

THEOREM 2.20. *(Burnside's formula) Let R be a set of regions to color and X be the set of colorings with k colors. Say that the set $\{Orbit(G;c) : c \in X\}$ has order m, then*

$$\mathcal{C}_R^G[k] = m.$$

PROOF. Let m be the number of elements in $\{Orbit(G;c) : c \in X\}$ and $c_1, c_2, \ldots, c_m$ be a set of representative elements so that every $c \in X$ is in exactly one of the sets $Orbit(G;c_i)$.

$$\begin{aligned}
\mathcal{C}_R^G[k] &= \frac{1}{|G|}\sum_{g\in G} p_{\lambda_R(g)}[k] \\
&= \frac{1}{|G|}\sum_{g\in G}\sum_{\substack{c\in X\\ g(c)=c}} 1 \\
&= \frac{1}{|G|}\sum_{c\in X}\sum_{\substack{g\in G\\ g(c)=c}} 1 \\
&= \frac{1}{|G|}\sum_{i=1}^{m}\sum_{c\in Orbit(G;c_i)} |Stab(G;c)| \qquad (2.61)\\
&= \frac{1}{|G|}\sum_{i=1}^{m}\sum_{c\in Orbit(G;c_i)} |Stab(G;c_i)| \\
&= \frac{1}{|G|}\sum_{i=1}^{m} |Orbit(G;c_i)||Stab(G;c_i)| \\
&= \frac{1}{|G|}\sum_{i=1}^{m} |G| = m.
\end{aligned}$$

□

EXAMPLE 17. We computed $\mathcal{C}_R^{C_4}$ and $\mathcal{C}_R^{D_4}$ for R equal to the regions in figure 10. Theorem 1.20 says that the number of distinct ways of coloring this figure k colors when the group C_4 acts on the figure is $\frac{1}{4}(k^8 + 2k^2 + k^4)$ and when D_4 acts on the figure is $\frac{1}{8}(k^8 + 2k^2 + 5k^4)$.

Let us consider a figure which we can verify by hand very easily that this formula does in fact work as advertised.

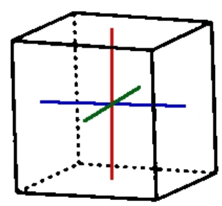
EXAMPLE 18. How many ways are there of coloring a cube with 2 colors such that two coloring are considered to be equal if they look the same by a rotation? The natural group of isometries which acts on this object is the group of rotations which permute the faces, edges and vertices of the cube.

It helps to have a cube to look at, if you can find a die or a Rubik's cube the calculations that we are about to do will seem easier.

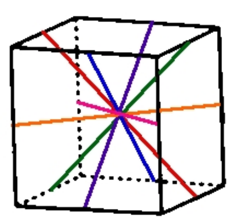
We note that the symmetry group of rotations the cube has 24 elements in it. This is easy to see because any rotation of the cube moves one of 6 faces to the face which is on top and then there are 4 choices for the face which is in front. The question is, what are these 24 group elements which act on the cube?

The first element to consider is the identity element. There is only 1 of these.

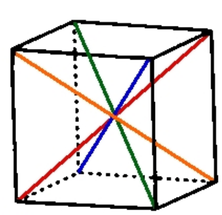
Next, there are 9 rotations of the cube with two faces left fixed. In the figure below this corresponds to the rotations about one of the three lines which pass through the center of the cube perpendicular to exactly two faces.



There are also 6 rotations of the cube by 180 degrees that leave two edges fixed. This corresponds to a rotation around one of the the six lines in the figure below that pass through the center of the cube and are perpendicular to two edges.



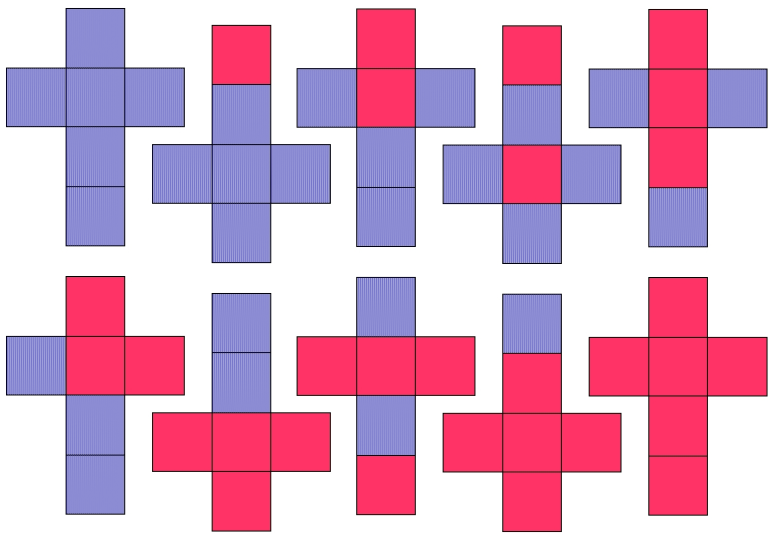
Finally there are 8 rotations by 120 degrees or 240 degrees around two opposite corners of the cube. This will be a rotation around one of the four lines in the figure below that pass through the center of the cube and connect two of the corners.



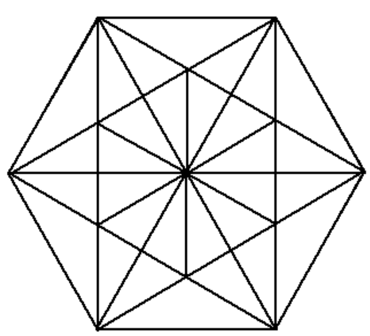
Let G be this group of isometries and R be the 6 faces of the cube. Calculating $\mathcal{C}_R^G$ is as easy as looking at each of the four types of permutations listed above. By looking at a cube and noticing what the cycle type of each of these permutations of the faces is we calculate

$$\frac{1}{24}\left(p_1^6 + 6p_4p_1^2 + 3p_2^2p_1^2 + 6p_2^3 + 8p_3^2\right).$$

Therefore if we wish to know the number of distinct ways of coloring the cube with 2 colors, it will be $\frac{1}{24}\left(2^6 + 6 \cdot 2^3 + 3 \cdot 2^4 + 6 \cdot 2^3 + 8 \cdot 2^2\right) = 10$. This number is something we can easily determine by exhaustively listing the 10 possible colorings. In the figure below we have folded out flat the 6 faces of the cube and colored them either red or blue.



Example 19. How many distinct ways are there of coloring the following figure with 3 colors where the group that acts on it is generated by a reflection across the vertical line and a rotation by 60 degrees?



The group acting on this figure is the dihedral group of order 12 since it is generated by two elements $x, y \in D_6$ satisfying $x^6 = y^2 = e$ and $xy = yx^5$. This means that $D_6 = \{e, x, x^2, x^3, x^4, x^5, y, yx, yx^2, yx^3, yx^4, yx^5\}$. There are 30 regions in the figure and are permuted by the elements of this group. We compute the following cycle structures of these group elements on the regions of the figure.

$$\lambda_R(e) = (1^{30})$$

$$\lambda_R(x) = \lambda_R(x^5) = (6^5)$$

$$\lambda_R(x^2) = \lambda_R(x^4) = (3^{10})$$

$$\lambda_R(x^3) = \lambda_R(yx) = \lambda_R(yx^3) = \lambda(yx^5) = (2^{15})$$

$$\lambda_R(y) = \lambda_R(yx^2) = \lambda(yx^4) = (2^{14}, 1^2)$$

These calculations determine

$$\mathcal{C}_R^{D_6} = \frac{1}{12}\left(p_1^{30} + 2p_6^5 + 2p_3^{10} + 4p_2^{15} + 3p_2^{14}p_1^2\right)$$

It follows then that the number of ways of coloring this figure with 3 colors is equal to

$$\mathcal{C}_R^{D_6}[3] = \frac{1}{12}\left(3^{30} + 2\cdot 3^5 + 2\cdot 3^{10} + 4\cdot 3^{15} + 3\cdot 3^{16}\right) = 17157609895752.$$

Example 20. A simple graph is a set of vertices V (which we shall take as the set $= \{1, 2, \ldots n\}$) together with a set of edges $E \subseteq \{\{u,v\} : u, v \in V \text{ and } u \neq v\}$. The symmetric group acts on V by permuting the vertex set and on edges by $\sigma\{u,v\} = \{\sigma(u), \sigma(v)\}$. In our context, two graphs (V,E) and (V,E') will be isomorphic if there is a permutation $\sigma \in Sym_n$ such that $\sigma E = E'$.

A simple graph is represented by a set of labeled points and a line between two points u and v if $\{u,v\}$ is an edge in E.

We wish to count non-isomorphic simple graphs and this can be done by thinking of a graph (V,E) as a coloring of the two element subsets $R_n = \{\{i,j\} : 1 \leq i < j \leq n\}$ with two colors say white and black. The black edges will represent those which are in E and the white ones will represent those that are not in E.

This means that $\mathcal{C}_{R_n}^{Sym_n}$ will be a symmetric function of degree $\binom{n}{2}$. Lets compute the number of non-isomorphic graphs on with 2, 3, 4 and 5 vertices. To determine these symmetric functions we need to determine the action for each σ on the two element subsets but it is only necessary to look at one of each cycle type (we will list calculate λ_R of each of the group elements listed in cycle notation).

For $n = 2$, $\lambda_{R_2}((1)(2)) = (1)$ and $\lambda_{R_2}((12)) = (1)$. Therefore $\mathcal{C}_{R_2}^{Sym_2} = \frac{1}{2}(p_1 + p_1) = p_1$.

For $n = 3$, there are 3 two element subsets and $\lambda_{R_3}((1)(2)(3)) = (1,1,1)$, $\lambda_{R_3}((12)(3)) = (2,1)$ and $\lambda_{R_3}((123)) = (3)$. We have, $\mathcal{C}_{R_3}^{Sym_3} = \frac{1}{6}\left(p_{(1^3)} + 3p_{(2,1)} + 2p_{(3)}\right) = h_3$.

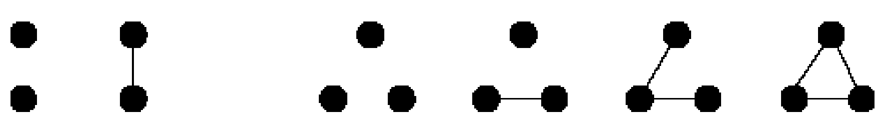
For $n = 4$, there are 6 two element subsets. $\lambda_{R_4}((1)(2)(3)(4)) = (1^6)$, $\lambda_{R_4}((12)(3)(4)) = (2,2,1,1)$, $\lambda_{R_4}((12)(34)) = (2,2,1,1)$, $\lambda_{R_4}((123)(4)) = (3,3)$, $\lambda_{R_4}((1234)) = (4,2)$. The cycle index symmetric function is

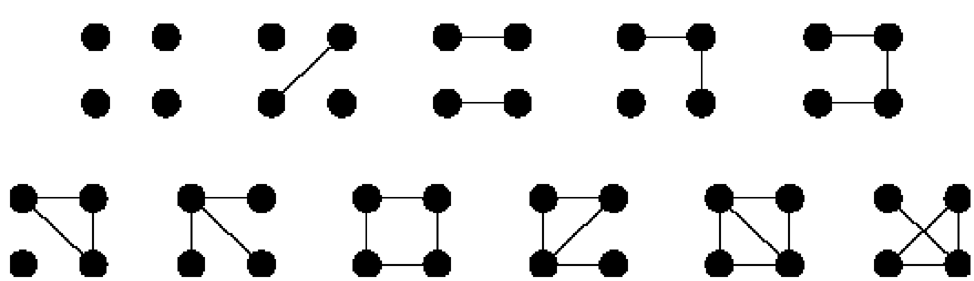
$$\mathcal{C}_{R_4}^{Sym_4} = \frac{1}{24}\left(p_{(1^6)} + 6p_{(2,2,1,1)} + 3p_{(2,2,1,1)} + 8p_{(3,3)} + 6p_{(4,2)}\right).$$

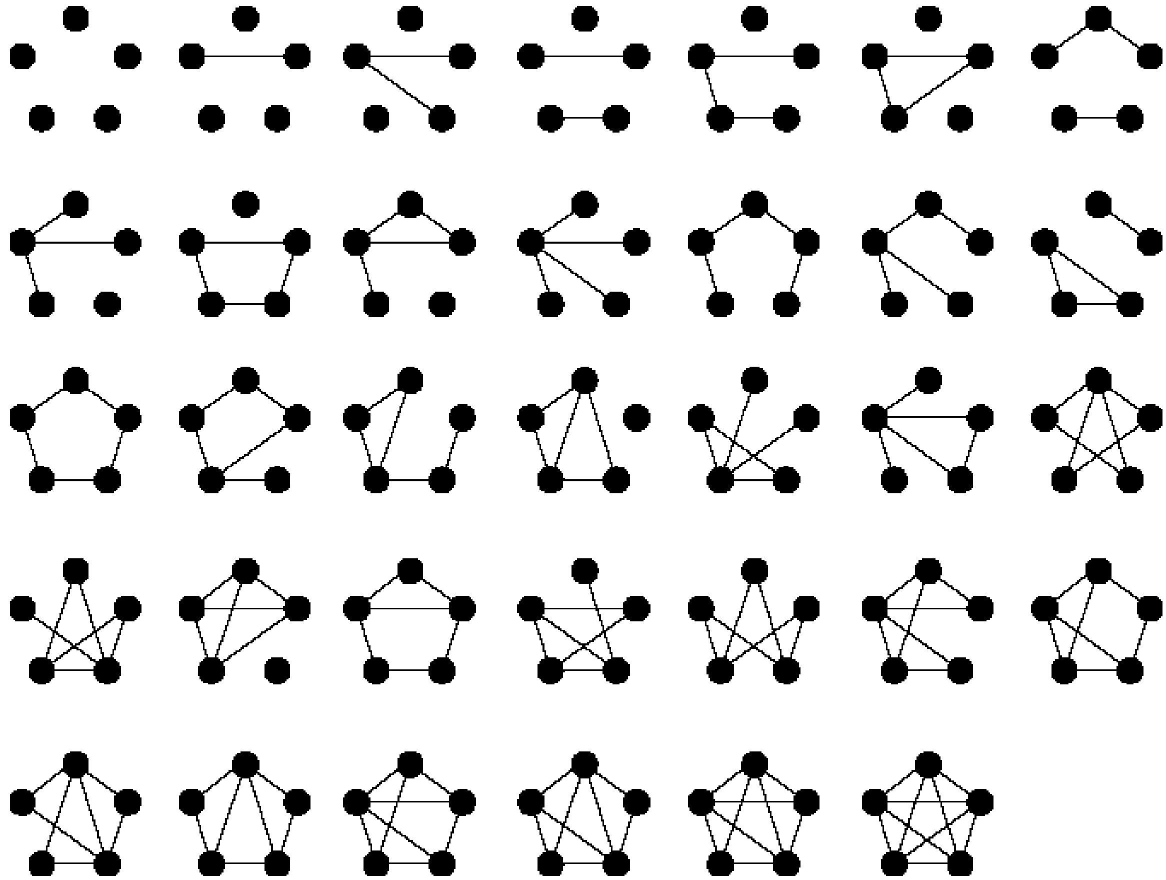
For $n = 5$ there are 10 two element subsets. For the most part, the order of the subset $\{u, v\}$ will be the l.c.m. of the length of the cycle that u is in and the length of the cycle that v is in (the exception being the sets $\{1, 3\}$ and $\{2, 4\}$ under the action of the element $(1234)(5)$). $\lambda_{R_5}((1)(2)(3)(4)(5)) = (1^{10})$, $\lambda_{R_5}((12)(3)(4)(5)) = (2^3, 1^4)$, $\lambda_{R_5}((12)(34)(5)) = (2^4, 1, 1)$, $\lambda_{R_5}((123)(4)(5)) = (3^3, 1)$, $\lambda_{R_5}((123)(45)) = (6, 3, 1)$, $\lambda_{R_5}((1234)(5)) = (4, 4, 2)$, $\lambda_{R_5}((12345)) = (5, 5)$.

$$\mathcal{C}_{R_5}^{Sym_5} = \frac{1}{120}\left(p_{(1^{10})} + 10p_{(2^3,1^4)} + 15p_{(2^4,1,1)} + 20p_{(3^3,1)} + 20p_{(6,3,1)} + 30p_{(4,4,2)} + 24p_{(5,5)}\right).$$

Now that we have the cycle index symmetric functions, it is quite easy to determine the number of non-isomorphic simple graphs there are, we just evaluate them at the value 2. This means that there are $\mathcal{C}_{R_2}^{Sym_2}[2] = 2$ graphs on two vertices, $\mathcal{C}_{R_3}^{Sym_3}[2] = 4$ graphs on three vertices, $\mathcal{C}_{R_4}^{Sym_4}[2] = 11$ graphs on four vertices, and $\mathcal{C}_{R_5}^{Sym_5}[2] = 34$ graphs on five vertices. Each of these values are not too difficult to verify by hand so we draw the sets of graphs below to see that they agree with the theory.







We turn our attention now to a more specific type of enumeration problem, that of counting the number of ways of coloring a figure using a prescribed number of colors. We gave an example of such a question when above when we posed the question how many ways the figure 10 could be colored with 6 blue regions and 2 green regions.

Let G be a group with an action on a set of regions R. For any coloring $c : R \to \mathbb{N}$, we call the *weight function* the map w with $w(c) = \prod_{i=1}^{|R|} x_{c(r_i)}$ where the r_i are the elements of R listed in some order (since our variables commute this expression is independent of the order). The weight function sends a coloring to a monomial in $\mathbb{Q}[x_1, x_2, x_3, \ldots]$ The generating function $\sum_c w(c)$ where the sum is over all distinct colorings of R under the group G is called the *pattern inventory.*

The pattern inventory is a generating function which contains all the information necessary to count the number of patterns with given number of colors appearing, we need only take a coefficient in this generating function of $x_1^{a_1} x_2^{a_2} x_3^{a_3} \cdots$ to find the number of colorings with color 1 appearing a_1 times, color 2 appearing a_2 times, color 3 appearing a_3 times, etc.

This generating function also encodes the total number of colorings using k colors. We can recover the information by setting $x_1 = x_2 = \cdots = x_k = 1$ and $x_{k+1} = x_{k+2} = \cdots = 0$. This give a clue to identifying the formula for the pattern inventory.

Theorem 2.21. *(Pòlya's theorem) Let G be a group which acts on a set R and let C be a set of colorings mapping R to the set $\mathbb{N}$ which are all distinct under the action of G.*

$$\mathcal{C}_R^G[X] = \sum_{c \in C} w(c).$$

Proof. This means □

2.4. Exercises:

(1) Show
$$e_k\left[[n]_q\right] = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$$
where $[n]_q = \frac{1-q^n}{1-q}$, $[n]_q! = [n]_q[n-1]_q \cdots [1]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[n-k]_q![k]_q!}$.

(2) Show
$$h_k\left[[n]_q\right] = \begin{bmatrix} n+k-1 \\ k \end{bmatrix}_q.$$

(3) Show
$$p_k\left[[n]_q\right] = \frac{[nk]_q}{[k]_q}.$$

(4) Show directly (without appealing to a formula which we have derived for these values) that the coefficient of m_ν in $m_\lambda m_\mu$ will be the same as the coefficient of $h_\lambda \otimes h_\mu$ in the expression $\Delta(h_\nu)$.

(5) Show that $e_k[X - z] = \sum_{i=0}^{k} (-z)^i e_{k-i}[X]$ and $h_k[X - z] = h_k[X] - zh_k[X]$.

(6) Show $\Omega * f = f * \Omega = f$ for all $f \in \Lambda$.

(7) Show that $\Omega\left[\frac{x-y}{1-q}\right] = \prod_{i \geq 0} \frac{1-xq^i}{1-yq^i}$.

(8) Show that if $f(X_n)$ is in the linear span of the symmetric polynomials $m_\lambda[X_n]$ over $\mathbb{Z}$ then when it is expressed in the $\{e_\lambda[X_n]\}_{\lambda_1 \leq n}$ basis it has coefficients in $\mathbb{Z}$ and when it is expressed in the $\{h_\lambda[X_n]\}_{\lambda_1 \leq n}$ basis it has coefficients in $\mathbb{Z}$. Show that in general if it is expressed in the $\{p_\lambda[X_n]\}_{\lambda_1 \leq n}$ basis the coefficients will be in $\mathbb{Q}$.

(9) Show that
$$\langle h_\mu, e_{\lambda'} \rangle \quad \begin{cases} > 0 & \text{if } \mu < \lambda \\ = 1 & \text{if } \lambda = \mu \\ = 0 & \text{otherwise.} \end{cases}$$

(10) Determine some sort of formula for the coefficient of h_λ in m_μ and the coefficient of e_λ in m_μ in terms of the other coefficients which we have already determined a formula.

Bibliography

[1] G. Andrews, Euler's Pentagonal Number Theorem, Mathematics Magazine, **56** no. 5 (1983), 279–284.

[2] G. Chrystal, Algebra, Part II, 2nd ed., Black, London, 1931.

[3] O. Egecioglu and J. B. Remmel: "Brick tabloids and the connection matrices between bases of symmetric functions," Discrete Applied Math, 34 (1991), 107-120.

[4] L. Euler, Evolutio producti infiniti $(1-x)(1-xx)(1-x^3)(1-x^4)(1-x^5)(1-x^6)$ etc., Opera Omnia, (1) 3, 472–479.

[5] F. Franklin, Sur le développement du produit infini $(1-x)(1-x^2)(1-x^3)\cdots$, Comptes Rendus, **82** (1881).

[6] E. Grosswald, Topics in the Theory of Numbers, Macmillan, New York, 1966; Birkhäuser, Boston, 1983.

[7] H. Gupta, Selected Topics in Number Theory, Abacus Press, Tunbridge Wells, 1980.

[8] G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, 4^{th} ed., Oxford University Press, London 1960.

[9] I. Niven and H. Zuckerman, An Introduction to the Theory of Numbers, 3rd ed., Wiley, New York, 1973.

[10] H. Rademacher, Topics in Analytic Number Theory, Grundlehren series, vol 169, Springer, New York, 1973.