MMM1:

An Introduction to Ordinary Generating Functions

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Introduction

1

This chapter is discussion about the goals of this book.

Section 1 tells what the book is about and section 2 talks about some of the background the reader will need to know to do the exercises.

About this ebook

For a video summary of this section:

http://garsia.math.yorku.ca/~zabrocki/MMM1/summary11.mov

Mini-Mathematical Monographs

This book is a short introduction to generating functions from a a computational perspective. It is an experiment with new technology.

My goal is to cover only the topic of *ordinary generating functions* and only a quick introduction. It does not extend beyond the computational aspects and does not cover more combinatorial applications of generating functions. These are topics that I hope to cover in other short books.

I would like this ebook to be a model for what can be done for explaining mathematics. What I am trying to develop here that is different than most texts:

1. brief and focused

2. visual

3. an introduction to the use of computers (in particular, Sage)

This is not a textbook in the usual sense where a large body of mathematics is summarized and contextualized. One aspect that I am attempting to experiment with in this textbook is to present a topic that one can potentially read and understand in a single day. The video summaries of each of the sections are supposed to facilitate this.

I am using Apple's iBook Author technology to write this book because I would like to use some of the aesthetic and video technology that this program offers. I realize that there is a great risk of obsolescence and limitations in the availability of the format, but it is an experiment.

This is a topic that I have covered in a number of courses that I teach at York University (in Toronto, Canada). In particular, this topic appears in an expanded form a course on number theory and combinatorics for teachers in a part time M.A. program for Teachers offered at York University. It is also a topic that I usually cover in a 4th year combinatorics class. At most this material might cover what I do over a period of 1-3 weeks of a course.

This topic is appropriate for any student that has basic algebra and (some) calculus skills. Generating functions are an important tool for manipulation of all kinds of sequences. This book might be appropriate for an advanced high school student.

I am not looking to do a comprehensive introduction to generating functions. Instead I would like a presentation which will serve as an introduction and give the reader the ability to understand and compute several advanced examples.

The main goal is to summarize the subject of generating functions so that the reader is able to prove all of the summation formulas in the last chapter and more generally recognize that summation formulas can be proven by using generating functions using a method which is even more systematic than induction.

In most of my courses where I cover generating functions, I would cover at least a "Part II" to this book which is a bridge between combinatorics and generating functions which would be an additional 1-3 weeks. More advanced topics would require at least two additional parts.

I hope to do something different than the usual mathematics textbook by adding video animations which summarize the text.

Notes about recommended background of the reader

For a video summary of this section:

http://garsia.math.yorku.ca/~zabrocki/MMM1/summary12.mov

This section is mainly written with my students in mind. I know that before introducing this topic to my first and second year undergraduate students I would spend some time introducing notation and ensuring that they have the requisite background.

I won't cover that material here, just recall what background I expect students to have before beginning.

Algebra, Symbolic notation and Pattern Matching

To my eye, one of the most important skills that a reader will have to have in order to appreciate the contents of this book is the skill to manipulate algebraic and symbolic expressions.

University level mathematics requires (often unspoken) skills of unpacking, packing and parsing symbolic expressions. This is a skill that teachers try to transmit starting in grade school and by university level we expect students to have mastered recognizing formulas. These skills are challenging to develop. Moreover they mainly taught indirectly by providing example after example.

For example, we will use summation notation

(1.2.1.1) $$\sum_{i \geq 0} a_i = a_0 + a_1 + a_2 + a_3 + \cdots$$

where the a_i represents some expression which depends on the index i.

Now, say that we were to encounter the following sum:

$$1 \cdot 2 - 2 \cdot 3x + 3 \cdot 4x^2 - 4 \cdot 5x^3 + \cdots \tag{1.2.2.2}$$

To say that this is an expression of the same form as the right hand side of equation (1.2.1.1) means that

$$a_0 = 1 \cdot 2,\ a_1 = -2 \cdot 3x,\ a_2 = 3 \cdot 4x^2,\ a_3 = -4 \cdot 5x^3,\ \ldots$$

and arriving at a formula for a_i in general is not necessarily obvious.

Asking someone to turn this into an expression involving a summation notation requires a lot of practice and intuition (and an answer is not at all unique either). A reader would need to notice that in order to make the terms alternate between positive and negative values they might need to know that $(-1)^i$ is 1 if i is even and -1 if i is odd. In which case, a more compact way of writing the sum in equation (1.2.2.2) is

$$\sum_{i \geq 0} (-1)^i (i+1)(i+2)x^i. \tag{1.2.2.3}$$

There are an infinite number of ways of writing the same expression and another person might find it equally helpful to write equations (1.2.2.2) or (1.2.2.3) as

$$\sum_{n \geq 1} (n^2 + n) \cdot \cos((n+1)\pi) x^{n-1}. \tag{1.2.2.4}$$

Do you see how these two expressions are equal? If the answer is yes, then you probably know enough algebra to proceed in this book. If not, beware! There are algebra skills which will be used in the rest of this book that may be challenging.

Calculus and Taylor series

Calculus is the study of the infinite and the infinitesimal.

We will be studying manipulations of infinite series and although we won't be asking the same questions that one would ask about series in a calculus class (we will not at all be concerned about the convergence of such series), we will be using operations that one learns in a calculus class.

In order to manipulate generating functions we will on occasion use the derivative and integral operators. We won't use all of their properties, but we will need to know the derivatives of basic functions such as x^n, the product rule, the chain rule, partial fraction decomposition. Maybe if you are clever you will see a way of using integration by parts in some of the exercises.

More frequently we will require the single variable Taylor's theorem. It relates the coefficient of x^n in the series expansion of $f(x)$ and the n^{th} derivative of the function evaluated at 0 (which is denoted $f^{(n)}(0)$). Taylor's theorem states that a function $f(x)$ has a series expansion in the variable x expanded about the point $x = 0$ given by

$$f(x) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} x^n \text{ or}$$

(1.2.2.1) $$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \cdots .$$

In this form the theorem is sometimes called the Maclaurin expansion of a function.

Geometric series

The starting point for many of the generating functions that we will consider in this book is the geometric series

(1.2.3.1) $$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

This is probably the first infinite series that most people will encounter. If you are not familiar with this series, you should ask yourself why the left hand side of equation is equal to the right hand side.

The formal proof of this fact starts by assuming that the right hand side makes sense. That is,

Step 1: Let $A(x) = 1 + x + x^2 + x^3 + \cdots$

Step 2: Multiply $A(x)$ by $(1 - x)$ and expand the expression

$$(1-x) \cdot A(x) = (1-x) + (1-x)x + (1-x)x^2 + (1-x)x^3 + \cdots$$

Step 3: Expand the expression further and cancel terms

$$1 - x + x - x^2 + x^2 - x^3 + x^3 - x^4 + \cdots = 1$$

Conclude: $(1-x) \cdot A(x) = 1$, hence $A(x) = \dfrac{1}{1-x}$.

The proof above is a bit disingenuous. It ignores the fact that sometimes when we do an infinite number of manipulations of symbols (as we have done) that sometimes things go very wrong. The proof above implicitly assumed that it is possible to do an infinite number of regrouping of terms (applications of the associative law) and an infinite number of cancellations and the result is what we expect it to be.

WARNING: It is not always the case that you can perform an infinite number of operations in two different ways and get the same answer.

For instance, consider

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

Now add two times the negative terms and subtract off just as much

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \cdots\right)$$

now expand the second sum and this is

$$= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots\right) = 0$$

Therefore $\ln 2 = 0$ and we have a serious problem. However, if you payed close attention to the operations, you will notice that we added and subtracted a sum which was infinite so it shouldn't be surprising that something went wrong.

What we are subtly doing when we work with generating functions is grading the operations that we do into parts of finite degree and then working on just the parts of a finite degree and then saying that we are doing those operations on all degrees. It saves us from doing infinite operations that break mathematical rules.

A more general form of the geometric series is with real numbers a and b that we will encounter many times is the expression

$$\frac{a}{1-bx} = a + abx + ab^2x^2 + ab^3x^3 + \cdots. \tag{1.2.3.2}$$

It is probably a good idea to go through the proof in the case that $a = b = 1$ that we have stated above to see that it also works for any a and b.

Computers - Sage!

One feature I would like this book to have is a lot of examples that can be computed by hand, and a number of examples that require computations which cannot easily be done by hand and are more suited to a computer.

Computer Algebra Systems (CAS) are advanced calculators. They are computer languages wrapped around functionality that one would like to have in order to do calculations in mathematics.

Fortunately there is an open source mathematical software package called Sage available that will do the types of calculations that we would like to for this book. The examples that I will put in the text will all be in this language but similar commands will work in Maple, Mathematica or other CAS.

I find that many of my students shy away from the computer. Many students have expressed to me that they are intimidated by programming languages or computers beyond the use of the internet. These are technical skills that require a large investment of time to gain.

I have two answers to this:

(1) you don't need to be an expert to use a computer to do certain calculations, but you have to be willing to experiment.
(2) the computer skills are worth your investment of time. Start here and now. Pick something you would like to do with a computer and learn to do it. The most important skill you can learn is how to learn on your own.

At the very least follow the examples in the book, parse the input commands and copy them into a running copy of Sage and verify that you get the same answer. Next, try to change the input and do a few calculations of your own and some of the exercises.

Sage is freely available and there are hundreds of mathematicians working to improve it's capabilities. It is a fantastic way of sharing mathematical programs for computation.

It is not necessary to learn a lot to start using Sage as a calculator. Do a computer search for "sage mathematics" and you will find the site for the open source mathematics program Sage. You can either download the program onto your computer or log into an online site and enter the commands in the white boxes in the text.

In the white text the word `sage:` indicates this is the computer prompt at the beginning of the line. It is there to express "Sage is waiting for you to enter text." You do not need to type this word. You may not see this in the version of Sage you are using unless you are running it from the command line, but it is a convenient way of indicating to a reader that the command starts here.

The bold face text that follows can be entered in the text box.

Except for very few examples the text will be commands to expand the Taylor series of some expression. The text

`taylor(expression, x, 0, 10)` indicates to the program Sage that it should compute the Taylor expansion of the "expression" in the variable x up to degree 10 (so that there 11 terms in total). If you change the 0 to another value a you will see that it will expand the series about the variable $x - a$.

The text that is not in bold in the white box examples is what Sage will respond with once the command is evaluated. For example, if I want to find the expansion of the first 10 terms of the generating function $\dfrac{1-\sqrt{1-4x}}{2x}$.

To do this you give Sage the command to give the Taylor expansion of this function.

```
sage: taylor((1-sqrt(1-4*x))/(2*x),x,0,10)
16796*x^10 + 4862*x^9 + 1430*x^8 + 429*x^7 + 132*x^6 + 42*x^5 +
14*x^4 + 5*x^3 + 2*x^2 + x + 1
```

This happens to be the generating function for the Catalan numbers. This sequence is one of the most important in combinatorics. If you read of the coefficients you see that they are

$$1, 1, 2, 5, 14, 42, 132, 429, \ldots$$

and this sequence is called the Catalan numbers. We will see this sequence again in Section 4.1.2.

There are some disadvantages to using Sage over commercial software. The error messages probably leave something to be desired, and the language has a steep learning curve. However, if you are just doing a few short computations there are only a few things that will go wrong.

If you are willing to experiment and read the error messages carefully, Sage is a great program to begin to learn how computers are used to do mathematics.

Generating functions neither generate, nor are they functions

2

This chapter includes a gentle introduction to the topic of this book.

The first section is a discussion about what generating functions are good for, the second has four starter examples.

The three W's of generating functions

For a video summary of this section:

http://garsia.math.yorku.ca/~zabrocki/MMM1/summary21.mov

What?

Say that

$$a_0, a_1, a_2, a_3, \ldots$$

is a sequence of numbers, then the generating function of this sequence in the formal parameter x is

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots$$

When?

Whenever you have *any* sequence of numbers make the generating function and see if you can find a formula for the series.

Why?

One motivating question that I encounter all the time is the following formula that my students see in high school or their first year proofs class:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

They see this equation and say "I can prove this by induction after you give me the right hand side of the equation, but I don't think that I could have guessed at the right hand side of the equation myself and I don't see how to derive it." The same applies for any of dozens of summation formulas that I throw at them when we learn induction. While induction shows them that the equation is true, it doesn't show them how to come up with what the sum is equal to.

My answer to their question is "give me 2 hours to teach you generating functions and you will be able to arrive at this formula yourself." (Note: the answer to their question is in Section 3.2.5)

Because the generating function is an algebraic expression that encodes the sequence and allows you to manipulate it in ways that are not possible in other forms. Many times if the sequence you are looking at is "interesting" (and this word has lots of interpretations), the generating function has a short simple form.

The generating function allows you to derive formulas for the sequence, identities involving the sequence, estimate the values and so much more.

The real answer to "why?" will come after seeing many examples and the power of what generating functions are able to do. Once you learn just a few basic skills you should be able to do the exercises in Chapter 4 and you will be able to derive and prove all sorts of mathematical identities.

Think of a sequence...

If you write down the generating function for a sequence that follows a pattern, it very likely has a relatively simple generating function.

$$1, 9, 25, 49, 81, 121, \ldots$$

The generating function for this sequence is

$$1 + 9x + 25x^2 + 49x^3 + 81x^4 + 121x^5 + \cdots$$

If you think about it, you will probably be able to guess at the next terms in the sequence for a number of reasons. The differences between consecutive ones follows a nice pattern and if you try and factor the terms you might guess at a formula for a_n.

There is also a nice formula for the generating function (which we will learn to calculate in later chapters) because it is equal to

$$\frac{1 + 6x + x^2}{(1 - x)^3}.$$

We can verify that the first few terms of this sequence agree by computing the Taylor series of this expression using Sage.

```
sage: taylor((1+6*x+x^2)/(1-x)^3,x,0,6)
169*x^6 + 121*x^5 + 81*x^4 + 49*x^3 + 25*x^2 + 9*x + 1
```

On the other hand, lets say that I were to consider the sequence

$$3, 7, 1.1, -2, \pi, 2011, 8, 1, -3, \dots$$

which really doesn't have any pattern or formula. The generating function is equal to

$$3 + 7x + 1.1x^2 - 2x^3 + \pi x^4 + 2011x^5 + 8x^6 + x^7 - 3x^8 + \cdots.$$

This generating function exists, but is unlikely to have a more compact formula.

One thing to *never* forget

The generating function is not the sequence and the sequence is not the generating function. They are not the same thing. One is a sequence, the other is an algebraic expression.

$$\textbf{sequence} \neq \textbf{generating function}$$

If you have a sequence you can say "the generating function of the sequence" to refer to the algebraic object. If you have a generating function you might say "the sequence of coefficients of the generating function" in order to refer to the sequence.

I emphasize this because it is easy to think about the sequence and exchange it with the generating function and vise versa, but they are two entirely different things.

I was only joking

The title of this chapter is a quote that I often use about generating functions. I say it to indicate that there is something misleading about the name because they don't really "generate" anything (at least not in any normal sense of the word).

We also don't really think of them as functions, although sometimes we specialize the parameter x and use algebraic operations as if they were functions. They are sometimes called 'formal power series' (but only very rarely).

So what are generating functions? I like to think of them as an infinite storage device for sequences of numbers. There is a good analogy that they are a clothesline for sequences of numbers where each power of x^n is a place to pin a number.

Lets just call them by their name and move on.

Examples

For a video summary of this section:

http://garsia.math.yorku.ca/~zabrocki/MMM1/summary22.mov

By the end of this chapter we would like to build up a library of examples that we can use and reuse. In this section I will give four examples which use some typical techniques for finding a formula for a generating function for a sequence.

In general, there isn't one technique that will work. In fact, for any random sequence it will not always be clear that there is a 'nice' formula for a generating function. The examples might be misleading in this way since they are chosen because for these sequences is possible to find a very simple and compact formula, while for a random sequence such a formula might not be possible.

In the first couple of examples we start with the geometric series and build up other examples by using differentiation and multiplication by x. In the last example of this section we use a typical technique to find a equation satisfied by a generating function and then use algebra to arrive at a formula.

A sequence of 1's

Lets try a simple example, the sequence consisting of all 1's:

(2.2.1.1) $$1, 1, 1, 1, 1, \ldots$$

The generating function is the geometric series

(2.2.1.2) $$1 + x + x^2 + x^3 + \ldots = \frac{1}{1-x}$$

In general, whenever we have a sequence that can be obtained by specializing the values a and b in the sequence

(2.2.1.3) $$a, ab, ab^2, ab^3, \ldots$$

then the generating function will be

(2.2.1.4) $$a + abx + ab^2x^2 + ab^3x^3 + \cdots = \frac{a}{1-bx}.$$

While most generating functions have an infinite number of terms, a computer can help us to compute a finite number of those to verify the the sequences of coefficients is correct up to a certain point, or to calculate a single coefficient.

```
sage: taylor(1/(1-x),x,0,10)
x^10 + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1

sage: taylor(2/(1-3*x),x,0,10)
118098*x^10 + 39366*x^9 + 13122*x^8 + 4374*x^7 + 1458*x^6 +
486*x^5 + 162*x^4 + 54*x^3 + 18*x^2 + 6*x + 2

sage: taylor(2/(1-3*x),x,0,100).coefficient(x,100)
1030755041464022662072922259531242545404215044002

sage: 2*3^100
1030755041464022662072922259531242545404215044002
```

The first command in the example above asks the computer to determine the first 11 terms (up to degree 10) of the series defined by `1/(1-x)`. The second command asks the computer to determine the first 11 terms in the expansion of the generating function `2/(1-3*x)`. This second example is equation (2.2.1.4) with $a = 2$ and $b = 3$.

Both of these calculations are easy enough to do by hand because I know for instance that the coefficient of x^{10} in the generating function $\frac{2}{1-3x}$ is equal to $2 \cdot 3^{10}$, but it might take me a while to work out what that number is without a calculator.

The computer is particularly useful for computing single coefficients. We will also use it regularly to test that the intuition we develop about generating agrees with direct calculations.

The positive integers

The next simplest example would be the positive integers:

(2.2.2.1) $$1,2,3,4,5,\ldots$$

This has a generating function

(2.2.2.2) $$1+2x+3x^2+4x^3+\ldots$$

Now observe that the derivative of the left hand side of equation (2.2.1.2) is equal to equation (2.2.2.2). This means we can say that equation (2.2.2.2) is equal to

$$\frac{d}{dx}\frac{1}{1-x}=1+2x+3x^2+4x^3+\cdots.$$

We conclude that

(2.2.2.3) $$\frac{1}{(1-x)^2}=\sum_{n\geq 0}(n+1)x^n$$

We've used a little calculus to show that the generating function for the sequence of positive integers is $\frac{1}{(1-x)^2}$. Now lets use the computer to verify that this is really the case for the first 9 terms.

```
sage: taylor(1/(1-x)^2,x,0,8)
9*x^8 + 8*x^7 + 7*x^6 + 6*x^5 + 5*x^4 + 4*x^3 + 3*x^2 + 2*x + 1
```

The squares of the positive integers

We can’t use the exactly same trick to figure out the generating function for the sequence

$$1,4,9,16,25,\ldots$$

because if we take the derivative of equation (2.2.2.2) then we do not quite have the square integers. But the clever reader will notice that if we first multiply equation (2.2.2.2) by x and then take the derivative then we have by equation (2.2.2.3) that

$$\begin{aligned} 1+4x+9x^2+16x^3+\cdots &= 1^2+2^2x+3^2x^2+4^2x^3+\cdots \\ &= \frac{d}{dx}(x+2x^2+3x^3+4x^4+\cdots) \\ &= \frac{d}{dx}(x(1+2x+3x^2+4x^3+\cdots)) \\ &= \frac{d}{dx}\left(\frac{x}{(1-x)^2}\right). \end{aligned}$$

Therefore,

(2.2.3.1) $$\sum_{n\geq 0}(n+1)^2x^n = \frac{1+x}{(1-x)^3}.$$

Lets briefly check that we have done this correctly by computing the first 11 terms of this series on the computer.

```
sage: taylor((1+x)/(1-x)^3,x,0,10)
121*x^10 + 100*x^9 + 81*x^8 + 64*x^7 + 49*x^6 + 36*x^5 + 25*x^4 +
16*x^3 + 9*x^2 + 4*x + 1
```

The Fibonacci numbers

A non-trivial example that one encounters when thinking about possible sequences is the one where $F_0 = F_1 = 1$ and then each subsequent integer is the sum of the previous two.

$$1,1,2,3,5,8,13,21,34,55,\ldots$$

This sequence is named in honor of a mathematician and banker who was instrumental in introducing the arabic numbering system to western society.

We will give the generating function for this sequence a name $F(x)$ so then

$$F(x) = \sum_{n\geq 0} F_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots$$

where $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Then we can see that

$$\begin{aligned} F(x) = 1 + x + \sum_{n\geq 2} F_n x^n &= 1 + x + \sum_{n\geq 2} (F_{n-1} + F_{n-2}) x^n \\ &= 1 + x + \sum_{n\geq 2} F_{n-1} x^n + \sum_{n\geq 2} F_{n-2} x^n \\ &= 1 + x + (x^2 + 2x^3 + 3x^4 + \cdots) + (x^2 + x^3 + 2x^4 + 3x^5 + \cdots) \\ &= 1 + xF(x) + x^2 F(x) \end{aligned}$$

Since we have figured out that $F(x) = 1 + xF(x) + x^2F(x)$, then

$$F(x) - xF(x) - x^2F(x) = 1$$

and this can be rewritten as

$$F(x)(1 - x - x^2) = 1$$

and hence

$$F(x) = \frac{1}{1 - x - x^2}. \tag{2.2.4.1}$$

It was always surprising to me that the generating function for the Fibonacci numbers has such a compact formula. In fact, even after I see the derivation I feel like maybe something isn't right and that somehow the Fibonacci numbers have disappeared. It helps me to see that they are still there by computing terms of this sequence and observe that we do see the Fibonacci numbers appearing in the expansion of the Taylor series.

```
sage: taylor(1/(1-x-x^2),x,0,10)
89*x^10 + 55*x^9 + 34*x^8 + 21*x^7 + 13*x^6 + 8*x^5 + 5*x^4 + 3*x^3
+ 2*x^2 + x + 1
```

Getting the most out of your generating functions

3

In this chapter we start to develop techniques for arriving at formulas for generating functions.

In the first section we look at the effect of algebraic operations on generating functions and then look at some examples.

Back and forth

For a video summary of this section:

http://garsia.math.yorku.ca/~zabrocki/MMM1/summary31.mov

It is not enough to go from the sequence to the generating function, one must also do the return trip.

Our goal is to manipulate a sequence by figuring out the generating function, perform algebra on the generating function and then recover the sequence.

Use the library

We have only a couple of examples under our belt, but we will start to make a list so that when we encounter a generating function of the form in our list, then we know what the coefficient of x^n is equal to.

$\frac{a}{1-bx} = \sum_{n\geq 0} ab^n x^n$ from (2.2.1.4)

$\frac{1}{(1-x)^2} = \sum_{n\geq 0} (n+1)x^n$ from (2.2.2.3)

$\frac{1+x}{(1-x)^3} = \sum_{n\geq 0} (n+1)^2 x^n$ from (2.2.3.1)

$\frac{1}{1-x-x^2} = \sum_{n\geq 0} F_n x^n$ from (2.2.5.1)

In the next chapter there is a sequence of exercises that will help us build up this library. Once you do the exercises you will have a more complete list of generating functions to put to use.

New generating functions from old

If you have two generating functions $A(x) = \sum_{n \geq 0} a_n x^n$ and $B(x) = \sum_{n \geq 0} b_n x^n$ for two sequences of integers $a_0, a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, b_3, \ldots$ then there are several ways that we can combine the sequences and get new generating functions for new sequences.

SUM: If we add the generating functions we have that $A(x) + B(x) = \sum_{n \geq 0} (a_n + b_n) x^n$ is a generating function for the sequence

(3.1.2.1) $$a_0 + b_0, a_1 + b_1, a_2 + b_2, a_3 + b_3, \ldots$$

PRODUCT: However if we multiply the the two generating functions we have that

$$\begin{aligned} A(x)B(x) &= (a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots)(b_0 + b_1 x + b_2 x^2 + b_3 x^3 + \cdots) \\ &= a_0 b_0 + (a_1 b_0 + a_0 b_1) x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \cdots \\ &= \sum_{n \geq 0} (a_n b_0 + a_{n-1} b_1 + \cdots + a_0 b_n) x^n. \end{aligned}$$

This can be summarized in the expression

(3.1.2.2) $$A(x)B(x) = \sum_{n \geq 0} \left(\sum_{i=0}^{n} a_{n-i} b_i \right) x^n.$$

One useful special case of this is the generating function $x^r A(x)$ (since this is the product of a generating function for the sequence $0,0,\ldots,0,1,0,0,\ldots$ (where the 1 is in the r^{th} position of a sequence of zeros) and the generating function for $a_0, a_1, a_2, a_3, \ldots$. The product has the effect of shifting the entries in the sequence by r entries higher in the sequence. More specifically

$$\begin{aligned} x^r A(x) &= a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + a_3 x^{r+3} + \cdots \\ &= \sum_{n \geq r} a_{n-r} x^n = \sum_{m \geq 0} a_m x^{r+m}. \end{aligned}$$

Another special case of the product of generating function is the product $\frac{1}{1-x} A(x)$. It is the product of two generating functions, the first one is the generating function in equation (2.2.1.2). By equation (3.1.2.2) the product of these is a generating function for the sequence

(3.1.2.3) $$a_0, a_0 + a_1, a_0 + a_1 + a_2, a_0 + a_1 + a_2 + a_3, \ldots.$$

One might ask what the generating function for the sequence $a_0 b_0, a_1 b_1, a_2 b_2, a_3 b_3, \ldots$ is in terms of the generating functions $A(x)$ and $B(x)$. Sometimes this is possible to do, but there is not always a really good answer for this question.

DERIVATIVE: We have already seen a couple examples of the use of the derivative in previous examples. If we take the derivative once of $A(x)$ then

$$A'(x) = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + 5a_5 x^4 + \cdots.$$

If we multiply the generating function by x then as a total effect it multiplies the coefficient of x^n which is a_n by n. Therefore

(3.1.2.4) $$xA'(x) = 0a_0 + 1a_1 x + 2a_2 x^2 + 3a_3 x^3 + \cdots = \sum_{n \geq 0} n a_n x^n$$

and, in case we want to multiply each term by $n+1$ instead,

(3.1.2.5) $$xA'(x) + A(x) = \frac{d}{dx}(xA(x)) = \sum_{n \geq 0} (n+1) a_n x^n.$$

We could repeatedly take the derivative and multiply our generating function by x to multiply the coefficient of x^n by n^2, n^3 or higher powers of n.

In fact, equation (3.1.2.5) says that the generating function for the sequence $1^r, 2^r, 3^r, 4^r, 5^r, \ldots$ is

(3.1.2.6) $$\left(\frac{d}{dx} x\right)^r \left(\frac{1}{1-x}\right) = \sum_{n \geq 0} (n+1)^r x^n.$$

So lets use the computer (partly to show how to do the calculation and partly because it is not easy to show the steps in text) to figure out what the generating function for the cubes of the positive integers is. We multiply x times $\frac{1+x}{(1-x)^3}$ and then differentiate then we should get the generating function for the positive integers cubed. We can then check that result by taking the Taylor expansion of the result.

```
sage: factor(diff(x*(1+x)/(1-x)^3,x))
(x^2 + 4*x + 1)/(x - 1)^4
sage: taylor((1+4*x+x^2)/(1-x)^4,x,0,6)
343*x^7 + 216*x^6 + 125*x^5 + 64*x^4 + 27*x^3 + 8*x^2 + x
```

This computer computation shows that

$$\frac{1+4x+x^2}{(1-x)^4} = \sum_{n\geq 0} (n+1)^3 x^n. \tag{3.1.2.7}$$

INTEGRAL: The inverse operation of derivation is integration and we will need to know that

$$\int A(x)dx = c + a_0 x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \frac{a_3}{4}x^4 + \cdots$$

for some constant c. One example of an equation that we will use in some of the exercises is the case when $a_i = 1$. From calculus and equation (2.2.1.2),

$$\int \frac{1}{1-x}dx = -\,ln(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots \tag{3.1.2.8}$$

These operations can be combined with operations such as shifting the indices 'up' by one with

$$(A(x) - a_0)/x = a_1 + a_2 x + a_3 x^2 + a_4 x^3 + \cdots = \sum_{n\geq 0} a_{n+1} x^n \tag{3.1.2.9}$$

and shifting 'down' by one with

$$xA(x) = a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \cdots = \sum_{n\geq 1} a_{n-1} x^n. \tag{3.1.2.10}$$

to multiply and divide coefficients by a factor of n as we did in equation (3.1.2.5).

Picking out the even and the odd terms from the generating function for a sequence

A useful technique is to build the generating function for the even terms in a sequence from the generating function for the whole sequence.

That is, take the generating function

$$A(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots$$

if we replace x with $-x$ then

$$A(-x) = a_0 - a_1x + a_2x^2 - a_3x^3 + a_4x^4 + \cdots$$

If we add these two generating functions we have

$$A(x) + A(-x) = 2a_0 + 2a_2x^2 + 2a_4x^4 + \cdots$$

if we then divide both sides of the equation by 2, then we have

$$\frac{A(x) + A(-x)}{2} = a_0 + a_2x^2 + a_4x^4 + \cdots$$

but this is the generating function for the sequence $a_0, 0, a_2, 0, a_4, 0, \ldots$. If we then replace x with $\sqrt{x}$ then we have the generating function

$$= a_0 + a_2x + a_4x^2 + a_6x^3 + a_8x^4 + \cdots.$$

Therefore we have

(3.1.3.1) $$\frac{A(\sqrt{x}) + A(-\sqrt{x})}{2} = \sum_{n\geq 0} a_{2n}x^n$$

which is the generating function for the sequence $a_0, a_2, a_4, a_6, a_8, \ldots$.

By taking the difference of $A(x)$ and $A(-x)$ and divide by 2 we have

$$= a_1x + a_3x^3 + a_5x^5 + a_7x^7 + a_9x^9 + \cdots$$

so then divide this generating function by x and then replace x with $\sqrt{x}$ to get the function

$$= a_1 + a_3x + a_5x^2 + a_7x^3 + a_9x^4 \cdots.$$

The result is that

(3.1.3.2) $$\frac{A(\sqrt{x}) - A(-\sqrt{x})}{2\sqrt{x}} = \sum_{n\geq 0} a_{2n+1}x^n$$

is the generating function for the odd terms. What is surprising is that when you do some algebraic computation that involves square roots then we should expect the result to also contain square roots, but usually if $A(x)$ doesn't, then neither will either of $\frac{A(\sqrt{x})+A(-\sqrt{x})}{2}$ or $\frac{A(\sqrt{x})-A(-\sqrt{x})}{2\sqrt{x}}$ after the equations are simplified.

For example, if we want the generating function for the odd integers we can use equation (2.2.2.3) and (2.2.1.2) to arrive at a formula.

$$\sum_{n\geq 0}(2n+1)x^n = 2\sum_{n\geq 0}(n+1)x^n - \sum_{n\geq 0}x^n \tag{3.1.3.3}$$
$$= 2\frac{1}{(1-x)^2} - \frac{1}{1-x}\,.$$

But then we can also use equation (3.1.3.1) and apply it to (2.2.2.3) and then we conclude

$$\sum_{n\geq 0}(2n+1)x^n = \frac{1}{2}\left(\frac{1}{\left(1-\sqrt{x}\right)^2} + \frac{1}{\left(1+\sqrt{x}\right)^2}\right). \tag{3.1.3.4}$$

It may not necessarily be clear (3.1.3.3) and (3.1.3.4) are equal and this can also be shown by finding using a bit of algebra or by using the computer.

```
sage: A = 2/(1-x)^2 - 1/(1-x)
sage: B = (1/(1-x^(1/2))^2 + 1/(1+x^(1/2))^2)/2
sage: factor(A-B)
0
```

The Computer and Taylor's theorem

Another way that we have to take the coefficient of x^n in a generating function is to use the formula in terms of the derivative of the function. We know by Taylor's theorem it will be equal to

$$\frac{f^{(n)}(0)}{n!}$$

where the $f^{(n)}$ is the n^{th} derivative of the function f and then this is evaluated at $x = 0$.

Sometimes finding a formula for the n^{th} derivative is not really possible, but it is how the computer can be used to determine a coefficient in a series.

So for instance if I wanted to compute the Fibonacci number F_{30} (the 31^{st} Fibonacci number) then I could do this on the computer with the following commands.

```
sage: diff(1/(1-x-x^2),x,30).subs(x=0)/factorial(30)
1346269
sage: diff(1/(1-x-x^2),x,5).subs(x=0)/factorial(5)
8
```

The first command takes the 30^{th} derivative of the generating function for the Fibonacci numbers and evaluates it at $x = 0$ and then divides by 30!. Since we don't know the value of F_{30} , this answer might not be right. We shouldn't trust the computer and our calculations implicitly. But we can at least convince ourselves that that we are on the right track, the second command just computes the 5^{th} Fibonacci number, F_4, which we can compute by hand and check the second command clearly agrees and so the first one probably does too.

We can compare our value of F_{30} that we computed using Taylor's theorem to another formula that we will arrive at in Section 3.2.2.

Examples

For a video summary of this section:

http://garsia.math.yorku.ca/~zabrocki/MMM1/summary32.mov

We have enough to build on now to use generating functions to prove results about sequences.

Sum the integers 1 through n+1

By equations (2.2.2.3) we know that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

is the generating function for the positive integers where the coefficient of x^n is $(n+1)$, therefore by equation (3.1.2.3) we know that $\frac{1}{(1-x)}\frac{1}{(1-x)^2}$ is a generating function for the sequence of the sum of the first n positive integers,

$$1, 1+2, 1+2+3, 1+2+3+4, \ldots.$$

In particular, the coefficient of x^n is $1 + 2 + 3 + \cdots + n + (n+1)$.

We also know by taking the derivative of (2.2.2.3) that

$$\frac{d}{dx}\frac{1}{(1-x)^2} = \frac{2}{(1-x)^3} = 2 \cdot 1 + 3 \cdot 2x + 4 \cdot 3x^2 + 5 \cdot 4x^3 + \cdots$$

Therefore if we divide this equation by two we have

(3.2.2.1) $$\frac{1}{(1-x)^3} = \sum_{n \geq 0} \frac{(n+1)(n+2)}{2} x^n.$$

It must be that the coefficient of x^n in $\dfrac{1}{(1-x)^3}$ is equal to $(n+1)(n+2)/2$, and it is equal to the sum of the first $n+1$ integers, so

(3.2.2.2) $1+2+3+\cdots+n+(n+1) = \dfrac{(n+1)(n+2)}{2}$.

Take that Gauss!

An explicit formula for Fibonacci numbers

We know how to calculate any given Fibonacci number recursively by adding the sum of the previous two Fibonacci numbers, so then we need the two before that, and the two before that, and so on... We stop at some point because we know that $F_0 = F_1 = 1$. That means in order to calculate the n^{th} Fibonacci number we kind of need to calculate all the Fibonacci numbers that come before.

Without explaining how to derive the algebra behind it, we note that if

$$\phi = \frac{1+\sqrt{5}}{2} \text{ and } \overline{\phi} = \frac{1-\sqrt{5}}{2}$$

then

$$(1-\phi x)(1-\overline{\phi}x) = 1-x-x^2.$$

If we work to find the partial fraction decomposition of

$$\frac{1}{1-x-x^2} = \frac{1}{(1-\phi x)(1-\overline{\phi}x)} = \frac{A}{1-\phi x} + \frac{B}{1-\overline{\phi}x}$$

then a little algebra shows that

$$\frac{\phi}{1-\phi x} - \frac{\overline{\phi}}{1-\overline{\phi}x} = \left(\frac{\phi - \phi\overline{\phi}x - \overline{\phi} + \phi\overline{\phi}x}{(1-\phi x)(1-\overline{\phi}x)}\right) = \frac{\sqrt{5}}{1-x-x^2}.$$

The coefficient of x^n in the right hand side of the equation is equal to $\sqrt{5}F_n$. The coefficient of the left hand side (since it is the sum of two geometric series, we can use equations (2.2.1.4) and (2.2.4.1)) is equal to $\phi^{n+1} - \overline{\phi}^{n+1}$.

This gives a formula for the n^{th} Fibonacci number which does not require us to calculate all of them in order order to calculate it, namely

$$F_n = \frac{\phi^{n+1} - \overline{\phi}^{n+1}}{\sqrt{5}}.$$

To demonstrate that this formula works as we say it does, we can use Sage to calculate the first 10 terms.

```
sage: phi = (1+sqrt(5))/2
sage: float(phi)
1.618033988749895
sage: phib = (1-sqrt(5))/2
sage: float(phib)
-0.6180339887498949
sage: [expand(phi^(n+1) - phib^(n+1))/sqrt(5) for n in range(10)]
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55]
sage: expand(phi^(31) - phib^(31))/sqrt(5)
1346269
```

You should notice that no induction was harmed in the making of this formula. One advantage of using generating functions is that it often allows us to use an explicit computation in place of an induction argument.

FIGURE 3.1 Pascal's triangle: The table of binary coefficients

1							
1	1						
1	2	1					
1	3	3	1				
1	4	6	4	1			
1	5	10	10	5	1		
1	6	15	20	15	6	1	
1	7	21	35	35	21	7	1

Binomial coefficients

Pascal's triangle is a diagram like the one below where the first and last entry in each row is a 1 and the entries in the middle of the triangle are determined by adding the value immediately above and the one above and just to the left.

The diagram continues in rows below the ones that are shown here. The numbers in this triangle are often referred to as "binomial coefficients" but they sometimes go by other names as well.

This triangle is also expressed symbolically so that the first row is $C_{0,0} = 1$ and the $(n+1)^{st}$ row is $C_{n,0}, C_{n,1}, C_{n,2}, \cdots, C_{n,n}$ where $C_{n,0} = C_{n,n} = 1$ and $C_{n,k} = C_{n-1,k-1} + C_{n-1,k}$ if $0 < k < n$. Instead of saying "the binomial coefficient indexed by n and k" it is more common to shorten the reference to $C_{n,k}$ as " n choose k."

Notice that if we compute by expanding the following powers of $(1+x)$ on the left hand side that we see the numbers which appear in Pascal's triangle on the right hand side.

Lets fix n and write down a generating function for the $(n+1)^{st}$ row of this table:

$C_n(x) = C_{n,0} + C_{n,1}x + C_{n,2}x^2 + \cdots + C_{n,n}x^n$. For a convention we can assume that $C_{n,K} = 0$ if $K > n$. Then direct calculation shows that if $n > 0$,

$$
\begin{aligned}
C_n(x) &= 1 + \sum_{k=1}^{n-1} C_{n,k}x^k + x^n \\
&= 1 + \sum_{k=1}^{n-1} (C_{n-1,k-1} + C_{n-1,k})x^k + x^n \\
&= 1 + \sum_{k=1}^{n-1} C_{n-1,k}x^k + \sum_{k=1}^{n-1} C_{n-1,k-1}x^k + x^n \\
&= C_{n-1}(x) + xC_{n-1}(x)
\end{aligned}
$$

We conclude that

$$C_n(x) = (1+x)C_{n-1}(x).$$

Now we can do some algebra because if we define

$$C(x,y) := \sum_{n,k \geq 0} C_{n,k}y^n x^k,$$

then this is what we would call a multivariate generating function. It works just as the other generating functions we have previously worked with except that it has two parameters.

In fact, each coefficient of y^n is itself a generating function $C_n(x)$ and each coefficient of x^k is also a generating function. So we can compute

$$C(x,y) = \sum_{n \geq 0} \sum_{k \geq 0} C_{n,k}y^n x^k$$

$$= \sum_{n\geq 0} C_n(x)y^n$$

$$= 1 + \sum_{n\geq 1} C_n(x)y^n$$

$$= 1 + \sum_{n\geq 1} (1+x)C_{n-1}(x)y^{n-1}y.$$

We can then factor out $(1+x)y$ from the summation and see

$$C(x,y) = 1 + y(1+x)C(x,y).$$

Therefore,

$$(1 - y(1+x))C(x,y) = 1$$

and hence

$$C(x,y) = \frac{1}{1-y(1+x)}. \tag{3.2.3.1}$$

This expression is of the form (2.2.1.4) and so we have the expansion,

$$C(x,y) = \sum_{n\geq 0}(1+x)^n y^n = \sum_{n\geq 0}\sum_{k\geq 0} C_{n,k}y^n x^k .$$

The expression $C(x,y)$ is a generating function for the binomial coefficients which are not just indexed by a single integer, but by a pair of integers.

To see this generating function using the computer we can take the Taylor expansion of this function about both the x and the y variable.

```
sage: y = var("y") # we have to declare variables other than x

sage: taylor(taylor(1/(1-y*(1+x)),x,0,10),y,0,5)
(x^5 + 5*x^4 + 10*x^3 + 10*x^2 + 5*x + 1)*y^5 + (x^4 + 4*x^3 + 6*x^2
+ 4*x + 1)*y^4 + (x^3 + 3*x^2 + 3*x + 1)*y^3 + (x^2 + 2*x + 1)*y^2 +
(x + 1)*y + 1
```

In particular, we look at the coefficient of y^n on both sides of this last equality and we see that

$$(1+x)^n = \sum_{k\geq 0} C_{n,k}x^k. \tag{3.2.3.2}$$

That is, if we expand the first few powers of $(1+x)$ then we will see the numbers in the table of Pascal's triangle appearing in as the coefficients in the expansion.

$(1+x)^2 = 1 + 2x + x^2$

$(1+x)^3 = 1 + 3x + 3x^2 + x^3$

$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$

$(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

$(1+x)^6 = 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 20x^5 + x^6$

The usual way to show this in a mathematics course would be to use induction. However this is a simple example where it is possible to use direct calculation with generating functions to avoid induction.

Taylor's theorem says how to get a formula for $C_{n,k}$. If we take the k^{th} derivative of $(1+x)^n$ then we get

$$\frac{d^k}{dx^k}(1+x)^n = n(n-1)\cdots(n-k+1)(1+x)^{n-k}.$$

Taylor's Theorem says that if we evaluate this at $x = 0$ and divide by $k!$ then I have a formula for $C_{n,k}$. That is,

$$C_{n,k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!k!}.$$

If we want to give the generating function for the sequences where the k is fixed in the $C_{n,k}$ then this will be the coefficient of x^k in

(3.2.3.3) $$C(x,y) = \sum_{n\geq 0}\sum_{k\geq 0} C_{n,k} y^n x^k = \frac{1}{1-y(1+x)}.$$

Using a bit of algebra we see that

$$C(x,y) = \frac{1}{1-y-yx} = \frac{\frac{1}{1-y}}{1-\frac{yx}{1-y}}.$$

Now this is another geometric series (see that it has the form of equation (2.2.1.4)) where $a = \dfrac{1}{1-y}$ and $b = \dfrac{y}{1-y}$. That means that the coefficient of x^k is equal to

$$\frac{1}{1-y}\frac{y^k}{(1-y)^k} = \frac{y^k}{(1-y)^{k+1}} = \sum_{n\geq 0} C_{n,k} y^n.$$

Therefore if we just want the numbers in the column of the table in Figure 3.1 starting with the first 1, we have

(3.2.3.4) $$\frac{1}{(1-y)^{k+1}} = \sum_{n\geq k} C_{n,k} y^{n-k} = \sum_{n\geq 0} C_{n+k,k} y^n.$$

In particular, the first column of Pascal's triangle is

$$\frac{1}{1-y} = 1 + y + y^2 + y^3 + y^4 + y^5 + y^6 + \cdots,$$

but we knew this because it is a geometric series and we saw it before in (2.2.1.2).

The next column is given by

$$\frac{1}{(1-y)^2} = 1 + 2y + 3y^2 + 4y^3 + 5y^4 + 6y^5 + \cdots = \sum_{n\geq 0} C_{n+1,1} y^n.$$

From equation (2.2.2.3), this is the generating function for the positive integers so we can conclude that $C_{n+1,1} = n + 1$.

If we set $k = 3$ in equation (3.2.3.4), we have

$$\frac{1}{(1-y)^3} = 1 + 3y + 6y^2 + 10y^3 + 15y^4 + 21y^5 + \cdots = \sum_{n\geq 0} C_{n+2,2}\, y^n.$$

This is the generating function that we saw in equation (3.2.2.1) so we can conclude that $C_{n+2,2} = \dfrac{(n+1)(n+2)}{2}$.

We have derived all of the basic facts about binomial coefficients and Pascal's triangle using generating functions starting from just the definition $C_{n,0} = C_{n,n} = 1$ and $C_{n,k} = C_{n-1,k-1} + C_{n-1,k}$ if $0 < k < n$.

A formula relating Fibonacci numbers and binomial coefficients

Notice that if we set $x = y$ in equation (3.2.3.1) we see that

(3.2.4.1) $$\frac{1}{1 - x - x^2} = \sum_{n\geq 0} \sum_{k\geq 0} C_{n,k}\, x^{n+k}.$$

We have already seen in equation (2.2.5.1) that the left hand side of this equation is the generating function for the Fibonacci numbers. Therefore if we take the coefficient of x^r on the left hand side we have the Fibonacci number F_r and on the right hand side we get a sum of binomial coefficients.

That is,

$$F_r = \sum_{n+k=r} C_{n,k}$$

where the sum is over all non-negative integers n and k that add up to r. By our convention that we used to define $C(x, y)$ we have that $C_{n,k} = 0$ if $k > n$ so we can express the right hand side as the sum

$$F_r = \sum_{k=0}^{\lfloor r/2 \rfloor} C_{r-k,k}\,.$$

The Fibonacci numbers are not obviously related to the binomial coefficients so this formula should be at least a little surprising.

For example,

$$F_8 = C_{8,0} + C_{7,1} + C_{6,2} + C_{5,3} + C_{4,4}$$

$$= 1 + 7 + \frac{6 \cdot 5}{2} + \frac{5 \cdot 4 \cdot 3}{3 \cdot 2} + 1$$

$$= 1 + 7 + 15 + 10 + 1 = 34.$$

Of course we already have other formulas for Fibonacci numbers, but this one is unexpected and follows very simply from the formulas that we needed for other applications.

We can use the computer to find the right hand side for the first 10 values of r and for F_{30} to see if it agrees with our other formulas.

```
sage: [sum(binomial(r-d,d) for d in range(r+1)) for r in range(10)]
[1, 1, 2, 3, 5, 8, 13, 21, 34, 55]

sage: sum(binomial(30-d,d) for d in range(16))
1346269
```

The first command in the box above creates a list of the sums of binomial coefficients $C_{r-d,d}$ where d is in the range from 0 to r. In this case if d is larger than $r/2$, then $C_{r-d,d} = 0$.

The second command adds $C_{30-d,d}$ where d is in the range from 0 to 15. This is a formula for F_{30} and agrees with our previous computations of this value.

The sum of the squares of positive integers

I also promised in Section 2.1.3 that I would show how to derive the equation

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \tag{3.2.5.1}$$

and I will be able to show that it is true without using induction.

Recall from equation (2.2.3.1) that

$$\frac{1+x}{(1-x)^3} = 1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \cdots = \sum_{n\geq 0}(n+1)^2x^2.$$

That means by equation (3.1.2.3) that

$$\frac{1}{1-x}\frac{1+x}{(1-x)^3} = 1^2 + (1^2+2^2)x + (1^2+2^2+3^2)x^2 + (1^2+2^2+3^2+4^2)x^3 + \cdots$$
$$= \sum_{n\geq 0}(1^2+2^2+\cdots+(n+1)^2)x^n.$$

Now by equation (3.2.3.4),

$$\frac{1+x}{(1-x)^4} = \frac{1}{(1-x)^4} + \frac{x}{(1-x)^4} = \sum_{n\geq 0} C_{n+3,3}x^n + \sum_{n\geq 0} C_{n+2,3}x^n$$

so that we have by taking the coefficient of x^n that

$$1^2 + 2^2 + 3^2 + \cdots + (n+1)^2 = C_{n+3,3} + C_{n+2,3} \tag{3.2.5.2}$$

$$= \frac{(n+1)(n+2)(n+3) + n(n+1)(n+2)}{6}$$

$$= \frac{(n+1)(n+2)(2n+3)}{6}$$

and this is more clearly equivalent to equation (3.2.5.1) if we replace n by $n-1$.

We can check this for a few values on the computer to make sure that all of our calculations are correct.

```
sage: taylor((1+x)/(1-x)^3,x,0,10)
121*x^10 + 100*x^9 + 81*x^8 + 64*x^7 + 49*x^6 + 36*x^5 + 25*x^4 +
16*x^3 + 9*x^2 + 4*x + 1
sage: taylor((1+x)/(1-x)^3/(1-x),x,0,10)
506*x^10 + 385*x^9 + 285*x^8 + 204*x^7 + 140*x^6 + 91*x^5 + 55*x^4 +
30*x^3 + 14*x^2 + 5*x + 1
sage: [sum(r^2 for r in range(1,n+2)) for n in range(0,10)]
[1, 5, 14, 30, 55, 91, 140, 204, 285, 385]
sage: [(n+1)*(n+2)*(2*n+3)/6 for n in range(0,10)]
[1, 5, 14, 30, 55, 91, 140, 204, 285, 385]
```

Examples and exercises

4

This is the chapter where you show off how much you learned from the rest of this book.

It includes a summary of the examples we have seen so far, exercises to fill in a more complete library, then more exercises that put the dictionary to good use.

Exercises to help build strong generating functions

For a video summary of this section:

http://garsia.math.yorku.ca/~zabrocki/MMM1/summary41.mov

A few warm ups

Verify that the answers to the following warm up exercises are correct in three ways

(a) by explicit computation and

(b) by using the computer,

(c) by comparing your answer to the table in Section 4.1.3.

You should be able to verify that all three answers agree.

(1) The Odd Square Positive Integers

Use the generating function for the square positive integers in equation (2.2.3.1) and the formula to pick out the even terms in that sequence (3.1.3.1) to give a formula for the generating function

$$\sum_{n\geq 0}(2n+1)^2x^n. \tag{4.1.1.1}$$

(2) Fibonacci Numbers Indexed By Even Integers

Use equation (2.2.5.1) and (3.1.3.1) to give a formula for

$$F^{even}(x) = \sum_{n\geq 0} F_{2n}x^n. \tag{4.1.1.2}$$

(3) Fibonacci Numbers Indexed By Odd Integers

Use equation (2.2.5.1) and (3.1.3.2) to give a formula for

(4.1.1.3) $$F^{odd}(x) = \sum_{n \geq 0} F_{2n+1} x^n.$$

(4) Lucas Numbers

The Lucas numbers are defined as $L_0 = 1$, $L_1 = 3$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Use the same techniques that were used to derive the generating function for the Fibonacci numbers in Section 2.2.4 in order to give a formula for the generating function for the Lucas numbers

(4.1.1.4) $$L(x) = \sum_{n \geq 0} L_n x^n.$$

(5) The Lucas Numbers Indexed By Even Integers

Given the result of the last problem and equation (3.1.3.1), find a formula for the generating function

(4.1.1.5) $$L^{even}(x) = \sum_{n \geq 0} L_{2n} x^n.$$

(6) The Lucas Numbers Indexed By Odd Integers

Given the result of question (4) and equation (3.1.3.2), find a formula for the generating function

(4.1.1.6) $$L^{odd}(x) = \sum_{n \geq 0} L_{2n+1} x^n \,.$$

(7) A Finite List Of 1's

Show that the generating function for the sequence consisting of k 1's followed by only 0's is

(4.1.1.7) $$1 + x + x^2 + \cdots + x^{k-1} = \frac{1 - x^k}{1 - x}.$$

Some computations to make you sweat

The answers to the following questions are provided in Section 4.1.3. Follow the instructions below about how to derive the formulas to give a proof of the equations given in the next section. These problems aren't technically difficult, but do sometimes require a lot of computation.

(1) Products Of Two Fibonacci Numbers

To find formulas for the generating functions

(4.1.2.1) $$D^{(0)}(x) = \sum_{n\geq 0} F_n^2 x^n$$

(4.1.2.2) $$D^{(1)}(x) = \sum_{n\geq 0} F_n F_{n+1} x^n$$

(4.1.2.3) $$D^{(2)}(x) = \sum_{n\geq 0} F_n F_{n+2} x^n$$

we can find a system of three equations and three unknowns and then solve for them algebraically.

The equation

(4.1.2.4) $$F_{n+2}^2 = F_{n+2}(F_{n+1} + F_n) = F_{n+2}F_{n+1} + F_{n+2}F_n$$

follows from the defining relation on the Fibonacci numbers. The generating function for the left hand side of (4.1.2.4) is

(4.1.2.5) $$\sum_{n\geq 0} F_{n+2}^2 x^n = \frac{1}{x^2}(D^{(0)}(x) - 1 - x)$$

The generating function for the right hand side of (4.1.2.4) is

(4.1.2.6) $$\frac{1}{x}(D^{(1)}(x) - 1) + D^{(2)}(x).$$

So we can conclude by combining (4.1.2.5) and (4.1.2.6) that

(4.1.2.7) $$D^{(0)}(x) - 1 - x = xD^{(1)}(x) - x + x^2 D^{(2)}(x).$$

Use the equations

(4.1.2.8) $$F_{n+2}F_{n+1} = (F_{n+1} + F_n)F_{n+1} = F_{n+1}^2 + F_{n+1}F_n$$

(4.1.2.9) $$F_{n+2}F_n = (F_{n+1} + F_n)F_n = F_{n+1}F_n + F_n^2$$

to show that

(4.1.2.10) $$D^{(1)}(x) - 1 = D^{(0)}(x) - 1 + xD^{(1)}(x)$$

and

(4.1.2.11) $$D^{(2)}(x) = D^{(1)}(x) + D^{(0)}(x).$$

Solve the three equations (4.1.2.7), (4.1.2.10) and (4.1.2.11) for the three unknowns $D^{(0)}(x)$, $D^{(1)}(x)$ and $D^{(2)}(x)$ to find explicit equations for $D^{(0)}(x)$, $D^{(1)}(x)$ and $D^{(2)}(x)$ that only depend on the variable x.

(2) One More Fibonacci Product

Find a formula for the generating function

(4.1.2.12) $$D^{(3)}(x) = \sum_{n\geq 0} F_{n+3}F_n x^n$$

using the relation

(4.1.2.13) $$F_{n+3}F_n = F_{n+2}F_n + F_{n+1}F_n$$

to show

(4.1.2.14) $$D^{(3)}(x) = D^{(2)}(x) + D^{(1)}(x).$$

Challenge: Conjecture and prove a formula for

(4.1.2.15) $$D^{(k)}(x) = \sum_{n\geq 0} F_{n+k}F_n x^n.$$

Find a formula for the generating function

(4.1.2.16) $$D(x, y) = \sum_{k\geq 0}\sum_{n\geq 0} F_{n+k}F_n x^n y^k.$$

(3) The Product Of Fibonacci Numbers And Lucas Numbers

Verify algebraically using the formulas that we have already derived for the generating functions for the Fibonacci and Lucas numbers that

(4.1.2.17) $$F(x) + x^2F(x) = 1 + xL(x)$$

Use this to conclude that for $n \geq 1$, that $L_n = F_{n-1} + F_{n+1}$.

Next, use your formulas from Exercise (1) of this section to give a formula for $\sum_{n\geq 0} L_nF_nx^n$, $\sum_{n\geq 0} L_{n+1}F_nx^n$ and $\sum_{n\geq 0} L_nF_{n+1}x^n$.

(4) Lucas Numbers Squared

Use the results of the last exercise and the fact that $L_n = F_{n-1} + F_{n+1}$ to give an expression for the generating function $\sum_{n\geq 0} L_n^2x^n$.

(5) Even And Odd Fibonacci Numbers Squared

Use the method in Section 3.1.3 and the result of Exercise (1) of this section to find a generating function for $\sum_{n\geq 0} F_{2n}^2x^n$ and $\sum_{n\geq 0} F_{2n+1}^2x^n$.

(6) The Central Binomial Coefficients

Give a formula for the n^{th} derivative of the equation $\frac{1}{\sqrt{1-4x}}$. Use it to show that the Taylor expansion of the generating function is

$$\frac{1}{\sqrt{1-4x}} = \sum_{n\geq 0} C_{2n,n} x^n.$$

(7) Catalan Numbers

The Catalan numbers are $\frac{1}{n+1}C_{2n,n}$ and the first few terms are

$$1, 1, 2, 5, 14, 42, 132, \ldots.$$

They appear often in combinatorics because these numbers count (for instance) the number of triangulations of an $(n+2)$ -gon.

Use the result of the last exercise and integration as we did in Section 3.1.2 to give a formula for the generating function for the Catalan numbers. When you integrate, you will need to set the constant of integration and divide by x to ensure that the coefficient of x^n is equal to $\frac{1}{n+1}C_{2n,n}$.

Cool down - a summary containing the answers

Here is a list of all of the generating functions that I have asked you to find in the previous sections. Keep this list handy because you will need it in the next section. The names that are given to these expressions in this section will be used in the exercises in the next section and the solutions to those exercises.

Geometric Series

See (2.2.1.2) and Section 1.2.3.

$$\sum_{n\geq 0} x^n = 1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

A Sequence Of k Ones

See Exercise (7) in Section 4.1.1

$$\sum_{n=0}^{k-1} x^n = 1 + x + x^2 + \cdots + x^{k-1} = \frac{1-x^k}{1-x}$$

Binomial Coefficients I

See equation (3.2.3.2)

$$P_n(x) = \sum_{k\geq 0} C_{n,k} x^k = C_{n,0} + C_{n,1}x + \cdots + C_{n,n}x^n = (1+x)^n$$

Binomial Coefficients II

See equation (3.2.3.4)

$$Q_k(x) = \sum_{n\geq 0} C_{n+k,k}x^n = C_{k,k} + C_{k+1,k}x + C_{k+2,k}x^2 + C_{k+3,k}x^3 + \cdots = \frac{1}{(1-x)^{k+1}}$$

Binomial Coefficients III

This is Exercise (6) in Section 4.1.2.

$$R(x) = \sum_{n\geq 0} C_{2n,n}x^n = 1 + 2x + 6x^2 + 20x^3 + 70x^4 + \cdots = \frac{1}{\sqrt{1-4x}}$$

Catalan Numbers

This is the result of Exercise (7) in Section 4.1.2.

$$C(x) = \sum_{n\geq 0} \frac{1}{n+1} C_{2n,n}x^n = 1 + x + 2x + 5x^2 + 14x^3 + \cdots = \frac{1-\sqrt{1-4x}}{2x}$$

Positive Integers

See equations (2.2.2.3) but it is also a special case of equation (3.2.3.4) with $k = 1$.

$$A(x) = \sum_{n\geq 0} (n+1)x^n = 1 + 2x + 3x^2 + 4x^3 + \cdots = \frac{1}{(1-x)^2}$$

One Over The Positive Integers

See equation (2.2.4.8)

$$\sum_{n\geq 1} \frac{1}{n}x^n = x + \frac{x^2}{2} + \frac{x^3}{3} + \cdots = -\,ln(1-x)$$

Odd Positive Integers

See equation (3.1.3.3) and (3.1.3.4)

$$A^{odd}(x) = \sum_{n\geq 0} (2n+1)x^n = 1 + 3x + 5x^2 + 7x^3 + 9x^4 + \cdots = \frac{1+x}{(1-x)^2}$$

Even Positive Integers

This is included for completeness since it is just two times the generating function for the positive integers.

$$A^{even}(x) = \sum_{n\geq 0} 2nx^n = 2 + 4x + 6x^2 + 8x^3 + \cdots = \frac{2}{(1-x)^2}$$

Square Of The Positive Integers

See equation (2.2.3.1)

$$A^{sqr}(x) = \sum_{n\geq 0} (n+1)^2x^n = 1 + 4x + 9x^2 + 16x^3 + \cdots = \frac{1+x}{(1-x)^3}$$

The Cubes Of The Positive Integers

See equation (2.2.4.6)

$$A^{cube}(x) = \sum_{n\geq 0} (n+1)^3 x^n = 1 + 8x + 27x^3 + 64x^3 + \cdots = \frac{1+4x+x^2}{(1-x)^4}$$

Fibonacci Numbers

See equation (2.2.5.1)

$$F(x) = \sum_{n\geq 0} F_n x^n = 1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots = \frac{1}{1-x-x^2}$$

Fibonacci Numbers Indexed By Even Integers

See Exercise (2) in Section 4.1.1

$$F^{even}(x) = \sum_{n\geq 0} F_{2n} x^n = 1 + 2x + 5x^2 + 13x^3 + 34x^4 + \cdots = \frac{1-x}{1-3x+x^2}$$

Fibonacci Numbers Indexed By Odd Integers

See Exercise (3) in Section 4.1.1

$$F^{odd}(x) = \sum_{n\geq 0} F_{2n+1} x^n = 1 + 3x + 8x^2 + 21x^3 + 55x^4 + \cdots = \frac{1}{1-3x+x^2}$$

Lucas Numbers

See Exercise (4) in Section 4.1.1

$$L(x) = \sum_{n\geq 0} L_n x^n = 1 + 3x + 4x^2 + 7x^3 + 11x^4 + \cdots = \frac{1+2x}{1-x-x^2}$$

Lucas Numbers Indexed By Even Integers

See Exercise (5) in Section 4.1.1

$$L^{even}(x) = \sum_{n\geq 0} L_{2n} x^n = 1 + 4x + 11x^2 + 29x^3 + \cdots = \frac{1+x}{1-3x+x^2}$$

Lucas Numbers Indexed By Odd Integers

See Exercise (6) in Section 4.1.1

$$L^{odd}(x) = \sum_{n\geq 0} L_{2n+1} x^n = 3 + 7x + 18x^2 + 47x^3 + \cdots = \frac{3-2x}{1-3x+x^2}$$

Squares Of Fibonacci Numbers

One of the equations in Exercise (1) of Section 4.1.2

$$D^{(0)}(x) = \sum_{n\geq 0} F_n^2 x^n = 1 + x + 4x^2 + 9x^3 + \cdots = \frac{1-x}{(1+x)(1-3x+x^2)}$$

Products Of Fibonacci Numbers With Indices Differing By 1

One of the equations in Exercise (1) of Section 4.1.2

$$D^{(1)}(x) = \sum_{n\geq 0} F_{n+1}F_n x^n = 1 + 2x + 6x^2 + \cdots = \frac{1}{(1+x)(1-3x+x^2)}$$

Products Of Fibonacci Numbers With Indices Differing By 2

One of the equations in Exercise (1) of Section 4.1.2

$$D^{(2)}(x) = \sum_{n\geq 0} F_{n+2}F_n x^n = 2 + 3x + 10x^2 + \cdots = \frac{2-x}{(1+x)(1-3x+x^2)}$$

Products Of Fibonacci Numbers With Indices Differing By 3

One of the equations in Exercise (2) of Section 4.1.2

$$D^{(3)}(x) = \sum_{n\geq 0} F_{n+3}F_n x^n = 3 + 5x + 16x^2 + \cdots = \frac{3-x}{(1+x)(1-3x+x^2)}$$

Products Of Fibonacci Number With A Lucas Number With The Same Index

One of the equations in Exercise (3) of Section 4.1.2

$$LF(x) = \sum_{n\geq 0} L_nF_n x^n = 1 + 3x + 8x^2 + 21x^3 + 55x^4 + \cdots = \frac{1}{1-3x+x^2}$$

Products Of A Fiboncacci Number With A Lucas Number With An Index Of One Lower

One of the equations in Exercise (3) of Section 4.1.2

$$LF^{(1)}(x) = \sum_{n\geq 0} L_nF_{n+1} x^n = 1 + 6x + 12x^2 + 35x^3 + \cdots = \frac{1+4x-2x^2}{(1+x)(1-3x+x^2)}$$

Products Of A Fiboncacci Number With A Lucas Number With An Index Of One Higher

One of the equations in Exercise (3) of Section 4.1.2

$$LF^{(-1)}(x) = \sum_{n\geq 0} L_{n+1}F_n x^n = 3 + 4x + 14x^2 + 33x^3 + \cdots = \frac{3-2x}{(1+x)(1-3x+x^2)}$$

The Square Of The Lucas Numbers

Exercise (4) of Section 4.1.2

$$L^{sqr}(x) = \sum_{n\geq 0} L_n^2 x^n = 1 + 9x + 16x^2 + 49x^3 + \cdots = \frac{1+7x-4x^2}{(1+x)(1-3x+x^2)}$$

Squares Of Fibonacci Numbers Indexed By Even Integers

One of the two equations from Exercise (5) of Section 4.1.2

$$F^{evensqr}(x) = \sum_{n\geq 0} F_{2n}^2 x^n = 1 + 4x + 25x^2 + 169x^3 + \cdots = \frac{1-4x+x^2}{(1-x)(1-7x+x^2)}$$

Squares Of Fibonacci Numbers Indexed By Odd Integers

One of the two equations from Exercise (5) of Section 4.1.2

$$F^{oddsqr}(x) = \sum_{n\geq 0} F_{2n+1}^2 x^n = 1 + 9x + 64x^2 + 441x^3 + \cdots = \frac{1+x}{(1-x)(1-7x+x^2)}$$

The Online Integer Sequence Database

In 1973, Neil Sloane published A Handbook of Integer Sequences. This was an interesting book because it allowed the reader to glance through all types of integer sequences, perhaps triggering mathematical ideas.

In 1995, Sloane and Plouffe updated the book to The Enclopedia of Integer Sequences and more than doubled the number of entries. People from all over the world submitted additional entries and the database grew and was turned into an on-line web database starting in 1996. Today it is an enormous and useful tool.

Before these references, if you were given a sequence of numbers such as

$$1, 1, 3, 12, 56, 288, 1584, 9152, \ldots$$

it would be nearly impossible to know if this sequence had been studied before. Now one can go to the website oeis.org, enter these numbers and see what, if anything is known about it.

This sequence in particular is titled "**Number of rooted bicubic maps: a(n)=(8n-4)a(n-1)/(n+2)**" and if you look at the entry under the heading "O.g.f" (for ordinary generating function) there is a formula given as

$$\frac{(1-8x)^{3/2}+8x^2+12x-1}{32x^2}.$$

Actually, at least three formulas for the generating function are given there, but the other two are more complicated or use notation that we have not introduced here.

If a sequence that you are interested in is not in the database, there is a way of submitting it along with a description and whatever related information about the sequence you might have.

Using generating functions to prove summation formulas

For a video summary of this section:

http://garsia.math.yorku.ca/~zabrocki/MMM1/summary42.mov

In the following exercises I hope to show you the power of generating functions to prove all sorts of equations relating coefficients. We saw a little of how this worked in Section 3.2 when we used a generating function to relate Fibonacci numbers with binomial coefficients. In these exercises you will do these calculations yourself.

In one set of exercsises you will take two expressions that you can show are equal using algebra and derive an identity relating coefficients in the left hand side of the equation with the coefficient in the right hand side of the equation and conclude that the coefficients must be equal.

In another set of exercises you will prove identities by writing down a generating function for the left hand side of the equation and a generating function for the right hand side of the equation and use algebra to show that they are equal.

You will use the table of generating functions for Section 4.1.3 to develop the generating functions for the left and the right hand side of the equation.

From generating functions to identities

In the following exercises you will be given an algebraic identity relating generating functions. There are methods for taking coefficients in products or sums of generating functions like equations (3.1.2.1) and (3.1.2.2). Since algebraically the left hand side is equal to the right hand side, their coefficients are also equal.

In the following exercises assume that a, b, n and k are all non-negative integers.

(1) Use the identity

$$(1+x)^n = x^n(1+1/x)^n$$

to show that $C_{n,k} = C_{n,n-k}$.

(2) Take the coefficient of x^k in both sides of the equation

$$(1+x)^a(1+x)^b = (1+x)^{a+b}$$

and use it to show that

$$\sum_{r=0}^{k} C_{a,r}C_{b,k-r} = C_{a+b,k}.$$

In particular, specialize the values of a and b and use the result from the previous exercise to show

$$\sum_{r=0}^{n} C_{n,r}^2 = C_{2n,n}.$$

(3) Take the coefficient of x^k in both sides of the equation

$$\frac{1}{(1-x)^{a+1}} \cdot \frac{1}{(1-x)^{b+1}} = \frac{1}{(1-x)^{a+b+2}}$$

and use it to derive another equation relating binomial coefficients.

(4) Assume that $a > b+1$ and then take the coefficient of x^k in both sides of the equation

$$\frac{1}{(1+x)^{b+1}} \cdot (1+x)^a = (1+x)^{a-b-1}$$

to derive an equation relating binomial coefficients. You will need to use the formula that

$$\frac{1}{(1+x)^{b+1}} = \frac{1}{(1-(-x))^{b+1}} = \sum_{k\geq 0} C_{b+k,k}(-x)^k.$$

(5) Take the coefficient of x^k for k an even number in the equation

$$\frac{1}{(1-x)^{a+1}} \cdot \frac{1}{(1+x)^{a+1}} = \frac{1}{(1-x^2)^{a+1}}.$$

to relate an alternating sum of binomial coefficients to a single binomial coefficient.

(6) Use the algebraic equation

$$\frac{1}{\sqrt{1-4x}} \cdot \frac{1}{\sqrt{1-4x}} = \frac{1}{1-4x}$$

to give an identity whose left hand side is a sum of products of central binomial coefficients and whose right hand side is 4^n.

(7) In Section 4.1.3 we saw that

$$\frac{1}{1-3x+x^2} = \sum_{n\geq 0} F_{2n+1}x^n$$

and

$$\frac{1}{(1+x)(1-3x+x^2)} = \sum_{n\geq 0} F_n F_{n+1} x^n.$$

Use the identity

$$\frac{1}{1+x}\cdot\frac{1}{1-3x+x^2} = \frac{1}{(1+x)(1-3x+x^2)}$$

to relate an alternating sum of odd Fibonacci numbers to a product of Fibonacci numbers.

(8) Look at the formulas for $D^{(0)}(x)$ and $D^{(1)}(x)$ that are given in Section 4.1.3. Show that

$$2D^{(1)}(x) = \frac{1}{1-x/2}D^{(2)}(x).$$

Find an equation relating the coefficients in the left hand side and the right hand side of the equation.

(9) Recall that the generating function for the cubes of positive integers is

$$\frac{1+4x+x^2}{(1-x)^4} = \sum_{n\geq 0}(n+1)^3x^n.$$

Relate this to the generating function $\frac{1}{(1-x)^5} = \sum_{n\geq 0} C_{n+4,4}x^n$ to give a formula for

$$1^3+2^3+3^3+\cdots+(n+1)^3.$$

From identities to generating functions

In the following exercises assume that $n \geq 0$. Use the tables of generating functions from Section 4.1.3 and write down a generating function for the left hand side and the right hand side of the equation and use algebra to show that they are equal and thus prove that the formula is true.

A large number of these identities were taken from a website by R. Knott that includes a collection of Fibonacci and Lucas identities. There will be very few identities on that website that you cannot prove using the same techniques that you learned here (warning: their Fibonacci numbers begin $F_0 = 0$ and $F_1 = 1$).

(1) $F_n L_n = F_{2n+1}$

(2) $F_0^2 + F_1^2 + F_2^2 + \cdots + F_n^2 = F_n F_{n+1}$

(3) $F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1}$

(4) $F_1 + F_3 + F_5 + \cdots + F_{2n+1} = F_{2n+2} - 1$

(5) $F_0F_1 + F_1F_2 + F_2F_3 + \cdots + F_{2n}F_{2n+1} = F_{2n+1}^2$

(6) $F_0F_1 + F_1F_2 + F_2F_3 + \cdots + F_{2n+1}F_{2n+2} = F_{2n+2}^2 - 1$

(7) $F_{n+1}^2 + 2F_nF_{n+1} = F_{2n+3}$

(8) $F_{n+2}^2 - F_n^2 = F_{2n+3}$

(9) $F_{n+1}^2 = F_nF_{n+2} + (-1)^{n+1}$

(10) $F_{n+1}F_{n+2} = F_nF_{n+3} + (-1)^{n+1}$

(11) $F_{n+1}L_{n+1} + F_nL_n = L_{2n+2}$

(12) $F_{n+1}L_{n+1} - F_nL_n = F_{2n+2}$

(13) $5(F_n^2 + F_{n+1}^2) = L_n^2 + L_{n+1}^2$

(14) $5F_n^2 - L_n^2 = 4(-1)^n$

(15) $L_n^2 - 2L_{2n+1} = -5F_n^2$

(16) $F_{n+3} - F_n = 2F_{n+1}$

(17) $F_{n+3} + F_n = 2F_{n+2}$

(18) $F_{n+4} + F_n = 3F_{n+2}$

(19) $F_{n+4} - F_n = L_{n+2}$

(20) $\dfrac{1}{1\cdot 2} + \dfrac{1}{2\cdot 3} + \dfrac{1}{3\cdot 4} + \cdots + \dfrac{1}{(n+1)(n+2)} = \dfrac{n+1}{n+2}$

(21) $C_{m+1,n} - C_{m+1,n-1} + C_{m+1,n-2} - \cdots + (-1)^n C_{m+1,0} = C_{m,n}$ for $m \geq 0$.

Generating functions for sequences defined by a recurrence

In the following exercises you will be given a sequence of numbers that are defined recursively by a formula and some initial terms. In this set of exercises I would like you to find the generating function.

The following exercises progress slightly in complexity from the first to the last. I have picked a selection which uses recurrences that are relatively easy to solve. You may need to use the quadratic formula for the roots of an equation to determine a solution for equations (6) through (9).

You should compute the first 5-10 terms and then

(1) when you find the generating function, use the computer to expand the first terms and compare the sequence to the coefficients in the Taylor series of the expression that you found.

(2) check to see if these sequences are in the Online Encyclopedia of Integer Sequences (see Section 4.1.4 and oeis.org).

I didn't include examples of sequences where the generating function satisfies an obvious differential equation or a more complex algebraic equation, but change the recurrence slightly and then one would need to develop more sophisticated techniques for solving these sorts of equations.

(1) $a_0 = 3$, $a_1 = 1$, $a_n = a_{n-1} + a_{n-2}$ for $n \geq 2$. Use the resulting generating function to find an expression for a_n in terms of the Fibonacci numbers for $n \geq 1$.

(2) $b_0 = 3$, $b_1 = 1$, $b_n = b_{n-1} - b_{n-2}$ for $n \geq 2$. Compare your resulting generating function to the formula for $C(-x, x)$ from equation (3.2.3.1). Use this to arrive a formula for b_n in terms of binomial coefficients.

(3) $c_0 = 2$, $c_n = 2c_{n-1} - 1$ for $n \geq 1$. Use the generating function to derive an equation for c_n in terms of powers of 2.

(4) $d_0 = 1$, $d_1 = 4$, $d_n = 2d_{n-1} - 1$ for $n \geq 2$. Use the generating function to derive an equation for d_n for $n \geq 1$ in terms of powers of 2.

(5) $e_0 = 1$, $e_1 = 4$, $e_n = 2e_{n-2} - 1$ for $n \geq 2$. Use the generating function to derive an equation for e_n in terms of powers of 2. You may need to handle the case with n even and n odd separately.

(6) $f_0 = 1, f_n = \sum_{i=0}^{n-1} f_i f_{n-i-1}$ for $r \geq 1$. Compare your answer with one of the known generating functions in the table from Section 4.1.3 to arrive at a formula for f_n.

(7) $g_0 = 1,\ g_n = \sum_{i=0}^{n-1} g_i g_{n-i-1} + 2$ for $n \geq 1$.

(8) $h_0 = 1,\ h_n = \sum_{i=0}^{n-1} h_i h_{n-i-1} + h_{n-1}$ for $n \geq 1$.

(9) $j_0 = 1,\ j_1 = 1,\ j_n = \sum_{i=0}^{n-1} j_i j_{n-i-1} + j_{n-1} - j_{n-2}$ for $n \geq 2$.

Solutions

Solutions from Section 4.2.1

In each of the exercises in this section I have taken the coefficient of x^k with the generating functions in the order stated.

Exercises (1) and (2) have solutions in the statement of the problem.

In Exercise (1), we compute

$$x^n(1+1/x)^n = x^n \sum_{k\geq 0} C_{n,k}x^{-k} = \sum_{k\geq 0} C_{n,k}x^{n-k}$$

$$= C_{n,n} + C_{n,n-1}x + C_{n,n-2}x^2 + \cdots + C_{n,0}x^n$$

while from equation (3.2.3.2) we have

$$(1+x)^n = C_{n,0} + C_{n,1}x + C_{n,2}x^2 + \cdots + C_{n,n}x^n.$$

In Exercise (2) you need only apply correctly equation (3.1.2.2) to find the coefficient of x^k in $(1+x)^a(1+x)^b$ is

$$\sum_{r=0}^{k} C_{a,r}C_{b,k-r}$$

and the coefficient of x^k in the expression $(1+x)^{a+b}$ is $C_{a+b,k}$.

In Exercise (3) you should conclude that

$$\sum_{r=0}^{k} C_{a+r,r} C_{b+k-r,k-r} = C_{a+b+1+k,k}.$$

In Exercise (4) the coefficient of x^k is equal to

$$\sum_{r=0}^{k} (-1)^r C_{b+r,r} C_{a,k-r} = C_{a-b-1,k}.$$

In Exercise (5), the coefficient of x^k in $1/(1-x^2)^{a+1}$ is 0 if k is odd, hence

$$\sum_{r=0}^{k} (-1)^{k-r} C_{a+r,r} C_{a+k-r,k-r} = \begin{cases} C_{a+k/2,k/2} & \text{if } k \text{ is even} \\ 0 & \text{if } k \text{ is odd} \end{cases}.$$

Exercise (6) relates the central binomial coefficients with a power of 4 and it shows

$$\sum_{r=0}^{k} C_{2r,r} C_{2k-2r,k-r} = 4^k.$$

From Exercise (7), deduce that

$$\sum_{r=0}^{k} (-1)^r F_{2k-2r+1} = F_k F_{k+1}.$$

In Exercise (8), after using algebra to show the algebraic relation between the generating functions, the coefficient of x^k is equal to

$$2F_k F_{k+1} = \sum_{r=0}^{k} F_{k-r} F_{k-r+2}/2^r.$$

In Exercise (9), one formula you can find for the sum of the positive integers cubed is

$$1^3 + 2^3 + 3^3 + \cdots + (k+1)^3 = C_{k+4,4} + 4C_{k+3,4} + C_{k+2,4}.$$

If you use the formula that we have for $C_{k,4}$, then you can show that the right hand side has the expression $\dfrac{(k+1)^2(k+2)^2}{4}$.

Solutions from Section 4.2.2

In each of the solutions below I write down two expressions which can be obtained directly from manipulating the generating functions in the table in Section 4.1.3. The full solution to this problem is to show using algebra that those two expressions are equal to each other.

(1) The generating functions for the coefficients $L_n F_n$ and F_{2n+1} are both (from Exercise (3) of Section 4.1.2 and Exercise (3) in Section 4.1.1)

$$LF(x) = \sum_{n\geq 0} L_n F_n x^n = \frac{1}{1-3x+x^2} = \sum_{n\geq 0} F_{2n+1} x^n = F^{odd}(x).$$

(2) The left hand side of the equation has generating function $\frac{1}{1-x}D^{(0)}(x)$ and the right hand side has generating function equal to $D^{(1)}(x)$.

(3) The left hand side had generating function $\frac{1}{1-x}F^{even}(x)$ and the right hand side has generating function $F^{odd}(x)$. Show that they are equal.

(4) The left hand side has generating function $\frac{1}{1-x}F^{odd}(x)$ and the right hand side is the generating function for one less than the even Fibonacci generating function with the index shifted up by 1 and so has generating function equal to $(F^{even}(x) - 1)/x - \frac{1}{1-x}$. Show that these two expressions are equal.

(5) and (6) are easier to show together than they are apart. The left hand side of both (5) and (6) have generating function $\frac{1}{1-x}D^{(1)}(x)$. The left hand side of (5) are the coefficients of x^{2n} and the left hand side of (6) are the coefficients of x^{2n+1}. Show that this is equal to $(D^{(0)}(x) - 1)/x - \frac{x}{1-x^2}$.

(7) The right hand side of the equation has generating function $(D^{(0)}(x) - 1)/x + 2D^{(1)}(x)$. The right hand side of the equation has generating function $(F^{odd}(x) - 1)/x$.

(8) The left hand side of the equation has generating function $(D^{(0)}(x) - 1 - x)/x^2 - D^{(0)}(x)$ and the right hand side has generating function (as in the previous question) $(F^{odd}(x) - 1)/x$.

(9) The left hand side of the equation has generating function equal to $(D^{(0)}(x) - 1)/x$. The right hand side has generating function equal to $D^{(2)}(x) - \frac{1}{1+x}$.

(10) The left hand side has generating function equal to $(D^{(1)}(x) - 1)/x$ and the right hand side has generating function $D^{(3)}(x) - \frac{1}{1+x}$.

(11) The left hand side has generating function equal to $((1+x)LF(x) - 1)/x$ and the right hand side has generating function $(L^{even}(x) - 1)/x$.

(12) The left hand side has generating function $((1-x)LF(x) - 1)/x$ and the right hand side has generating function $(F^{even}(x) - 1)/x$.

(13) Show that $5((1+x)D^{(0)}(x) - 1)/x$ is equal to $((1+x)L^{sqr}(x) - 1)/x$.

(14) Show that $5D^{(0)}(x) - L^{sqr}(x)$ is equal to $\frac{4}{1+x}$.

(15) The left hand side has generating function $L^{sqr}(x) - 2L^{odd}(x)$ while the right hand side has generating function $-5D^{(0)}(x)$.

(16) The left hand side has generating function $(F(x) - 1 - x - 2x^2)/x^3 - F(x)$ while the right hand side has generating function $2(F(x) - 1)/x$.

(17) The left hand side has generating function $(F(x) - 1 - x - 2x^2)/x^3 + F(x)$ while the right hand side has generating function $2(F(x) - 1 - x)/x^2$.

(18) The left hand side has generating function $(F(x) - 1 - x - 2x^2 - 3x^3)/x^4 + F(x)$ and the right hand side has generating function $3(F(x) - 1 - x)/x^2$.

(19) The left hand side has generating function $(F(x) - 1 - x - 2x^2 - 3x^3)/x^4 - F(x)$ and the right hand side has generating function $(L(x) - 1 - 3x)/x^2$.

(20) One way of creating the generating function for the left hand side is to integrate $\frac{1}{1-x}$ twice (ensuring that the constant term of the sequence is 0 each time) and then divide by x^2 and (using Equation (3.1.2.3)) multiply by $\frac{1}{1-x}$. This means that the left hand side will have generating function $\frac{1}{x^2(1-x)}\int\int\frac{1}{1-x}dxdx$. You will probably need to do an integration by parts to come up with the expression $\frac{(1-x)ln(1-x)+x}{x^2(1-x)}$. For the right hand side you can integrate $\frac{x}{(1-x)^2}$ and then divide by x^2 to get the generating function and if you are careful that when you integrate that you get $\int\frac{1}{(1-x)^2}+\frac{-1}{1-x}dx = \frac{1}{1-x} + ln(1-x) - 1$ before you divide by x^2 then you should then show that these two expressions are equal.

(21) Because there are two parameters in this expression it may not be clear which should be the best expression to take, but I meant for all exercises to have the generating function where n is the parameter indexing the sequence. If you solved the problem this way, then the left hand side has generating function $\frac{1}{1+x}(1+x)^{m+1}$ and the right hand side has generating function $(1+x)^m$.

Solutions from Section 4.2.3

Solutions in this section are given by the upper case letter corresponding to the lowercase letter of the sequence (e.g. $A(x) = \sum_{n\geq 0} a_n x^n$, $B(x) = \sum_{n\geq 0} b_n x^n$, etc.). The solutions in this section do not refer to any generating functions except those in Section 4.2.3.

$A(x) = \frac{3-2x}{1-x-x^2}$. $a_n = 3F_n - 2F_{n-1}$.

$B(x) = \dfrac{3-2x}{1-x+x^2}$. $b_n = \sum_{k\geq 0}(-1)^k 3C_{n-k,k} - \sum_{k\geq 0}(-1)^k 2C_{n-k-1,k}$ for $n \geq 1$.

$C(x) = \dfrac{2-3x}{(1-x)(1-2x)}$. $c_n = 2^n + 1$ for $n \geq 0$.

$D(x) = \dfrac{1+x-3x^2}{(1-x)(1-2x)}$. $d_n = 3 \cdot 2^{n-1} + 1$ for $n \geq 1$.

$E(x) = \dfrac{1+3x-5x^2}{(1-x)(1-2x)}$. $e_{2n} = 1$ and $e_{2n+1} = 3 \cdot 2^n + 1$ for $n \geq 0$.

$F(x) = \dfrac{1-\sqrt{1-4x}}{2x}$. $f_n = \dfrac{1}{n+1}C_{2n,n}$ for $n \geq 0$.

$G(x) = \dfrac{1-\sqrt{\frac{1-5x-4x^2}{1-x}}}{2x}$. $g_n =$ A110886

$H(x) = \dfrac{1-x-\sqrt{1-6x+x^2}}{2x}$. $h_n =$ A006318

$J(x) = \dfrac{1-x+x^2-\sqrt{1-6x+7x^2-2x^3+x^4}}{2x}$. $j_n =$?????

Final words

5

If you like generating functions you will want to find out more.

This chapter includes list of references and discussion about future ideas for topics I would like to write.

Where to find more

Where to go from here

After you have finished this book I hope that you have the basic tools to manipulate generating functions for sequences and that you are able to use them to prove basic identities. This is a very good starting point, but you will find that what I have presented here only a very small part of the subject and you will need to find other references to learn more advanced aspects.

I hope to write more exposition of my own, but I also suggest looking at some of the references in the next section.

I am experimenting with a model of a textbook that one could potentially digest in a single sitting. Here is a list of other topics which I think would make good subjects for similar publications in the future.

(1) Applications of generating functions to combinatorics

A typical question in combinatorics might ask, “how many ways are there of making change for 35 cents using pennies, nickels, dimes and quarters?”

This can be counted fairly quickly by listing out all possibilities, but it would be an impossibly difficult problem to solve with the same method if we asked how many ways are there of making change for $516.23 using pennies, nickels, dimes and quarters.

If we translate this into a generating function question, we would ask “what is the coefficient of x^{51623} in the generating function

$$\frac{1}{(1-x)(1-x^5)(1-x^{10})(1-x^{25})}?”$$

(2) Applications of generating functions to number theory

Some topics in number theory are related the Fibonacci and Lucas numbers and we covered some basic techniques which are helpful in working with those sequences.

Other questions involve integer partitions, and combinatorial techniques are also useful in working with partitions.

A third topic in number theory involves sequences which involve summations of the form

$$c_n = \sum_d a_d b_{n/d}$$

where the sum is over all integers d which divide evenly into n. There is a family of generating functions called Dirichlet series which are useful for computing with sequences that satisfy recurrences like this.

(3) Exponential generating functions

The sequences that we worked with in this book were all defined by a linear or a simple algebraic recurrence. Often if we see relations of the form

$$d_n = \sum_{k=0}^{n} C_{n,k} \cdot a_k b_{n-k}$$

where $C_{n,k}$ is the binomial coefficient. In that case it is better to work with a generating functions of the form

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}.$$

This is called the *exponential generating function* for the sequence $a_0, a_1, a_2, a_3, \ldots$.

Q: When do you use the exponential generating function rather than the ordinary generating function for a sequence?

A: When the exponential generating function “works” better.

Q: When does the exponential generating function work better than the ordinary generating function?

A: There are a few basic rules to help you decide, but one of those rules is ‘try both and see what works.’

There are many other directions to extend exposition about generating functions such as analytic aspects, relationships with languages, multivariate generating functions, the theory of species, Polyà's counting theory.

At the end of this section I will try to list some textbooks, papers and references that you might consider consulting for further information.

One particularly good exposition that also focuses almost entirely on generating functions and goes much further than this reference is Generatingfunctionology. Other references listed below dedicate only a part of the text to generating functions.

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