New approaches to the restriction problem

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Joint work with Sridhar Narayanan, Amritansu Prasad and Shraddha Srivastava.

The Applied Algebra Seminar, York University, 1st June, 2020.

This talk is based on

- S. Narayanan, D. Paul, A. Prasad, S. Srivastava
  Polynomial Induction and the Restriction Problem,
  (submitted) 2020, arxiv:2004.03928.
- S. Narayanan, D. Paul, A. Prasad, S. Srivastava
  Character Polynomials and the Restriction Problem,
  (submitted) 2020, arXiv:2001.04112.

Polynomial representation

Polynomial representation

A pair (ρ, W) where $\rho : GL_n(\mathbf{C}) \to GL(W)$ is a group homomorphism such that the entries of $\rho(A)$ are **polynomials** in the entries of $A \in GL_n(\mathbf{C})$.

Example

$\rho : GL_2(\mathbf{C}) \to GL_3(\mathbf{C})$ given by

$$\rho\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a^2 & 2ab & b^2 \\ ac & ad + bc & bd \\ c^2 & 2cd & d^2 \end{bmatrix}.$$

Weyl modules and Schur polynomials

Irreducible representations

Weyl modules: $\{W_\lambda(n) : len(\lambda) \leq n\}$ has dimension $=$ $|SSYT(\lambda, \leq n)|$

Characters: Schur polynomials

$$char(W_\lambda(n)) = trace(\rho(\text{diag}(x_1, \ldots, x_n)); W_\lambda(n)) = s_\lambda(x_1, \ldots, x_n)$$

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$\begin{smallmatrix}1&1\\2&\end{smallmatrix}$	$\begin{smallmatrix}1&1\\3&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\2&\end{smallmatrix}$	$\begin{smallmatrix}1&2\\3&\end{smallmatrix}$	$\begin{smallmatrix}1&3\\2&\end{smallmatrix}$	$\begin{smallmatrix}1&3\\3&\end{smallmatrix}$	$\begin{smallmatrix}2&2\\3&\end{smallmatrix}$	$\begin{smallmatrix}2&3\\3&\end{smallmatrix}$
$x_1^2x_2$	$x_1^2x_3$	$x_1x_2^2$	$x_1x_2x_3$	$x_1x_2x_3$	$x_1x_3^2$	$x_2^2x_3$	$x_2x_3^2$

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Goal

To understand the decomposition of the restriction of a polynomial representation of $GL_n(\mathbf{C})$ to the subgroup S_n:

$$\mathrm{Res}_{S_n}^{GL_n(\mathbf{C})} W_\lambda(n) \cong \bigoplus_{\mu \vdash n} V_\nu^{\oplus r_{\lambda,\mu}}.$$

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Open problem: Positive combinatorial interpretation for the multiplicities $r_{\lambda,\mu}$.

Littlewood's formula

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Mike Zabrocki, OPAC 2021

"This is an advance in the problem, but recasts the solution of one problem in terms of another for which we don't have a combinatorial formula".

Why polynomial induction?

- $r_{\lambda,\mu} = dim\,\mathrm{Hom}_{S_n}(\mathrm{Res}\,W_\lambda, V_\mu) = \langle s_\lambda, s_\mu[1 + h_1 + h_2 + \cdots]\rangle$
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- **Scharf–Thibon** gave a proof using Hopf algebra techniques
- The Frobenius reciprocity suggests

Question

Does there exist an induction functor $\mathsf{Ind}^d : \mathrm{Rep}S_n \to \mathrm{Rep}^d GL_n$ such that

$$\mathsf{Hom}_{S_n}(\mathrm{Res}\,W_\lambda, V_\mu) \cong \mathsf{Hom}_{GL_n}(W_\lambda, \mathsf{Ind}^d\, V_\mu)$$

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Our work

We adapt Mackey's construction to the setting of polynomial representations.

The construction of polynomial induction

Notation: M_n: Ring of $n \times n$ matrices with entries in $\mathbf{C}$.
$P^d(M_n)$: Space of homogeneous polynomials of degree d in the entries of matrices $Q \in M_n$.

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$$\mathrm{Ind}_{S_n}^{GL_n} V = \{f : P^d(M_n) \otimes V \mid f(wQ) = \rho(w) f(Q), \forall w \in S_n, Q \in M_n\}.$$

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We proved: $\mathrm{Ind}^d : \mathrm{Rep} S_n \to \mathrm{Rep}^d GL_n$ is right adjoint to the restriction functor.

Polynomial induction of trivial representation

- Let 1_n denote the trivial representation of S_n.

  $$\operatorname{Ind}^d 1_n = \{f \in P^d(M_n) \mid f(wQ) = f(Q) \text{ for all } w \in S_n,\ Q \in M_n\}$$

- $M(d,n)$: all matrices with entries in $\mathbf{N}$ that sum to d. For $A \in M(d,n)$ of the form $A = (a_{ij})$, let q^A denote the monomial $\prod_{1 \le i,j \le n} q_{ij}^{a_{ij}}$.
- Then $\{q^A \mid A \in M(d,n)\}$ is a basis of $P^d(M_n)$.
- Therefore $\operatorname{Ind}^d 1_n$ has a basis indexed by S_n-orbits in $M(d,n)$, where S_n acts by permutation of rows.

What is the character?

Theorem

For every positive integer n,

$$\mathit{char}\,\mathrm{Ind}^d\, 1_n = \sum_{\{\mathbf{x}\in\mathbf{N}^n : |\mathbf{x}|=d\}} p_n(\mathbf{x}) t^{\mathbf{x}}$$

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Sketch of the proof:

Let $A = (a_{ij}) \in M(d,n)/S_n$, then $g = \mathrm{diag}(t_1, \ldots, t_n)$ acts on $\mathrm{Ind}^d\, 1_n$ by

$g \cdot Q^A = (Qg)^A = t^{\mathbf{x}} Q^A$. where $\mathbf{x}$ is the sum of the columns of A.

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$g \cdot Q^A = (Qg)^A = t^{\mathbf{x}}Q^A$. where $\mathbf{x}$ is the sum of the columns of A. Thus the basis elements of $\mathrm{Ind}^d\,1_n$ that contribute to the monomial $t^{\mathbf{x}}$ in $\mathit{char}\,\mathrm{Ind}^d\,1_n$ are in bijection with vector partitions of $\mathbf{x}$ with at most n parts.

Main results in polynomial induction

For a representation (ρ, V) of S_n, let $(\operatorname{Ind}\rho, \operatorname{Ind} V)$ denote the family $\{(\operatorname{Ind}^d \rho, \operatorname{Ind}^d V)\}_{d\geq 0}$.

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Corollary (Representation theoretic view of Littlewood's identity)

$r_{\lambda,\mu} = \langle \mathrm{Res}\, W_\lambda, V_\mu\rangle_{S_n} = \langle s_\lambda, s_\mu[1 + h_1 + h_2 + \cdots]\rangle.$

Attempts so far ctd.

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$$s_\lambda = s_\lambda^\dagger + \sum_{|\mu|<|\lambda|} r_{\lambda\mu[n]} s_\mu^\dagger.$$

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Nate Harman "Representations of monomial matrices and restriction from GL_n to S_n".

Positivity of some Restriction coefficients

Question

For which λ is $r_{\lambda,(n)} = \langle \mathrm{Res} W_\lambda(n), V_{(n)} \rangle_{S_n} > 0$?

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2. *If λ has two columns then $r_{\lambda,(n)} > 0$ if and only if $\lambda_1' - \lambda_2' \leq 1$.*
3. *If $\lambda = (a+1, 1^b)$ then $r_{\lambda,(n)} > 0$ if and only if $a \geq \binom{b+1}{2}$.*

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Example (Standard representations)

$trace(w; V_{(n-1,1)}) =$ no. of fixed points of $w - 1 = X_1(w) - 1$.

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Character polynomial is a polynomial in the cycle-counting functions.

Note that P is a graded algebra when the variable X_i has degree i.

Character polynomials ctd.

Theorem (Binomial basis)

Given a partition $\alpha = 1^{a_1} 2^{a_2} \cdots$, *define* $\binom{X}{\alpha} := \prod_{i \geq 1} \binom{X_i}{a_i}$. *Then* $\left\{ \binom{X}{\alpha} \mid \alpha \text{ is a partition } \right\}$ *is a basis of* P.

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Representations with Polynomial Character

A family of representation $\{V_n\}$ of S_n is said to have *eventually polynomial character* if there exists $q \in P$ and a positive integer N such that, for each $n \geq N$ and each $w \in S_n$,

$$\mathit{trace}(w; V_n) = q(X_1(w), X_2(w), \dots)$$

Church, Ellenberg and Farb, "FI-modules and stability for representations of symmetric groups".

Character polynomial of Weyl modules

Specht module has eventually polynomial character, that is, for every partition λ there exist $q_\lambda \in P$ such that

$$trace(w; V_{n-|\lambda|,\lambda}) = q_\lambda(X_1(w), X_2(w), \dots).$$

for $n \geq \lambda_1 + |\lambda|$.

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To compute $\mathcal{S}_\lambda \in P$ such that

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Recipe is to find character polynomial of $\mathrm{Sym}^d(\mathbf{C}^n)$ or $Alt^d(\mathbf{C}^n)$ and apply Jacobi–Trudi identities.

Character Polynomials of *Sym* and *Alt*

Let us find $H_d(w) := \mathit{trace}(w; \mathrm{Sym}^d(\mathbf{C}^n))$ and $E_d(w) := \mathit{trace}(w; \mathit{Alt}^d(\mathbf{C}^n))$.

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Let us find $H_d(w) := trace(w; \mathrm{Sym}^d(\mathbf{C}^n))$ and $E_d(w) := trace(w; Alt^d(\mathbf{C}^n))$.

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$$H_d = \sum_{\alpha \vdash d} \left(\!\!\binom{X}{\alpha}\!\!\right),$$

$$E_d = \sum_{\alpha \vdash d} (-1)^{a_2 + a_4 + \cdots} \binom{X}{\alpha}.$$

Here $\left(\!\binom{n}{d}\!\right) = \binom{n+d-1}{d}$ and $\left(\!\!\binom{X}{\alpha}\!\!\right) := \prod_{i \geq 1} \left(\!\!\binom{X_i}{a_i}\!\!\right)$ when $\alpha = 1^{a_1} 2^{a_2} \cdots$.

Stable restriction coefficients

Recall $\mathcal{S}_\lambda = det(H_{\lambda_i+j-i}) = det(E_{\lambda'_i+j-i})$.
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Two bases of P

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Observation

$$\mathcal{S}_\lambda = \sum r_{\lambda,\mu[n]} q_\mu$$

where $\mu[n] := (n - |\mu|, \mu_1, \mu_2, \ldots)$.

Relation to symmetric functions

Let Λ denote the ring of symmetric functions. Define $\Phi : \Lambda \to P$ by

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Character polynomials and symmetric functions

$$\begin{array}{ccccc} s_\lambda & \overset{r_{\lambda\mu[n]}}{\text{———}} & s_\mu^\dagger & \qquad & \text{in } \Lambda \\ \Big\downarrow \Phi & & \Big\downarrow \Phi & & \\ S_\lambda & \underset{r_{\lambda\mu[n]}}{\text{———}} & q_\mu & \qquad & \text{in } P \end{array}$$

Moment of a character polynomial

For two representations V and W of S_n, we have

$$\langle V, W\rangle_n := dimHom_{S_n}(V, W) = \frac{1}{n!}\sum_{w\in S_n} trace(w; V)trace(w; W)$$

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Can be extended to this setting:

Definition (Moment)

Given character polynomials q_1, q_2 corresponding to two families of S_n representations $\{V_n\}, \{W_n\}$ respectively, define moment

$$\langle q_1 q_2\rangle_n := \sum_{\alpha\vdash n} \frac{1}{z_\alpha} q_1(\alpha)q_2(\alpha).$$

$$\alpha = 1^{a_1}2^{a_2}\cdots \qquad z_\alpha = \prod_i i^{a_i} a_i!.$$

Moments and the restriction coefficients

Theorem (NPPS)

For every integer partition $\alpha = 1^{a_1} 2^{a_2} \cdots$, *we have:*

$$\left\langle \binom{X}{\alpha} \right\rangle_n = \begin{cases} 0 & \text{if } n < |\alpha|, \\ 1/z_\alpha & \text{otherwise.} \end{cases}$$

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Moments of a character polynomial stablises beyond a certain n.

Recall

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- $ResW_\lambda \cong \bigoplus_\mu V_{\mu[n]}^{r_{\lambda\mu[n]}}$
- $\mathcal{S}_\lambda = \sum r_{\lambda,\mu[n]} q_\mu$.
- By definition, $r_{\lambda,\mu[n]} = \langle \mathcal{S}_\lambda q_\mu \rangle$.
- After expanding the product in the binomial basis, the moment can be computed and hence restriction coefficients.

Moment of Weyl character polynomial

Theorem (NPPS)

For every partition λ*,* $\langle \mathcal{S}_\lambda \rangle_n$ *is the coefficient of* $t^\lambda v^n$ *in*

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Thank You