SOME EXERCISES RELATED TO THE FIRST FOUR VIDEOS

(1) Calculate $h_2 s_{221}$ in three different ways:
   (a) Expand s_{221} as a determinant in the h-basis; multiply by h_2; find the lowest term $c_\mu h_\mu$ in lex order and subtract off $c_\mu s_\mu$; repeat this last step until you get 0.
   (b) Using the Pieri rule
   (c) Using SAGE symmetric function package

(2) Define $e_1 = h_1$ and $e_0 = h_0 = 1$ and $e_r = -\sum_{i=1}^{r}(-1)^i h_i e_{r-i}$ for $r > 1$. Compute e_2, e_3, e_4 and e_5. We will use similar notation to the h-basis. For a partitions $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_\ell)$, set $e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_\ell}$.

(3) Compute $e_2 s_{21}$ by expanding e_2 and s_{21} as polynomials in the h-basis and the re-expanding in the Schur basis.

(4) Conjecture a formula for $e_r s_\lambda$. Do this by computing more examples, such as $e_2 s_{22}$, $e_3 s_4$, $e_2 s_{41}$ and then drawing the partitions which you observe appear in this expansion. Draw the partitions as Young diagrams and compare the diagrams to the diagram for λ. Make a guess from these examples and the formula we "know" $e_1 s_\lambda = h_1 s_\lambda$. You should do the first few examples using the method of expanding in the h-basis by hand, then collect more data using the computer.

(5) Let $\omega(h_r) = e_r$ for all $r \geq 1$ and then extend this to all partitions by setting $\omega(h_\lambda) = e_\lambda$. Show that $\omega(e_r) = h_r$.

(6) Show that $e_r = s_{1^r}$ for $r \geq 0$.

(7) Compute examples of $\omega(s_\lambda)$ for λ a partition of 1,2,3,4. Conjecture a formula for $\omega(s_\lambda)$.

(8) Expand $3h_{31} + h_{22} - h_{1111}$ in terms of Schur functions. Describe how to do this in general.

(9) Expand the quantity

$$\prod_{1 \leq i < j \leq n} (x_i - x_j)$$

for $n = 1, 2, 3, 4$. If you just think of $\prod_{1 \leq i < j \leq n} (x_i + x_j)$ and set $x_k = 1$ (for all $1 \leq k \leq n$) you can see that you should in theory get up to $2^{n(n-1)/2}$ monomials, but with the expression $x_i - x_j$ you get fewer because some cancel with the sign. How many terms do you seem to get for each of $n = 1, 2, 3, 4$? Conjecture how many there are for all n.

(10) The symmetric functions are endowed with an inner product, where

$$\langle s_\lambda, s_\mu \rangle = \begin{cases} 1 & \text{if } \lambda = \mu \\ 0 & \text{otherwise} \end{cases}$$

and the scalar product is a bi-linear form (that is for symmetric functions F, G, H, K and for coefficients $a, b, c, d \in \mathbb{Q}$, $\langle aF + bG, cH + dK\rangle = ac\langle F, H\rangle + ad\langle F, K\rangle + bc\langle G, H\rangle + bd\langle G, K\rangle$. Compute the following examples:

(a) $\langle 4s_{31} - 3s_{22} + s_{1111} + 12s_{211} - 2s_4, s_{22}\rangle$
(b) $\langle 4s_{31} - 3s_{22} + s_{1111} + 12s_{211} - 2s_4, 4s_{31} - 3s_{22} + s_{1111} + 12s_{211} - 2s_4\rangle$
(c) $\langle 4h_{31} - 3h_{22} + h_{1111} + 12h_{211} - 2h_4, s_{22}\rangle$
(d) $\langle 4h_{31} - 3h_{22} + h_{1111} + 12h_{211} - 2h_4, 4h_{31} - 3h_{22} + h_{1111} + 12h_{211} - 2h_4\rangle$
(e) $\langle h_1, h_1\rangle, \langle h_{11}, h_{11}\rangle, \langle h_{111}, h_{111}\rangle, \langle h_{1111}, h_{1111}\rangle$

(11) The scalar product can be used to expand a symmetric function in the Schur basis. Prove that for any symmetric function F,

$$F = \sum_{\lambda} \langle F, s_\lambda\rangle s_\lambda$$

(12) Compute the h-expansion of s_{221} using the Pieri rule following this procedure.

(a) Compute $h_2 s_{21} = s_{221}+$ terms higher
(b) Rearrange this equation to read $s_{221} = h_2 s_{21}-$ terms higher
(c) Now recursively use the Pieri rule to expand each of the terms on the right hand side of this equation by computing $h_{\lambda_1} s_{(\lambda_2,\lambda_3,\ldots,\lambda_\ell)} = s_\lambda+$ terms higher.

Example: $h_{21} = h_2 s_1 = s_{21} + s_3$ so then we can solve for the Schur function s_{21} and find that $s_{21} = h_{21} - s_3 = h_{21} - h_3$. Now this last expression is the same as $det \begin{bmatrix} h_2 & h_3 \\ 1 & h_1 \end{bmatrix}$.

(13) We can then use this scalar product to define the operation which is "dual" to multiplication. For two symmetric functions F and G define $F^\perp G$ to be the symmetric function such that $\left\langle F^\perp G, s_\lambda\right\rangle = \langle G, F \cdot s_\lambda\rangle$. Find a formula for the Schur expansion of $h_r^\perp s_\lambda$.