Combinatorial Structures: Mombasa

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1 Permutations

A **permutation** of the integers $\{1, \ldots, n\}$ is a one-to-one map of the integers onto themselves: $i \to \sigma_i$. There are several ways to denote such an action.

Two line notation: $\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \end{pmatrix}$ **One line notation:** $\sigma = (\sigma_1\, \sigma_2\, \ldots\, \sigma_n)$

The set of all permutations of the integers $\{1, \ldots, n\}$ forms a group called the **symmetric group**, S_n, under composition of maps.

$$S_3 = \{(1\,2\,3), (1\,3\,2), (3\,1\,2), (2\,1\,3), (2\,3\,1), (3\,2\,1)\}$$

$$\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \text{ and } \sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \implies \tau\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$$

The symmetric group is not an abelian group since the elements do not necessarily commute.

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

The **inverse** of $\sigma \in S_n$ is the element σ^{-1} where $\sigma^{-1}\sigma = id$. Since the inverse of a permutation σ is the permutation that undoes the action of σ, it can easily be computed by:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \end{pmatrix} \quad \implies \quad \sigma^{-1} = \begin{pmatrix} \sigma_1 & \sigma_2 & \sigma_3 & \cdots & \sigma_n \\ 1 & 2 & 3 & \cdots & n \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \sigma'_1 & \sigma'_2 & \sigma'_3 & \cdots & \sigma'_n \end{pmatrix}$$

$$\sigma = (3\,2\,5\,1\,4) \quad \implies \quad \sigma^{-1} = \begin{pmatrix} 3 & 2 & 5 & 1 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 5 & 3 \end{pmatrix}$$

A permutation that is equal to its inverse is called an **involution**. Note $\sigma = \sigma^{-1} \iff \sigma\sigma = id$. The involutions of S_3 are $(1\,2\,3)$, $(2\,1\,3)$, $(3\,2\,1)$, $(1\,3\,2)$. For example, check $(2\,1\,3)$:

$$\begin{pmatrix} 2 & 1 & 3 \\ 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

There is a third notation used in the study of permutations called **cycle notation**. In this case, $\sigma = (a_1, \ldots, a_\ell) \cdots (b_1, \ldots, b_j)$ means $\sigma(a_i) = a_{i+1}$ and $\sigma(a_\ell) = a_1, \ldots,$ $\sigma(b_i) = b_{i+1}$ and $\sigma(b_j) = b_1$.

$$\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} = (2\,1\,3) = (1,2)(3)$$

The **cycle structure** of a permutation $\sigma \in S_n$ is the non-increasing vector of integers $\lambda = (\lambda_1, \ldots, \lambda_\ell)$ determined by the number of elements in the disjoint cycles of σ. For example,

$$\text{one-line notation} = \{(1\,2\,3),\ (2\,1\,3),\ (3\,1\,2),\ (2\,3\,1),\ (3\,2\,1),\ (1\,3\,2)\}$$

$$\text{cycle notation} = \{(1)(2)(3),\ (1,2)(3),\ (1,3,2),\ (1,2,3),\ (1,3)(2),\ (1)(2,3)\}$$

$$\text{cycle structures} = \{(1,1,1),\ (2,1),\ (3),\ (3),\ (2,1),\ (2,1)\}$$

Recall that the conjucacy class of a permutation σ is the set of all permutations, $\{\gamma^{-1}\sigma\gamma : \gamma \in S_n\}$. It turns out that the conjugacy class of a given permutation can be determined by finding all permutation with the same cycle structure. For instance, since the cycle structure of $(3\,2\,1)$ is (2,1), the elements in the conjugacy class of $(3\,2\,1)$ are $(3\,2\,1)$, $(1\,3\,2)$, and $(2\,1\,3)$.

2 Partitions

The conjugacy classes are one of many examples of objects that can be indexed by weakly decreasing vectors of positive integers. This is one of many reasons to closely study such vectors. These vectors are called partitions. That is, a **partition** is a vector of integers $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_1 \geq \lambda_1 \geq \cdots \geq \lambda_n > 0$. The partitions of 4 are

$$(4), (3,1), (2,2), (2,1,1), (1,1,1,1)$$

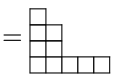
We can express partitions in an alternative notation that emphasizes the number of times a given part occurs. That is, $\lambda = (\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_\ell^{m_\ell})$ where there are m_i occurences of λ_i. In this case, the partitions of 4 are denoted

$$(4),\ (3,1),\ (2^2),\ (2,1^2),\ (1^4)$$

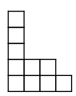
The **degree** or **order** of a partition λ is $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_n$. We say "$\lambda$ is a partition of $|\lambda|$" and denote this by $\lambda \vdash |\lambda|$. For example, $\lambda = (2, 1, 1)$ is a partition of 4, or $\lambda \vdash 4$, or $|\lambda| = 4$. The **length** of a partition is the number of parts. The length of λ is denoted $\ell(\lambda)$. $\lambda = (2, 1, 1) \implies \ell(\lambda) = 3$. The **conjugate** of a partition λ is $\lambda' = (\lambda'_1, \lambda'_2, \ldots, \lambda'_n)$ where $\lambda'_i =$ number of parts of $\lambda \geq i$. 3 parts of (2,1,1) are ≥ 1 implies $\lambda'_1 = 3$, 1 part of (2,1,1) is ≥ 2 implies $\lambda'_2 = 1$. Thus $\lambda' = (3, 1)$.

2.1 Ferrers Diagrams

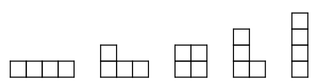
The **Ferrers diagram** of a partition λ is the collection of stacked boxes, arranged in left-justified rows, so that number of boxes in each row is λ_i.

$\lambda = (5, 2, 2, 1) =$ 

Everything translates into the notation of shapes. The total number of boxes gives the degree of the partition. The partition in the previous example has degree 10. The number of rows is the length of the partition, $\ell(\lambda) = 4$. The reflection of the diagram about $y = x$ is the conjugate partition

$\lambda' =$  $\implies \quad \lambda' = (4, 3, 1, 1, 1) = (4, 3, 1^3).$

The partitions of 4 are

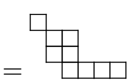


2.2 Skew diagrams

If $\mu \subseteq \lambda$ as Ferrers diagrams, then a *skew diagram* can be defined by taking the set of cells:

$$\lambda/\mu = \{c : c \in \lambda \text{ and } c \notin \mu\}.$$

For example, with $\lambda = (6, 3, 3, 1)$ and $\mu = (2, 1, 1)$ we have

$\lambda/\mu =$ 

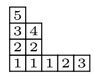
Note that partitions are simply a special case of skew diagrams. That is, when $\mu = \emptyset$, the skew diagram λ/μ reduces to a usual partition λ.

3 Tableaux

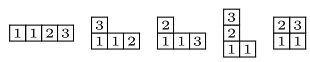
The symmetric group is linked to beautiful combinatorics. In particular, a combinatorial structure obtained by putting numbers into Ferrers diagrams is of utmost importance. A few examples of the role these objects (called tableaux) play are:

- There is a bijection between permutations and pairs of standard tableaux with the same shape (or more generally between two rowed arrays and pairs of tableaux).
- The number of involutions of $\{1, 2, \ldots, n\}$ is the number of standard tableaux of n
- The tableaux form an associative monoid that facilitates the study of symmetric functions.

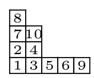
A **semi-standard tableau** of shape λ and content μ is a filling of the Ferrers diagram for λ with μ_1 ones, μ_2 twos,... so that the numbers are weakly increasing in rows and strictly increasing in columns.



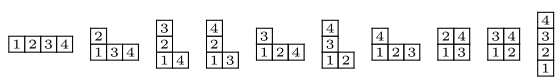
is a tableau of shape $\lambda = (5, 2, 2, 1)$ and content (3,3,2,1,1). All the semi-standard tableaux of degree 4 with content (2,1,1) are



A **standard tableaux** of shape λ is a semi-standard tableau with content $(1, 1, \ldots, 1)$. In a standard tableau, the integers $\{1, 2, \ldots, |\lambda|\}$ each occur once in the Ferrers diagram filled strictly increasing in rows and columns.



is a standard tableau of shape $\lambda = (5, 2, 2, 1)$ and degree 10. The set of all standard tableaux of degree 4 is

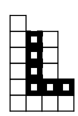


3.1 Counting standard tableaux

The number of standard tableaux of a given shape can be determined using the "hook formula". If $s = (i, j)$ is a cell in the diagram of λ, then it has a **hook**

$$H_s = H_{i,j} = \{(i, j')|j' \geq j\} \cup \{(i', j)|i' \geq i\}$$

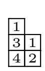
with corresponding **hook-length** $h_s(\lambda) = |H_s|$. For example $h_{2,2}(4^2, 3^3, 1) = 6$ since $H_{2,2}$ are the outlined cells in



The hook formula: The number of standard tableaux of shape λ (f^λ) is $n!$ divided by the product of all hook-lengths;

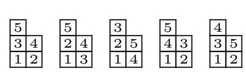
$$f^\lambda = \frac{n!}{\prod_{s \in \lambda} h_s}$$

The number of standard tableaux of shape $(2, 2, 1)$ is determined by finding each hook-lengths h_s of λ (written in the corresponding cells s):



$$\Longrightarrow \quad f^{2,2,1} = \frac{5!}{4 \cdot 3 \cdot 2 \cdot 1 \cdot 1} = 5$$

The five standard tableaux of shape (2,2,1) are



4 Robinson-Schensted Correspondence

4.1 Insertion/deletion algorithm

A fundamental operation on tableaux is given by the **Schensted insertion algorithm**, which sends a tableau T and a positive integer x to a new tableau that has one more box than T. This new tableau is denoted $T \leftarrow x$.

If x is weakly larger than every entry in the bottom row of T, add n to the end of this row.
If not, find the leftmost entry e in the bottom row that is strictly larger than x. Replace this entry by x. Repeat the process on the next row with the letter e.

$$\begin{array}{llll} 5 \\ 3&3&4 \\ 1&1&2&3 \end{array} \leftarrow 3 \quad = \quad \begin{array}{lllll} 5 \\ 3&3&4 \\ 1&1&2&3&3 \end{array}$$

$$\begin{array}{llll} 3&6 \\ 2&5 \\ 1&1&3&4 \end{array} \leftarrow 3 \quad = \quad \begin{array}{llll} 3&6 \\ 2&5 \\ 1&1&3&3 \end{array} \leftarrow 4 \quad = \quad \begin{array}{llll} 3&6 \\ 2&4 \\ 1&1&3&3 \end{array} \leftarrow 5 \quad = \quad \begin{array}{llll} 3&5 \\ 2&4 \\ 1&1&3&3 \end{array} \leftarrow 6 \quad = \quad \begin{array}{llll} 6 \\ 3&5 \\ 2&4 \\ 1&1&3&3 \end{array}$$

Note that this algorithm will send any word to a tableau by successively inserting the letters using Schensted's insertion algorithm.

$$\emptyset \leftarrow 234412232 = \begin{array}{l} 2 \end{array} \leftarrow 34412232 = \begin{array}{ll} 2&3 \end{array} \leftarrow 4412232 = \cdots = \begin{array}{llll} 2&3&4&4 \end{array} \leftarrow 12232$$

$$= \begin{array}{llll} 2 \\ 1&3&4&4 \end{array} \leftarrow 12232 = \begin{array}{llll} 2&3 \\ 1&2&4&4 \end{array} \leftarrow 232 = \begin{array}{llll} 2&3&4 \\ 1&2&2&4 \end{array} \leftarrow 32 = \begin{array}{llll} 2&3&4&4 \\ 1&2&2&3 \end{array} \leftarrow 2 = \begin{array}{llll} 4 \\ 2&3&3&4 \\ 1&2&2&2 \end{array}$$

Given that $\widehat{T} = T \leftarrow x$ has a new cell in position c, we can easily reverse the process of insertion starting from the entry in position c. The **deletion algorithm** will send a given tableau T and a corner c of T to a tableau of the shape of T with the cell c removed.

Delete the entry x from corner c. If corner c was in the bottom row of T, we are done.
If not, find the rightmost entry e smaller than x in the row below that with c and replace e by x. Repeat this process with entry e.

If c is the cell containing 6 in the tableau T:

$$T = \begin{array}{llll} 6 \\ 3&5 \\ 2&4 \\ 1&1&3&3 \end{array} \longrightarrow \begin{array}{llll} 3&5 \\ 2&4 \\ 1&1&3&3 \end{array} \leftarrow 6 \longrightarrow \begin{array}{llll} 3&6 \\ 2&4 \\ 1&1&3&3 \end{array} \leftarrow 5 \longrightarrow \begin{array}{llll} 3&6 \\ 2&5 \\ 1&1&3&3 \end{array} \leftarrow 4 \longrightarrow \begin{array}{llll} 3&6 \\ 2&5 \\ 1&1&3&4 \end{array}$$

4.2 The correspondence

A bijection between permutations and pairs of standard tableaux with the same shape arises from the insertion algorithm. The **Robinson-Schensted algorithm** is a method for constructing a pair of tableaux of the same shape from a permutation.

From any $\sigma \in S_n$, the first tableau in our pair is constructed from σ using the Schensted insertion algorithm. However, during this construction when a new box is added by inserting σ_i, put the letter i into this box in the second tableau. The first tableau is called the **insertion tableau** and the second is the **recording tableau**.

To find the pair of tableaux associated to $\sigma = 4\,1\,5\,3\,2\,6$, start by row inserting $\sigma_1 = 4$ into $\emptyset$;

$$\emptyset \leftarrow 4 = \begin{array}{l} 4 \end{array} \qquad \begin{array}{l} 1 \end{array}$$

$$\begin{array}{l} 4 \end{array} \leftarrow 1 = \begin{array}{l} 4 \\ 1 \end{array} \qquad \begin{array}{l} 2 \\ 1 \end{array}$$

$$\begin{ytableau} 4 \\ 1 \end{ytableau} \leftarrow 5 = \begin{ytableau} 4 \\ 1 & 5 \end{ytableau} \qquad \begin{ytableau} 2 \\ 1 & 3 \end{ytableau}$$

$$\begin{ytableau} 4 \\ 1 & 5 \end{ytableau} \leftarrow 3 = \begin{ytableau} 4 & 5 \\ 1 & 3 \end{ytableau} \qquad \begin{ytableau} 2 & 4 \\ 1 & 3 \end{ytableau}$$

$$\begin{ytableau} 4 & 5 \\ 1 & 3 \end{ytableau} \leftarrow 2 = \begin{ytableau} 4 \\ 3 & 5 \\ 1 & 2 \end{ytableau} \qquad \begin{ytableau} 5 \\ 2 & 4 \\ 1 & 3 \end{ytableau}$$

$$\begin{ytableau} 4 \\ 3 & 5 \\ 1 & 2 \end{ytableau} \leftarrow 6 = \begin{ytableau} 4 \\ 3 & 5 \\ 1 & 2 & 6 \end{ytableau} \qquad \begin{ytableau} 5 \\ 2 & 4 \\ 1 & 3 & 6 \end{ytableau}$$

Note that the recording tableau is a standard tableau of n since Q is formed by adding elements in increasing order to cells on the periphery.

Since the Schensted insertion algorithm is reversible, this process can be inverted to obtain a permutation of S_n from a pair of standard tableaux of the same shape. That is, starting with the entry e in the insertion tableau that lies in position n determined by the recording tableaux, we use the deletion process to remove this letter. Then repeat starting with $n-1$. Thus, the **Robinson-Schensted Correspondence** is a bijection between elements of S_n and the set of ordered pairs (P, Q) of standard tableaux of n having the same shape.

One direct consequence of the RS-correspondence is that:

Theorem 1. *The number of permutations of $\{1, 2, \ldots, n\}$ is equal to the number of pairs of standard tableaux of the same shape λ as λ varies over all partitions of n.*

$$\sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2 = n!$$

When we consider only permutation that are involutions, another beautiful identity holds. Notice the number of involutions of S_4:

$$(1\,2\,3\,4), (2\,1\,3\,4), (3\,2\,1\,4), (4\,2\,3\,1), (1\,3\,2\,4), (1\,4\,3\,2), (1\,2\,4\,3), (2\,1\,4\,3), (3\,4\,1\,2), (4\,3\,2\,1)$$

equals the number of standard tableaux of degree 4:

$$\begin{ytableau} 1 & 2 & 3 & 4 \end{ytableau} \quad \begin{ytableau} 2 \\ 1 & 3 & 4 \end{ytableau} \quad \begin{ytableau} 3 \\ 2 \\ 1 & 4 \end{ytableau} \quad \begin{ytableau} 4 \\ 2 \\ 1 & 3 \end{ytableau} \quad \begin{ytableau} 3 \\ 1 & 2 & 4 \end{ytableau} \quad \begin{ytableau} 4 \\ 3 \\ 1 & 2 \end{ytableau} \quad \begin{ytableau} 4 \\ 1 & 2 & 3 \end{ytableau} \quad \begin{ytableau} 2 & 4 \\ 1 & 3 \end{ytableau} \quad \begin{ytableau} 3 & 4 \\ 1 & 2 \end{ytableau} \quad \begin{ytableau} 4 \\ 3 \\ 2 \\ 1 \end{ytableau}.$$

This turns out to be true in general:

Theorem 2. *The number of involutions on $\{1, \ldots, n\}$ is the number of standard tableaux on n letters.*

The argument relies on an interesting result: If σ corresponds to the pair of standard tableaux (P, Q) under the RS-correspondence, then σ^{-1} corresponds to the pair (Q, P). Thus for $\sigma = \sigma^{-1}$ an involution, $(P, Q) = (Q, P)$. That is, σ corresponds to a pair of the same standard tableaux.

5 Plactic monoid

Another important idea linked closely to Schensted insertion is the study of tableaux as a monoid. Recall that an associative **monoid** is a set G containing an identity e ($x \cdot e = e \cdot x = x$ for all $x \in G$) and having a binary operation (called multiplication) for which the associative law holds. That is, a monoid is a group without an inverse.

The collection of all words on a given alphabet forms a monoid called the **free monoid**, where multiplication is defined by word juxtaposition:

$$w = (3416647) \quad \text{and} \quad w' = (2155) \quad \text{give} \quad ww' = (34166472155)$$

The empty word $\emptyset$ is the identity element. In particular, we have the monoid F on letters $\{1, \ldots, m\}$. Using the obvious structure of the free monoid, a connection between words and tableaux that will enable us to induce a monoid on tableaux.

5.1 Knuth equivalence

The **row word** of tableau T, denoted $w(T)$, is the sequence of letters read from a tableau like a book in English.

$$w\left(\begin{array}{|c|c|c|c|c|}\hline 3&5\\\hline 2&4\\\hline 1&1&2&2&6\\\hline\end{array}\right) = 35241126$$

Note that **every tableau** $T \to$ **a word** $w(T)$. The original tableau is recovered from its row word by noting that the rows end at each **descent** – any position in a word w where $w_i > w_{i+1}$:

$$35|24|1126 \quad \longrightarrow \quad \begin{array}{|c|c|c|c|c|}\hline 3&5\\\hline 2&4\\\hline 1&1&2&2&6\\\hline\end{array}$$

However, not every word comes from a tableaux. Only when a word that is cut at its descents has pieces of weakly increasing length, and when stacked have columns that are strictly increasing does the word form a tableau. Otherwise, we may find the following scenario:

$$357241126 \to 357\,|\,24\,|\,1126 \to \begin{array}{|c|c|c|c|c|}\hline 3&5&7\\\hline 2&4\\\hline 1&1&2&2&6\\\hline\end{array}$$

Some **words** $\to$ **a tableau**.

To reconcile that there are "too many" words, we shall classify certain words as "the same". The **Knuth relation** A is move on a three letter word:

$$y\,z\,x \Rightarrow y\,x\,z \iff \begin{array}{|c|c|}\hline y&z\\\hline\end{array} \leftarrow x = \begin{array}{|c|c|}\hline y\\\hline x&z\\\hline\end{array} \quad \text{if } x < y \leq z$$

The **Knuth relation** B is move on a three letter word:

$$x\,z\,y \Rightarrow z\,x\,y \iff \begin{array}{|c|c|}\hline x&z\\\hline\end{array} \leftarrow y = \begin{array}{|c|c|}\hline z\\\hline x&y\\\hline\end{array} \quad \text{if } x \leq y < z$$

For example,

$$3\,4\,2 \overset{A}{\equiv} 3\,2\,4$$

An **elementary Knuth transformation** on a word applies one of A of B or their inverses to three successive letters of a word. Thus, words w and w' are **Knuth equivalent** if w can be obtained by applying a sequence of elementary Knuth transformations to w'. For example,

$$1\,2\,2\,3\,3\,2\,4 \equiv 1\,2\,3\,2\,2\,3\,4 \qquad \text{since} \quad 1\,2\,2\,3\,3\,2\,4 \overset{A}{\equiv} 1\,2\,2\,3\,2\,3\,4 \overset{B}{\equiv} 1\,2\,3\,2\,2\,3\,4$$

The set $M = F/R$ where R is the equivalence relation generated by the Knuth relations (the Knuth equivalence classes of words) is closed under multiplication (juxtaposition). That is, $w, u \in M$ are such that $wu \equiv w'u'$ if $w \equiv w'$ and $u \equiv u'$. Thus the multiplication on F descend to a multiplication on the set M. This makes M into an associative monoid called the **plactic monoid**.

It will develop that **each equivalence class** $\to$ **tableau**. Then, a bijection between equivalence classes and tableaux will reveal that tableaux form a monoid that is isomorphic to the plactic monoid.

5.2 Knuth slow insertion

Recall, the insertion algorithm produces a tableau $T \leftarrow x$ from a tableau T and letter x. This algorithm can instead be achieved systematically by applying a sequence of Knuth transformations A and B to the left of $w(T)x$.

Consider only the row word of the bottom row of T, $w = u_1 \cdots u_p x' v_1 \cdots v_q x$ where $x' > x \geq u_p$. Check that x is strictly smaller than its two preceeding letters. If so, we move x forward by transposing with the entry to its right since $v_q \geq v_{q-1} > x$ implies the Knuth operation A moves x to the left of v_q.

$$1\,2\,2\,3\,3\,4\,\underline{2} \overset{A}{\equiv} 1\,2\,2\,3\,3\,\underline{2}\,4 \overset{A}{\equiv} 1\,2\,2\,3\,\underline{2}\,3\,4$$

This continues until x is no longer smaller, then x rests while x' is pushed forward by transposing with the entry to its left. That is, use transformation A until the configuration $u_1 \cdots u_p x' x v_1 \cdots v_q$ is reached. Then $u_p \leq x$ and $x' > x$ implies that operation B moves x' to the left of u_{p-1}.

$$1\,2\,2\,\underline{3}\,2\,3\,4 \overset{B}{\equiv} 1\,2\,\underline{3}\,2\,2\,3\,4 \overset{B}{\equiv} 1\,\underline{3}\,2\,2\,2\,3\,4 \overset{B}{\equiv} \underline{3}\,1\,2\,2\,2\,3\,4$$

This continues until x' has been moved to the beginning of the row word w of the first row of T. As such, the letter x has been Schensted inserted into the bottom row and x' has been bumped. This process is then repeated with x' at the end of the row word of the second row of T, amounting exactly to the insertion of x' into the second row, etc. Thus, the Schensted insertion of x into T can be achieved by Knuth transformations on $w(T)x$. That is,

Proposition 3. *For any tableau T and positive integer x,*

$$w(T \leftarrow x) \equiv w(T)x$$

More generally, given any word $w = x_1 \cdots x_\ell$ and starting from the empty tableau $T = \emptyset$ we have $w(\emptyset \leftarrow x_1) \equiv x_1$, then with $T = \emptyset \leftarrow x_1$, $w(T \leftarrow x_2) \equiv x_1\, x_2$, and so. Thus proving:

Proposition 4. *Every word is Knuth equivalent to the word of a tableau. In particular,*

$$x_1\, x_2 \cdots x_\ell \equiv w(\emptyset \leftarrow x_1 \leftarrow \cdots \leftarrow x_\ell)$$

implying that the word $w = x_1 \cdots x_\ell$ is Knuth equivalent to the tableau $T = \emptyset \leftarrow w$.

In fact a stronger result holds[1]. ...

Theorem 5. *Every word is Knuth equivalent to the word of a unique tableau. That is, each equivalence class contains exactly one row word of a tableau.*

All words that are Knuth equivalent can be associated to the same tableau and no others:

$$2\,3\,5\,5\,1\,2\,2\,3\,2 \equiv 2\,3\,5\,5\,1\,2\,3\,2\,2 \equiv 2\,3\,5\,5\,1\,3\,2\,2\,2 \equiv 2\,3\,5\,5\,3\,1\,2\,2\,2$$

$$\equiv 2\,3\,5\,3\,5\,1\,2\,2\,2 \equiv 2\,5\,3\,3\,5\,1\,2\,2\,2 \equiv 5\,2\,3\,3\,5\,1\,2\,2\,2 \equiv w\left(\begin{array}{|c|c|c|c|}\hline 5 \\ \cline{1-1} 2 & 3 & 3 & 5 \\ \hline 1 & 2 & 2 & 2 \\ \hline \end{array}\right)$$

6 The tableaux monoid and the plactic algebra

We have now seen that the map w from a tableau to to the row word of this tableau is a bijection between the set of tableaux and the Knuth equivalence classes of words M. Further, Schensted insertion is the inverse of w:

> Given any tableau T, $w(T) = e$ for some word e. To see that $\emptyset \leftarrow e = T$, note that $w(\emptyset \leftarrow e) \equiv e$ by Proposition **??**. That is, $e \equiv w(\emptyset \leftarrow e) \equiv w(T)$. Since a word is Knuth equivalent to the word of only one tableaux, $\emptyset \leftarrow e = T$. On the other hand, if a word e gives rise to the tableau $T = \emptyset \leftarrow e$, then $w(T) = w(\emptyset \leftarrow e) \equiv e$.

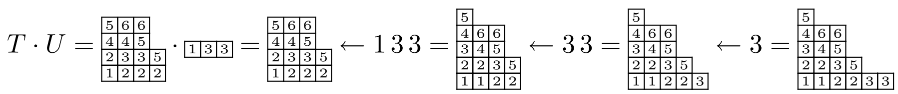
[1] The proof is much more complex and is relegated to the Appendix

This bijection between the associative monoid M and the set of tableaux implies that the set of tableaux forms a monoid under the multiplication defined on tableaux by the **insertion product**: $T \cdot U := T \leftarrow w(U)$. That is, for tableaux T, U, V:

- The empty tableau $\emptyset$ is such that $T \cdot \emptyset = \emptyset \cdot T = T$.
- $T \cdot U$ is a tableau.
- $(T \cdot U) \cdot V = T \cdot (U \cdot V)$

6.1 Multiplication in the monoid

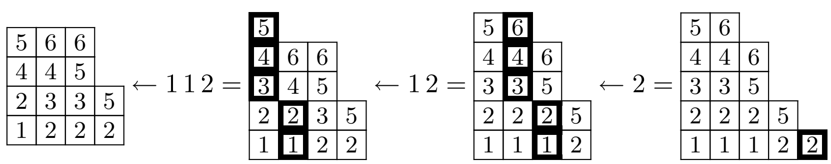
In general, the insertion product masks the outcome of multiplying two given tableaux, however there are circumstances under which some information is revealed. For example, given tableaux T and U, it is clear the the shape of $T \cdot U$ must contain the shape of T since each insertion adds boxes to the periphery of T. In the case that U is a tableau of row or column shape, even more can be said about the resulting shape of $T \cdot U$. Examine the following example:



Notice that in the process of multiplying $T \cdot U$, at most one cell has been added to any column of T. The following important lemma about Schensted insertion will lead to such properties on the product of tableaux.

6.2 A result on bumping

The proof relies on the **bumping route**, the sequence of all cells affected during an insertion.

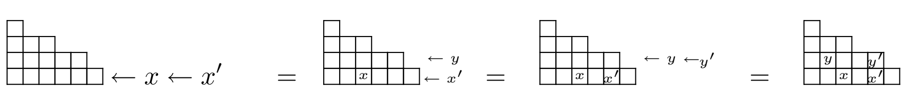


Observe that the bumping route of the first insertion lies to the left of that of the second.

The row bumping lemma: Insert x into the tableau T and denote the bumping route by R. Then insert x' into the resulting tableau and denote its bumping route by R'. Let c and c' denote the cells added to $T \leftarrow x$ and $T \leftarrow x \leftarrow x'$, respectively.

- If $x \leq x'$, then every cell of R is strictly left of those in R' and c is strictly left and weakly above c'.
- If $x > x'$, then every cell of R' is weakly left of those in R and c' is weakly left and strictly above c.

Proof. Consider the case that $x \leq x'$ bumps an element y from the first row. If an element y' is bumped by x' from the first row, then y' lies strictly to the right of y.



Repeating this argument gives that the bumping route R must lie strictly to the left of R'. The bumping route R ends at a cell c which must be strictly left and weakly above c'. Similarly we can show the second assertion. □

Following from this lemma: If ν is the shape of $T \cdot U$ and λ is the shape of T. Let ν/λ denote the **skew diagram** obtained by removing the cells of λ from ν. Notice that no two cells lie in the same column. This holds in general, following from an important lemma about Schensted insertion.

Proposition 6. *If λ is the shape of T and ν is the shape of $T \cdot U$, then no two boxes in the skew diagram ν/λ lie in the same column if U is a row. Conversely, if X is a tableau of shape ν and $\lambda \subseteq \nu$ such that no two cells of ν/λ lie in the same column, then there is a tableau T of shape λ and a tableau U whose shape is a row such that $X = T \cdot U$.*

Proposition 7. *If the shape of U is a column, then no two boxes of ν/λ lie in the same row. Conversely, if X is a tableau of shape ν and $\lambda \subseteq \nu$ such that no two cells of ν/λ lie in the same row, then there is a tableau T of shape λ and a tableau U whose shape is a column such that $X = T \cdot U$.*

When U is a row, we say $\nu = \lambda+$ a horizontal n-strip where n is the number of cells in U and when U is a columns, $\nu = \lambda+$ a vertical n-strip.

6.3 The plactic algebra and Pieri rule

From any monoid, we can form a ring by considering the set of linear combinations of the elements in the monoid with coefficients in a fixed ring. Multiplication of two basis elements is determined by the monoid structure which then extends by bilinearity to all linear combinations. We will denote **the tableaux ring:** $R_{[m]}$ to be the ring associated to the tableau monoid.

- $R_{[m]}$ is a $\mathbb{Z}$-module with the tableaux in letters $\{1, \ldots, m\}$ as a basis
- Multiplication in the ring can be defined by insertion product
- Associative, not commutative.

In the tableau ring, we will be especially interested in $S_\lambda \in R_{[m]}$, the sum of all tableaux of shape λ. There are beautiful combinatorial formulas for the product $S_\lambda \cdot S_\mu$. For example, in the case that μ is a row or column we are equipped to prove

Theorem 8.

$$S_\lambda \cdot S_{(r)} = \sum_{\nu=\lambda+\textit{horizontal } r\textit{-strip}} S_\nu$$

$$S_\lambda \cdot S_{(1^r)} = \sum_{\nu=\lambda+\textit{vertical } r\textit{-strip}} S_\nu$$

Proof. The product of a tableau of shape λ and a row-tableau U gives a term occuring exactly once in one of the summands on the right. On the other hand, given any tableau X occuring in one of the S_ν, it can be written uniquely as $X = T \cdot U$ where T is a tableau of shape λ and U is a tableau whose shape is (r). The second identity follows similarly. □

7 Appendix

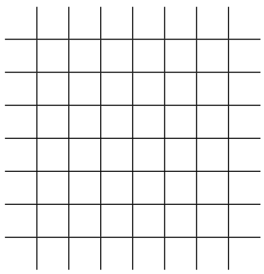
7.1 Viennot's construction

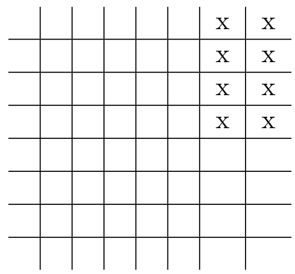
Viennot's construction is an alternative method for obtaining the RS-insertion and recording tableaux from a given permutation. This method has the advantage of clearly showing that if σ corresponds to the pair of standard tableaux (P, Q) under the RS-correspondence, then σ^{-1} corresponds to the pair (Q, P).

We start by identifying each permutation $\sigma = (\sigma_1 \cdots \sigma_n)$ with the set of lattice points in $\mathbb{N} \times \mathbb{N}$:

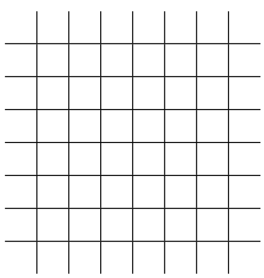
$$\{(i, \sigma_i) : i = 1, \ldots, n\}$$

The *shadow* of any lattice point (a, b) is the region to the right of $x = a$ and above $y = b$. For example with $\sigma = (6\,2\,3\,1\,5\,4\,7)$, we obtain:





The shaded region depicts the shadow of $(6, 4)$. The *special points* in a set of lattice points $\mathcal{S}$ are those that do not lie in the shadow of any other point in $\mathcal{S}$. For example, the special points in the set $\mathcal{S} = \{(i, \sigma) : i = 1, \ldots, n\}$ are $\{(1, 6), (2, 2), (4, 1)\}$. Given a set of lattice points $\mathcal{S}$, the *shadow line* of $\mathcal{S}$ is the boundary of the region obtained by casting shadows of the special points. Taking all the points in set $\mathcal{S}$, the shadow line is:



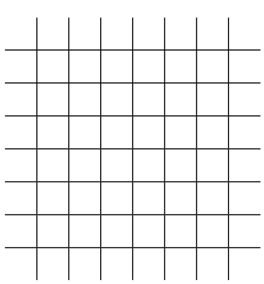
Since every shadow is cast upward and to the right, a shadow line must be a series of vertical and horizontal steps moving to the right and downward.

Furthermore,

Remark 9. Lower corners $\lfloor\_\_$ in the shadow line of $\mathcal{S}$ are determined by the special points of $\mathcal{S}$.

Remark 10. If the shadow line is horizontal at $y = \sigma_i$ then it remains horizontal until reaching an x-coordinate where $\sigma_x < \sigma_i$.

We shall consider the set of shadow lines $\{L_1, L_2, \ldots\}$, where L_1 is the shadow line obtained from the set $\mathcal{S}$ of all points determined by σ. L_2 is the shadow line of $\mathcal{S}^2$ – the set of points in $\mathcal{S}$ minus the special points. In this manner, L_i is the shadow line of the set $\mathcal{S}^i$ obtained by deleting the special points from $\mathcal{S}^{i-1}$. The *shadow diagram of* σ is the set of shadow lines $\{L_1, L_2, \ldots\}$. In our example, the shadow diagram is:



Since a shadow line is vertical only at a special point, at each vertical line $x = i$, every shadow line in the shadow diagram is horizontal except the line on which (i, σ_i) lies. There are two possible configurations at $x = i$:

In summary, we have

Remark 11. A new shadow line begins at $x = i$ only if the bottom of every shadow line at $x = i-1$ is below $y = \sigma_i$.

Proof. (i, σ_i) is the lower corner of some shadow line L. If L begins before $x = i$, it must contain a point higher then $y = \sigma_i$. □

Remark 12. A shadow line has a vertical step at $x = i$ if the bottom of some shadow line at $x = i-1$ lies at $y = \sigma_j > \sigma_i$.

Proof. If $y = j > \sigma_i$ at $x = i - 1$ then a new shadow line does not begin at $x = i$ by the previous remark. However, (i, σ_i) is a special point (and thus a lower corner) of some line which must drop down from some higher y-coordinate. □

Examine the insertion tableau at each step in the RS-correspondence for $\sigma = (6\,2\,3\,1\,5\,4\,7)$:

$$P_1 = \begin{array}{|c|}\hline 6\\\hline\end{array} \quad P_2 = \begin{array}{|c|}\hline 6\\\hline 2\\\hline\end{array} \quad P_3 = \begin{array}{|c|c}\hline 6 & \\\hline 2 & \multicolumn{1}{c|}{3}\\\hline\end{array} \quad P_4 = \begin{array}{|c|c}\hline 6 & \\\hline 2 & \\\hline 1 & \multicolumn{1}{c|}{3}\\\hline\end{array} \quad P_5 = \begin{array}{|c|cc}\hline 6 & & \\\hline 2 & & \\\hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5}\\\hline\end{array} \quad P_6 = \begin{array}{|c|cc}\hline 6 & & \\\hline 2 & \multicolumn{1}{c|}{5} & \\\hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4}\\\hline\end{array} \quad P_7 = \begin{array}{|c|ccc}\hline 6 & & & \\\hline 2 & \multicolumn{1}{c|}{5} & & \\\hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{4} & \multicolumn{1}{c|}{7}\\\hline\end{array} \quad \text{and} \quad Q = \begin{array}{|c|ccc}\hline 4 & & & \\\hline 2 & \multicolumn{1}{c|}{6} & & \\\hline 1 & \multicolumn{1}{c|}{3} & \multicolumn{1}{c|}{5} & \multicolumn{1}{c|}{7}\\\hline\end{array}$$

Compare the bottom row in any P_i to the lowest y-coordinates on each shadow line that is crossed by the vertical line $x = i$. Compare the bottom row of Q to the x-coordinates marking the start of each shadow line. To be precise:

Lemma 13. *Consider $\sigma \in S_n$ and its insertion tableau $P_i = \sigma_1 \leftarrow \cdots \leftarrow \sigma_i$ for any $i = 1, \ldots, n$. The first row of P_i contains the entries $y_1\, y_2 \ldots y_\ell$ that record the lowest y-coordinates of shadow lines crossed by the vertical line $x = i$. Further, the bottom row of Q can be obtained by reading the x-coordinates marking the start of the set of shadow lines.*

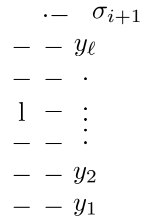
Proof. Note that P_1 is simply the tableau $\boxed{\sigma_1}$. Since $(1, \sigma_1)$ is the leftmost point, it never falls in the shadow of another point and thus the lowest point of L_1 crossed by $x = 1$ will be σ_1. Thus assume by induction that the bottom row of P_i is filled with $y_1 \ldots y_\ell$, the lowest y-coordinates of shadow lines at $x = i$.

$$P^i = \begin{array}{|c|c|cccc}\hline & & & & & \\\hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & & \\\hline y_1 & y_2 & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c|}{y_\ell}\\\hline\end{array} \qquad \begin{array}{l} - \; y_\ell \\ - \; \cdot \\ \mathrm{l} \;\; \vdots \\ - \; \cdot \\ - \; y_2 \\ - \; y_1 \end{array}$$

We shall prove that the bottom row of P_{i+1} contains the lowest y-coordinates of shadow lines at $x = i+1$. Note that $P_{i+1} = P_i \leftarrow \sigma_{i+1}$. In the case that $\sigma_{i+1} > y_\ell$, Schensted insertion implies that a new cell is added to the bottom row of P_{i+1}, filled with letter σ_{i+1}. Note this implies that letter $i+1$ is placed in the bottom row of the recording tableau Q.

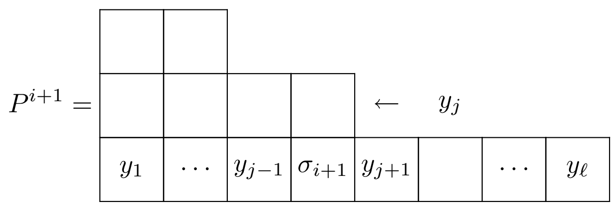
$$P^{i+1} = \begin{array}{|c|c|ccccc}\hline & & & & & & \\\hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & & & \\\hline y_1 & y_2 & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c|}{\cdot} & \multicolumn{1}{c|}{y_\ell} & \multicolumn{1}{c|}{\sigma_{i+1}}\\\hline\end{array} \qquad Q^{i+1} = \begin{array}{|c|c|ccccc}\hline & & & & & & \\\hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & & & \\\hline & & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{} & \multicolumn{1}{c|}{i+1}\\\hline\end{array}$$

In the shadow diagram of σ, $\sigma_{i+1} > y_\ell > \cdots > y_1$ implies that the lowest points on all shadow lines are lower then $y = \sigma_{i+1}$ and thus a new shadow line begins at $x = i+1$ with bottom corner $(i+1, \sigma_{i+1})$.



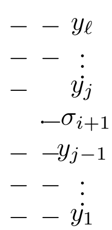
Therefore, the bottom row of P_{i+1} contains the lowest y-coordinates of shadow lines at $x = i+1$.

On the other hand, if there is some j where $y_1 < \cdots < y_{j-1} < \sigma_{i+1} < y_j$, then row insertion implies that the entry y_j is replaced by σ_{i+1} in the bottom row of $P_{i+1} = P_i \leftarrow \sigma_{i+1}$.



Note that since a cell has not been added to the bottorm row, $i+1$ does not occur in the bottom row of Q.

In the shadow diagram when $y_{j-1} < \sigma_{i+1} < y_j$, $(i+1, \sigma_{i+1})$ lies on the shadow line that was at height y_j when $x = i$. Thus, the lowest coordinate y_j of this line is replaced by σ_{i+1} (depicted by a vertical move) while all others remain at the same height (since they are horizontal moves).



Therefore, the bottom row of P_{i+1} contains exactly the lowest y-coordinates of the shadow lines at $x = i+1$ as claimed.

In summary, the shadow diagram of σ depicts the changes to the first row of the insertion tableau of $\emptyset \leftarrow \sigma$. That is, a vertical segment at $x = i$ indicates that a letter is bumped from the bottom row when σ_i is inserted, and if the vertical piece starts a new shadow line, nothing is bumped and σ_i is added to the end of the first row.

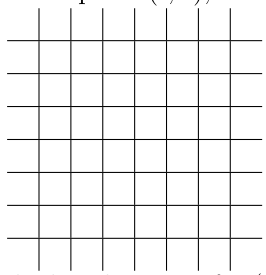
Since the recording tableau puts letter i in the cell added to P_{i-1} after inserting σ_i, and a cell is added to the bottom row only when a new shadow line is started, the bottom row of the recording tableau Q is determined by the coordinates $x_1, \ldots, x_n$ that mark the start of the shadow lines $L_1, \ldots, L_n$.

$\square$

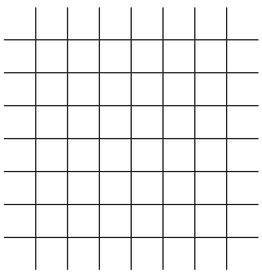
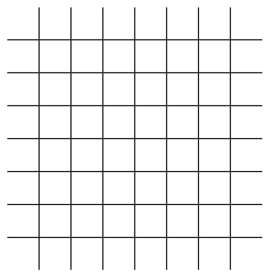
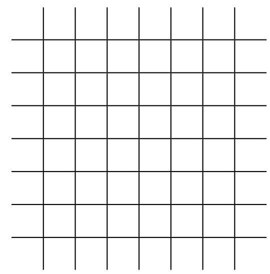
In particular, if we examine the vertical line $x = n$, we find that the bottom row of $P = \emptyset \leftarrow \sigma$ is given by the lowest coordinates $y_1, \ldots, y_n$ of the shadow lines $L_1, \ldots, L_n$. And if we examine the horizontal line $y = n$, we find that the bottom row of Q is given by the leftmost x-coordinates of the shadow lines $L_1, \ldots, L_n$.

If $i \to \sigma_i$ under σ then $\sigma_i \to i$ under σ^{-1}. Thus, the shadow diagram for σ^{-1} is simply the reflection of the shadow diagram for σ about the line $y = x$ (i.e. $(i, \sigma_i) \to (\sigma_i, i)$). Since we can obtain the bottom row of P and Q in $\sigma \leftrightarrow (P, Q)$ by reading the appropriate x and y-coordinates from the shadow diagram for σ, it is thus clear that with $\sigma^{-1} \leftrightarrow (\bar{P}, \bar{Q})$, the bottom row of $\bar{P}$ is given by the bottom row of Q and the bottom row of $\bar{Q}$ is given by the bottom row of P.

To construct the remainder of the tableaux (P, Q) from the shadow diagram of σ, note that if an upper corner on any shadow diagram is the point (i, a), then the letter a was bumped from the first row of P^i with the insertion of σ_{i+1}.



Thus, if $y_1, \ldots, y_r$ denote the set of y-coordinates of the upper corners in the shadow diagram of σ (where y_1 is the leftmost such corner and y_r the rightmost), the tableau P (minus the bottom row) is exactly $\emptyset \leftarrow y_1 \leftarrow \cdots \leftarrow y_r$.

If we construct a new shadow diagram using the set of upper corners, we can apply the previous result to obtain the second row of P and Q. By iterating this procedure, we can construct the tableaux (P, Q) and have proven:

Theorem 14. *For $\sigma \in S_n$, if $\sigma \leftrightarrow (P, Q)$ then $\sigma^{-1} \leftrightarrow (Q, P)$.*