Math 1505G, 2013
Graphs and Matrices

September 27, 2013

These are some notes for the short talk I gave the other day. We'll discuss an interesting application of matrix algebra. This is outside what will be tested in the course, but gives an idea of some other ways that math (and in particular linear algebra) shows up in the world.

1 Graphs

Suppose we have, say, a billion Facebook users and the information about who is friends with who. We can represent this information using a *graph*. For our graph, we'll have a *vertex* (or point) for each person, and we'll put an *edge* connecting two people if they are Facebook friends. If there were only four Facebook users, we might get a graph that looks like Figure 1.

We can actually use graphs to keep track of all kinds of information: Any time we have things connected to other things, there's a graph to be made. For example, there's the 'internet graph' of all computers and their connections to the internet. (The computer in your apartment is connected to a computer at your Internet Service Provider, is connected to some big server on the internet backbone, is connected to a compute at Facebook, for example.) There's the graph of cities in Canada, where the edges are roads. And so on. Graphs like this are useful in many, many contexts.

Returning to graphs of social networks, there's an idea of 'degrees of separation:' What is the least number of firendships to get from one person to another? In our picture, person 0 is two degrees of separation from person 2, for example. There's a hypothesis that say everyone in the world is separated by at most 6 degrees of separation. We're going to see how we can use matrix algebra to get an idea of whether that's true or not.

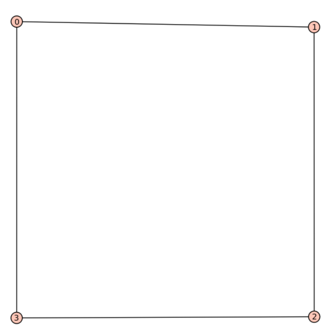

Figure 1: A graph of four people, numbered 0 through 3. Person 0 is friends with person 1 and person 3, but not person 2.

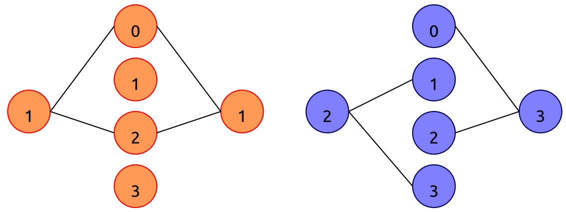

Figure 2: This is an illustration of two of the dot products, red and blue, we used when computing M^2. The first dot product counts ways to go from person 1 to someone else to person 1. The second dot product counts ways of going from person 2 to someone to person 3.

1.1 Encoding a graph in a matrix.

We can keep track of all of the information in the graph in a matrix! With n vertices, we make an $n \times n$ matrix, where each row and column represent a vertex. The graph in Figure 1 has matrix:

$$M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

The first row and first column correspond to person 0, the second row and second column correspond to person 1, and so on. We put a '1' wherever there's a connection between two people: For example, the 1 in the first row and second column says that there's a connection between person 0 and person 1.

As an experiment, let's look at what happens when we square the matrix M.

$$M^2 = M \cdot M = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{pmatrix}.$$

This matrix is actually encoding some interesting information: How many ways are there to get from one person to another person using two steps in the graph for M? In the graph for M, there are two ways to get from person 0 to person 2 using two steps: one via person 1 and another via person 3. There are also two ways to get from person 0 to person 0 using two steps: just go to person 1 or 3 and then back to person 0. Meanwhile, there's *no way* to get from person 0 to person 1 using exactly two steps: perhaps we go to person 1 with the first step, and then have to move using the second step. Similar reasoning works for all the other entries in the matrix.

Let's look at how matrix multiplication works to see why this happens. The entry of M^2 in row i and column j comes from taking the dot product of row i of M with column j of M. Here's how we get the entry 2 in the second row and second column of M^2. I've highlighted in red the second row of the first copy of M and the second column of the second copy of M. The dot product of those two is 2.

$$M^2 = M \cdot M = \begin{pmatrix} 0 & 1 & 0 & 1 \\ {\color{red}1} & {\color{red}0} & {\color{red}1} & {\color{red}0} \\ {\color{blue}0} & {\color{blue}1} & {\color{blue}0} & {\color{blue}1} \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & {\color{red}1} & 0 & {\color{blue}1} \\ 1 & {\color{red}0} & 1 & {\color{blue}0} \\ 0 & {\color{red}1} & 0 & {\color{blue}1} \\ 1 & {\color{red}0} & 1 & {\color{blue}0} \end{pmatrix} = \begin{pmatrix} 2 & 0 & 2 & 0 \\ 0 & {\color{red}1} & 0 & 2 \\ 2 & 0 & 2 & {\color{blue}0} \\ 0 & 2 & 0 & 2 \end{pmatrix}.$$

Let's look at that dot product: $1 \cdot 1 + 0 \cdot 0 + 1 \cdot 1 + 0 \cdot 0$. The first summand has two ones, one for the edge from person 1 to person 0; the second 1 comes from a way to get from person 0 to person 1. Together, they give a way to go from person 1 to person 0 and back to person 1.

Let's look also at one of the zero entries. Consider the row and column highlighted in blue. The dot product gives $0 \cdot 1 + 1 \cdot 0 + 0 \cdot 1 + 1 \cdot 0$. The first $0 \cdot 1$ says 'there's no way to get from person 2 to person 0, and there is one way to get from person 0 to person 3. Together, that makes no ways to get from person 2 to person 3.' And the rest of the terms of the dot product say similar things. They sum to zero. (You can see this represented visually in Figure 2.)

Similar reasoning holds for every term of M^2, so the entries of M^2 are counting the number of ways to get from person i to person j.

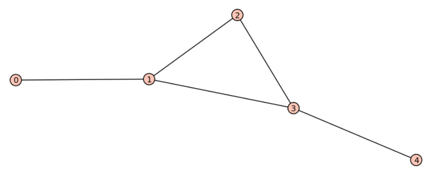

Figure 3: A graph of five people, labelled 0 through 4, as encoded in the matrix N.

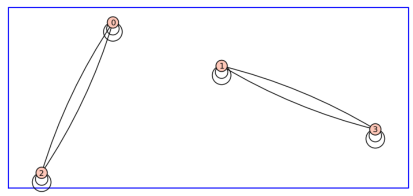

Figure 4: The graph for M^2.

1.2 Graphs from Matrices

We can also start with a matrix and reconstruct a graph. For example, the matrix:

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

corresponds to the graph in Figure 3.

We can also get a graph for M^2, but we need to decide what the 2's mean. We could have graphs with more than one edge between some people; if we draw the corresponding graph, it looks like Figure 4. This graph tells has one edge for each way to get from one person to another using *two* steps. We also see that this graph has *loops*, ways to get from person i to person i using a step. We'll come back to think about loops shortly.

What happens if we lok at M^3? We can think og $M^3 = M \cdot M^2$. The matrix multiplication then says, 'How many ways are there to get from person i to some person x with one step (using M), and then from person x to person j using 2 steps (using M^2)?'

$$M^3 = M \cdot M^2 = \begin{pmatrix} 0 & 1 & 0 & 1 \\ \color{red}{1} & \color{red}{0} & \color{red}{1} & \color{red}{0} \\ \color{blue}{0} & \color{blue}{1} & \color{blue}{0} & \color{blue}{1} \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 2 & \color{red}{0} & 2 & \color{blue}{0} \\ 0 & \color{red}{2} & 0 & \color{blue}{2} \\ 2 & \color{red}{0} & 2 & \color{blue}{0} \\ 0 & \color{red}{2} & 0 & \color{blue}{2} \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 & 4 \\ 4 & \color{red}{0} & 4 & 0 \\ 0 & 4 & 0 & \color{blue}{4} \\ 4 & 0 & 4 & 0 \end{pmatrix}.$$

The red and blue dot products are again represented visually in Figure 5. This matrix gives the number of ways of getting from one person to another using *three* steps. For example, to get from person 2 to person 3, there's one way to go from person 2 to person 1, and then two ways to go from person 1 to person 3 using two steps. This makes two ways; going via person 3 gives another two ways, to make four ways in all.

I can use the same logic again and again, and conclude that if I'm interested in how many ways there are to get from point i to point j using n steps, I should then look at the i, j entry of M^n!

1.3 Back to the Facebook Problem

But what if I want to know if there's a way to get from one person to another using *at most* 6 steps; I should be allowed to take four steps or two steps or however many I like, so long as the total is less than six. One way to do it is to include loops on all of the points in the original graph. This means we can 'hang around' for a while when we make a path with six steps. A loop on every vertex means putting a 1 in every diagonal

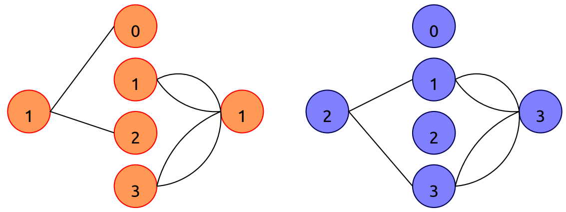

Figure 5: This is an illustration of two of the dot products, red and blue, we used when computing M^2. The first dot product counts ways to go from person 1 to someone else to person 1. The second dot product counts ways of going from person 2 to someone to person 3.

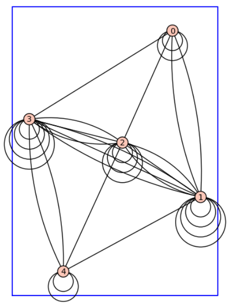

Figure 6: The graph for $(N+1)^2$.

entry of the graph's matrix. This is the same as adding the identity matrix to M! Then M with loops is the same as $M+1$.

Let's look again at the graph in Figure 4, whose matrix is N. Then:

$$(N+1)^2 = \begin{pmatrix} 2 & 2 & 1 & 1 & 0 \\ 2 & 4 & 3 & 3 & 1 \\ 1 & 3 & 3 & 3 & 1 \\ 1 & 3 & 3 & 4 & 2 \\ 0 & 1 & 1 & 2 & 2 \end{pmatrix}.$$

The graph that goes with this matrix is shown in Figure 6. Numerically, we see 0's in the matrix in entries the corners. These are saying that there's no way to go from person 0 to person 4 in the graph using two steps. We can see this in the picture of $(N+1)^2$ because there's no edge connecting vertex 0 and vertex 4.

So to solve the Facebook problem, we can take the one billion-by-one billion matrix of information about Facebook friends. Call this matrix F. Then the six degrees of separation hypothesis is true if $(F+1)^6$ has no zero entries!

1.4 Other Applications

Once we have this perspective, there are many, many other applications available, possibly using different kinds of mathematics alongside what we've already done. For example, suppose we have a population of people and a graph of who spends time with who regularly. If one person has a (say) airborne disease, then there's a probability that they'll spread the disease to their friends. We can use similar methods to what we've seen here to figure out the chances that different people will be infected after some time goes by.

Similarly, people have studied the spread of AIDS by examining sexual networks: by making graphs of sexual partners and calculating transmission chances, we can make a model of how a sexually transmitted disease could spread through the population. Places with different kinds of sexual networks end up having different vulnerability to thte AIDS epidemic.

On the social science side, we could study how different bits of cultural information known as YouTube videos 'go viral.' We have social networks, through which people share cultural information. How does the structure of the network affect the spread of cultural information? Again, we could study this using graphs and matrices.

1.5 A Few Problems

To check your understanding, try the following:

1. Draw a graph with four vertices. Write down the matrix M for that graph.
2. Compute the matrix M^2, and verify that it counts paths of length 2 in the graph that you drew.
3. Compute the matrix $M^5 = M^2 \cdot M^3$. Draw a diagram showing the number of ways of getting from some particular vertex to another using an edge from M^2 followed by an edge from M^3. Verify that the entries of M^5 correspond to paths of length 5 in the graph.
4. Use $(M + 1)^2$ to find which vertices can reach which using a path with at most two steps.