Introduction to Symmetric Functions
Chapter 5

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Abstract. A development of the symmetric functions using the plethystic notation.

CHAPTER 1

The Littlewood-Richardson Rule

We started by developing the symmetric functions and developed three different 'multiplication' operations: product, the Kronecker or inner product and plethysm. These were defined on the power basis because there they are the easiest to state and understand.

$$p_\lambda \cdot p_\mu = p_{\lambda \uplus \mu}$$

$$\frac{p_\lambda}{z_\lambda} * \frac{p_\mu}{z_\mu} = \frac{p_\nu}{z_\nu}$$

$$p_\lambda \circ p_\mu = \prod_{i=1}^{\ell(\lambda)} p_{(\lambda_i\mu_1, \lambda_i\mu_2, \ldots, \lambda_i\mu_{\ell(\lambda)})}$$

Each of these operations have an interpretation in terms of representation theory that makes them a curious object to study in the theory of symmetric functions. The definition of these operations leads to three natural questions that arise. What is the coefficient of s_ν in the expression $s_\lambda \odot s_\mu$ where $\odot$ is one of the operations $\cdot$, $*$ or $\circ$. This means that we are looking for some expression/formula/combinatorial interpretation/ algorithm or method of computation for the following three expressions:

$$\langle s_\lambda s_\mu, s_\nu \rangle, \langle s_\lambda * s_\mu, s_\nu \rangle, \langle s_\lambda \circ s_\mu, s_\nu \rangle . \tag{1.1}$$

Since we have developed means of expanding the Schur function in terms of the power basis in equations (**??**), a formula for each of these expressions can be found by expanding the Schur functions in the expressions above and giving an expression for these quantities in terms of sums of expressions involving the coefficients $\chi^\mu(\lambda)$. This is our starting point for each of these coefficients. This formula however is fairly unsatisfactory because it does little to explain why the coefficients are positive integers or what they might represent.

A goal of exposition will be to arrive at a 'satisfactory' explanation for the coefficients above.

One of the most important aspects of the Schur functions are the coefficients which appear when a product of two Schur functions are again expanded in the Schur basis.

EXAMPLE 1. We may compute the product $s_{(21)}s_{(32)}$ by expanding these Schur functions in terms of the homogeneous basis and then converting this expression into the back into the Schur basis by computing the Schur functions of degree 8 which appear in this expression. We find that

$$\begin{aligned} s_{(21)}s_{(32)} = {} & s_{(53)} + s_{(521)} + s_{(44)} + 2\, s_{(431)} + s_{(422)} \\ & + s_{(4211)} + s_{(332)} + s_{(3311)} + s_{(3221)} \end{aligned}$$

It is not obvious from the method which we used to compute this expansion that the coefficients which appear should be non-negative integers. This is a property which always occurs in the expansion of a product of Schur functions and the purpose of this section will be to arrive at a combinatorial rule for computing each coefficient. This combinatorial rule is known as the Littlewood-Richardson rule.

The coefficient of s_μ in the product of Schur functions s_λ and s_ν is typically denoted as $c^\mu_{\lambda\nu}$, that is,

$$s_\lambda s_\nu = \sum_{\mu \vdash |\lambda|+|\nu|} c^\mu_{\lambda\nu} s_\mu. \tag{1.2}$$

This means that $c^\mu_{\lambda\nu} = \langle s_\lambda s_\nu, s_\mu \rangle$ and by the definition of $s^\perp_\lambda$ this means as well that $c^\mu_{\lambda\nu} = \left\langle s_\nu, s^\perp_\lambda s_\mu \right\rangle$ and hence

$$s^\perp_\lambda s_\mu = \sum_{\nu \vdash |\mu|-|\lambda|} c^\mu_{\lambda\nu} s_\nu \tag{1.3}$$

Since the product of two Schur functions is commutative, we note that it must be the case that $c^\mu_{\lambda\nu} = c^\mu_{\nu\lambda}$ and since $\omega(s_\lambda) = s_{\lambda'}$ then it follows that $c^{\mu'}_{\lambda'\nu'} = c^\mu_{\lambda\nu}$ since $\langle s_{\lambda'} s_{\nu'}, s_{\mu'} \rangle = \langle \omega(s_\lambda s_\nu), \omega(s_\mu) \rangle = \langle s_\lambda s_\nu, s_\mu \rangle$. We also note that $c^\mu_{\lambda\nu}$ is non-zero only if $|\mu| = |\lambda| + |\nu|$.

To begin with, we will derive a combinatorial rule for computing the coefficient $c^\mu_{\lambda\nu}$. This rule is not the Littlewood-Richardson rule, but is instead a precursor since will be a sum which contains negative components and we will arrive at the Littlewood-Richardson rule by refining this combinatorial procedure.

Proposition 1.1.

$$s^\perp_\lambda s_\mu = \sum_{T \in CST_\lambda} s_{\mu - n(T)} \tag{1.4}$$

where the sum is over all column strict tableau of shape λ and with cells labeled with entries in $\{1, 2, \ldots, \ell(\mu)\}$ and $n(T) = (n_1(T), n_2(T), \ldots, n_{\ell(\mu)}(T))$ where $n_i(T)$ is the number of cells in T labeled with an i.

PROOF. Recall that by definition (**??**) we know that $s_\mu[X] = \Omega[XZ_n]\prod_{1\leq i<j\leq n}(1 - z_i/z_j)\Big|_{z_\mu}$. Since we have in general that $(g^\perp f)[X] = \langle g[Y], f[X+Y]\rangle$ then

$$\begin{aligned}
(s_\lambda^\perp s_\mu)[X] &= \left\langle s_\lambda[Y], \Omega[(X+Y)Z_n]\prod_{1\leq i<j\leq n}(1-z_i/z_j)\Big|_{z_\mu}\right\rangle_Y \\
&= \langle s_\lambda[Y], \Omega[YZ_n]\rangle_Y\, \Omega[XZ_n]\prod_{1\leq i<j\leq n}(1-z_i/z_j)\Big|_{z_\mu} \\
&= s_\lambda[Z_n]\Omega[XZ_n]\prod_{1\leq i<j\leq n}(1-z_i/z_j)\Big|_{z_\mu} \qquad (1.5)\\
&= \sum_{T\in CST_\lambda} z_n^{n(T)}\Omega[XZ_n]\prod_{1\leq i<j\leq n}(1-z_i/z_j)\Big|_{z_\mu} \\
&= \sum_{T\in CST_\lambda} s_{\mu-n(T)}[X]
\end{aligned}$$

This last equality follows by using the extended definition of the Schur functions indexed by a composition since as we remarked in the proof of the equivalence of definitions (**??**), (**??**) and (**??**) that this relation holds for Schur functions indexed by an arbitrary sequence. □

Although it seems like this is a sum over a positive set of objects, $\mu - n_i(T)$ is not always a partitions and hence some of the terms in the sum could be negative when expanded in the Schur basis. We will give an example to demonstrate this combinatorial formula.

EXAMPLE 2. Let us show how this equation can be used to compute the expression $s^\perp_{(21)}s_{(442)}$. The formula says that there is one term in this sum for each column strict tableau of shape $(2,1)$ with labels in the set $\{1,2,3\}$. Below we list all 8 tableau as well as the composition of integers representing the content and the Schur function indexed by $(4,4,2)$ minus the content composition.

$\begin{array}{\|c\|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{\|c\|}{1} \\ \hline \end{array}$	$\begin{array}{\|c\|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{\|c\|}{1} \\ \hline \end{array}$	$\begin{array}{\|c\|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{\|c\|}{2} \\ \hline \end{array}$	$\begin{array}{\|c\|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{\|c\|}{3} \\ \hline \end{array}$
$(2,1,0)$	$(2,0,1)$	$(1,2,0)$	$(1,0,2)$
$s_{(232)} = 0$	$s_{(241)} = -s_{(331)}$	$s_{(322)}$	$s_{(340)} = 0$
$\begin{array}{\|c\|c} \hline 3 & \\ \hline 2 & \multicolumn{1}{\|c\|}{2} \\ \hline \end{array}$	$\begin{array}{\|c\|c} \hline 3 & \\ \hline 2 & \multicolumn{1}{\|c\|}{3} \\ \hline \end{array}$	$\begin{array}{\|c\|c} \hline 3 & \\ \hline 1 & \multicolumn{1}{\|c\|}{2} \\ \hline \end{array}$	$\begin{array}{\|c\|c} \hline 2 & \\ \hline 1 & \multicolumn{1}{\|c\|}{3} \\ \hline \end{array}$
$(0,2,1)$	$(0,1,2)$	$(1,1,1)$	$(1,1,1)$
$s_{(421)}$	$s_{(430)} = s_{(43)}$	$s_{(331)}$	$s_{(331)}$

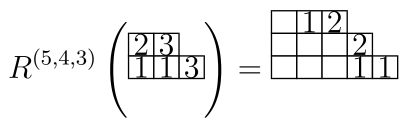
This implies that the Schur function expansion of $s^\perp_{(21)}s_{(442)}$ is given by $s_{(322)} + s_{(421)} + s_{(43)} + s_{(331)}$.

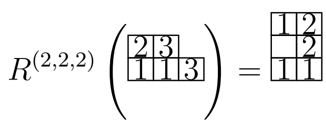
Notice that the one term that represents a negative Schur function when indexed by a partition in this example cancels with one of the two terms coming from the tableaux of content $(1,1,1)$. It is not clear from the way that we have presented this example which terms survive in this sum but add an additional combinatorial element and we will be able to identify exactly what the terms are which survive.

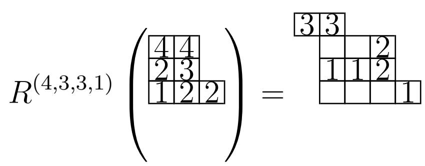
For a partition μ and a column strict tableau T we define the recording diagram for T with respect to the partition μ which we will denote by $R^\mu(T)$. $R^\mu(T)$ will have $n_i(T)$ cells in the i^{th} row and these cells will be right justified so that the rightmost cell lies in the μ_i^{th} column of the diagram (we will allow these cells to trickle into the $(-x, +y)$-quadrant if necessary). In the i^{th} row the cells will be labeled in increasing order and will contain a label k for each label i in the k^{th} row of the tableau T.

This is best demonstrated with a few examples.

Example 3.





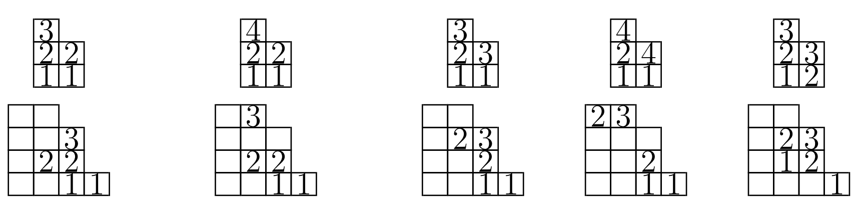


The Littlewood-Richardson rule can now be stated as follows.

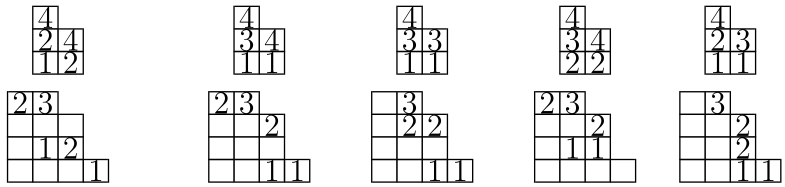
Theorem 1.2. *$c^\mu_{\lambda\nu}$ is the number of column strict tableaux of shape λ such that $R^\mu(T)$ is a column strict tableau of shape μ/ν.*

Before we proceed with the proof we will give a second example of a computation with equation (??) and this time we will list all of the column strict tableaux as well as the recording tableau to demonstrate that the Littlewood-Richardson rule works as advertised.

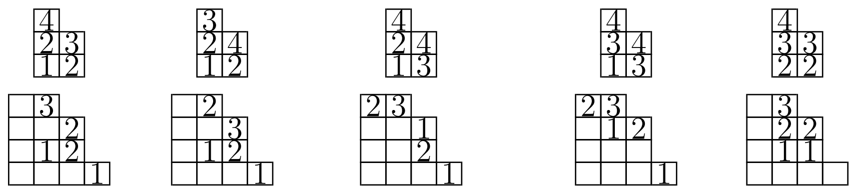
Example 4. As an example we will compute $s^\perp_{(221)} s_{(4332)}$. There are 20 tableaux of shape $(2,2,1)$ with content in the labels $\{1,2,3,4,5\}$.



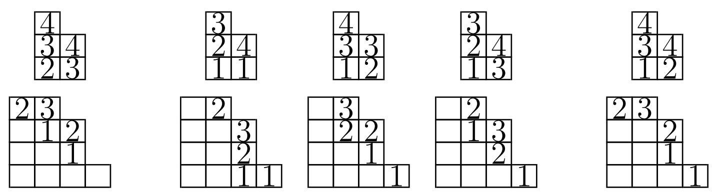
$s_{(2122)} = 0$ $s_{(2131)} = -s_{(2221)}$ $s_{(2212)} = 0$ $s_{(2230)} = 0$ $s_{(3112)} = 0$



$s_{(3130)} = -s_{(322)}$ $\quad s_{(2320)} = 0$ $\quad s_{(2311)} = 0$ $\quad s_{(4120)} = 0$ $\quad s_{(2221)}$



$s_{(3121)} = 0$ $\quad s_{(3121)} = 0$ $\quad s_{(3220)} = s_{(322)}$ $\quad s_{(3310)} = s_{(331)}$ $\quad s_{(4111)}$



$s_{(3310)} = s_{(331)}$ $\quad s_{(4111)}$ $\quad s_{(2221)}$ $\quad s_{(3211)}$ $\quad s_{(3220)} = s_{(322)}$

There are two terms in this collection with negative weight and two terms with positive weight such that $R^{(4332)}(T)$ is not a column strict tableau and these negative and positive terms cancel. 9 of the 20 terms in this sum have 0 weight and the only terms which survive are those such that $R^{(4332)}(T)$ is a skew column strict tableau. This calculation shows that

$$s^{\perp}_{(221)} s_{(4332)} = s_{(421)} + s_{(4111)} + s_{(331)} + s_{(322)} + 2s_{(3211)} + s_{(2221)}$$