Midterm Solutions (TMTOWTDI)

① Solve $\tan\alpha \sin\alpha + \sqrt{3}\sin\alpha = 0$

$\sin(\alpha)(\tan\alpha + \sqrt{3}) = 0$

$\Rightarrow \sin\alpha = 0$ or $\tan\alpha + \sqrt{3} = 0$.

By special angles, this gives

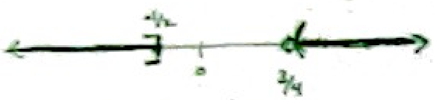
$\alpha = \frac{\pi}{2}$ or $\frac{3}{2}\pi$, or $\alpha = \frac{2\pi}{3}$ or $\frac{5\pi}{3}$.

---

② a) Find domain for $f = \sqrt{\frac{2x+1}{4x-3}}$

- No div. by 0, so $x \neq \frac{3}{4}$.
- Need $\frac{2x+1}{4x-3} > 0$. So numerator, denominator both greater or less than 0.

$\begin{matrix} 2x+1>0 \\ 4x-3>0 \end{matrix} \Rightarrow \begin{matrix} x > -\frac{1}{2} \\ x > \frac{3}{4} \end{matrix}$ or $\begin{matrix} 2x+1<0 \\ 4x-3<0 \end{matrix} \Rightarrow \begin{matrix} x < -\frac{1}{2} \\ x < \frac{3}{4} \end{matrix}$



Then $x \in (-\infty, -\frac{1}{2}] \cup (3/4, \infty)$.

---

b) Find inverse for $f = \sqrt{\frac{2x+1}{4x-3}}$, with domain.

Solve for x.

$y^2 = \frac{2x+1}{4x-3} \xrightarrow{\text{ALGEBRA}} x = \frac{1+3y^2}{4y^2-2}$

Domain: $\mathbb{R} \setminus \{\pm\frac{1}{\sqrt{2}}\}$

---

③ Find all solutions for: $\begin{cases} x+y-2z=-9 \\ 3x-5y+2z=-11 \\ 4x-7y+3z=-14 \end{cases}$

Gaussian Elimination yields:

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is a consistent system with infinitely many solutions. Let z be a free variable.

Then $y - z = -2 \Rightarrow \boxed{y = z-2}$

$x - z = -7 \Rightarrow \boxed{x = z-7}$

---

④ Limits. a) $\lim_{x\to 1} \frac{x^2-1}{x+1}$. Function is cts at $x=1$.

So limit is $\frac{1^2-1}{1+1} = \frac{0}{2} = \boxed{0}$

b) $\lim_{x\to 0} \frac{1-\cos^2(2x)}{3x^2} = \lim \frac{\sin^2(2x)}{3x^2}$. Set $u = 2x$, so $x^2 = u^2/4$

$= \lim_{u\to 0} \frac{\sin^2(u)}{\frac{3}{4}u^2} = \lim_{u\to 0} \frac{4}{3}\frac{\sin^2(u)}{u^2} = \boxed{\frac{4}{3}}$

c) $\lim_{x\to 0} \frac{\sqrt{2x^2+3x+1} - \sqrt{2x^2+x+1}}{x}$

Set $A = 2x^2+3x+1$, $B = 2x^2+x+1$.

Then $\lim_{x\to 0} \frac{\sqrt{A} - \sqrt{B}}{x} = \lim_{x\to 0} \frac{A-B}{x(\sqrt{A}+\sqrt{B})}$

$(A-B = 2x$, so$) = \lim_{x\to 0} \frac{2x}{x(\sqrt{A}+\sqrt{B})}$

$(\lim_{x\to 0} A = 1, \lim_{x\to 0} B = 1$, so$) = \lim_{x\to 0} \frac{2}{\sqrt{A}+\sqrt{B}} = \frac{2}{2} = 1$

d) $\lim_{x\to 0} \frac{e^{2x}-1}{1-e^x} = \lim_{x\to 0} \frac{(e^x)^2-1}{-(e^x-1)} = \lim_{x\to 0} \frac{(e^x+1)\cancel{(e^x-1)}}{-\cancel{(e^x-1)}}$

$= \lim_{x\to 0} -e^x - 1 = \boxed{-2}$

e) $\lim_{x\to 0} x^2 \sin(\frac{1}{x})$

~~Set ... lim~~

$-x^2 \leq x^2\sin(\frac{1}{x}) \leq x^2$

By sandwich theorem, $\lim_{x\to 0} x^2\sin(\frac{1}{x}) \to \boxed{0}$

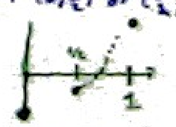
f) $\lim_{x\to\infty} \frac{\ln(x)+1}{\ln(x)+5} = \lim_{x\to\infty} \frac{1+\frac{1}{\ln(x)}}{1+\frac{5}{\ln(x)}} \to \frac{1+0}{1+0} = \boxed{1}$

---

5) $f(x) = x^5 + x - 1$. Solution in $(0, \frac{1}{2})$ or $(\frac{1}{2}, 1)$?

$f(0) = -1$, $f(1) = 1$.

$f(\frac{1}{2}) = \frac{1}{32} + \frac{1}{2} - 1 < 0$



By continuity, $\exists\, c \in [\frac{1}{2}, 1]$ such that $f(c) = 0$.

---

6) $f(x) = |x|$ a) Show f cts at $x=0$.

$f(0) = 0$. For $x<0$, $f(x) = -x \Rightarrow \lim_{x\to 0^-} |x| = 0$.

For $x>0$, $f(x) = x \Rightarrow \lim_{x\to 0^+} |x| = 0$

Limits are equal, and equal to evaluation, so f is cts at $x=0$.

b) Prove f not diff'ble at 0.

The $\lim_{x\to 0^-} f'(x) = -1$, $\lim_{x\to 0^+} f'(x) = 1$.

These are not equal, so f not diff'ble at 0.

---

⑦ Derivatives! a) $\frac{\sqrt{3}}{t^2} - \frac{3}{\sqrt{t}} + \ln(33)$

$f = \sqrt{3}t^{-2} - 3t^{-1/2} + \ln(33)$

$\Rightarrow \boxed{\frac{df}{dt} = -2\sqrt{3}t^{-3} + \frac{3}{2}t^{-3/2} + 0}$

b) $f = e^{\arcsin(x)}$

$\frac{df}{dx} = e^{\arcsin(x)} \cdot \frac{d}{dx}\arcsin(x) = \boxed{e^{\arcsin(x)} \cdot \frac{1}{\sqrt{1-x^2}}}$

c) $f = \frac{\tan x + 1}{\sec x + 2} = \frac{\sin x + \cos x}{1 + 2\cos(x)}$

$\Rightarrow \frac{df}{dx} = \frac{(\cos(x)-\sin(x))(1+2\cos(x)) + (\sin x + \cos x)(2\sin x)}{(1+2\cos(x))^2}$

d) $f = \log_7[(x^2+2)^{1/x}] = \frac{1}{\ln(7)} \cdot \ln[(x^2+2)^{1/x}]$

$\rightarrow f = \frac{1}{x\ln(7)} \cdot \ln(x^2+2)$

$\Rightarrow \boxed{\frac{df}{dx} = \frac{-1}{x^2\ln(7)} \cdot \ln(x^2+2) + \frac{1}{x\ln(7)} \cdot \frac{1}{x^2+2} \cdot 2x}$

e) $f = x^3\sqrt{5x^4+3}$

$\boxed{\frac{df}{dx} = 3x^2\sqrt{5x^4+3} + x^3\frac{1}{2}(5x^4+3)^{-1/2} \cdot 20x^3}$

---

⑧ Eqn of tangent line

for $x^3 + 2xy - xy^2 + y^2 = 11$ at $(2,1)$

$3x^2 + 2y + 2x\frac{dy}{dx} - y^2 - 2xy\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

Solve for $\frac{dy}{dx} = \frac{3x^2+2y-y^2}{2xy-2y-2x}$.

Then at $(2,1)$, $\frac{dy}{dx} = \frac{3\cdot4+2-1}{4-2-4} = \frac{13}{-2} = -\frac{13}{2}$

Then $\boxed{y = -\frac{13}{2}(x-2)+1}$ is the eqn of the tangent line.

Midterm Solutions II

(9) $f = 2x^2+x+1$, $x \geq -\frac{1}{4}$. Find slope of tangent for $y = f^{-1}(x)$ at $x=4$.

This problem is a bit confusingly worded. Rewrite:

Find slope of tangent to $x = f^{-1}(y)$ at $y=4$.

(Thus, f takes x's to y's, f^{-1} takes y's to x's.)

First, when $y=4$, $y = f(x) \Rightarrow 4 = 2x^2+x+1 \Rightarrow 2x^2+x-3=0$.

Solve w/ quadratic formula to get $x=1$ or $x=-\frac{3}{2}$.

Since $x \geq -\frac{1}{4}$, take solution $\boxed{x=1}$.

For slope of tangent line, use: $f(f^{-1}(y)) = y$

So $\frac{d}{dy}\left(f(f^{-1}(y))\right) = 1 \Rightarrow \frac{df}{dy}(f^{-1}(y)) \cdot \frac{d}{dy} f^{-1}(y) = 1$

$$\Rightarrow \frac{df^{-1}}{dy}(y) = \frac{1}{\frac{df}{dy}(f^{-1}(y))} = \frac{1}{\frac{df}{dy}(x)}$$

Have $f' = 4x+1$. At $x=1$, we get 5.

So $\frac{df^{-1}}{dy}(y) = \frac{1}{5}$.

(10) $f = (x^4+2x+1)^{x+1}$. Find df/dx.

$\ln(f) = (x+1)\ln(x^4+2x+1)$

$\Rightarrow \frac{1}{f} \cdot \frac{df}{dx} = \ln(x^4+2x+1) + \frac{x+1}{x^4+2x+1}$

$\Rightarrow \frac{df}{dx} = \left[\ln(x^4+2x+1) + \frac{x+1}{x^4+2x+1}\right] \cdot (x^4+2x+1)^{x+1}$.

(11) $V(r) = 100\pi r^{1/2}$, $r=2$, $\frac{dr}{dt} = \frac{1}{2}$. Find $\frac{dV}{dt}$.

$\frac{dV}{dt} = \frac{100\pi}{2} \cdot r^{-1/2} \cdot \frac{dr}{dt} \rightsquigarrow \boxed{50\pi \cdot \frac{1}{4} \cdot \frac{1}{2}}$ at $r=2$, $\frac{dr}{dt} = \frac{1}{2}$.

(12) $V(r) = 100\pi r^{1/2}$

$r_0 = 4$, percent error in $r < 2$

Have percent error in V: $\left|\frac{V(r) - V(r_0)}{V(r_0)}\right|$

$V(r) \sim V'(r_0)(r-r_0) + V(r_0)$ (linearizing), so

percent error in $V \sim \left|\frac{V'(r_0)\cdot(r-r_0) + \cancel{V(r_0)} - \cancel{V(r_0)}}{V(r_0)}\right|$

$$= \left|\frac{V'(r_0)(r-r_0)\, r_0}{V(r_0)\, r_0}\right| = \left|\frac{V'(r_0)\cdot r_0}{V(r_0)}\right| \cdot \underbrace{\left|\frac{r-r_0}{r_0}\right|}_{\text{percent error in } r \leq 2}$$

$V'(r_0) = 50\pi \cdot \frac{1}{2}$

$V(r_0) = 100\pi \cdot 2$

So percent error $\leq \left|\frac{50\pi \cdot \frac{1}{2} \cdot \cancel{4}^{\,2}}{100\pi \cdot \cancel{2}}\right| \cdot \cancel{2} = 1$