What does it mean for something to be random?

An event is called *random* if the process which produces the outcome is sufficiently complicated that we are unable to predict the precise result and are instead able to determine just a range of possible outcomes. It is quite easy to come up with examples of random processes in gambling (e.g. dealing from a shuffled deck of cards, rolling a pair of dice, spinning a roulette wheel), but there are plenty of examples we encounter daily where the mechanism that is generating the outcome is too complicated for us to predict with precision the outcome (e.g. how long it takes to make a commute, the weather forecast, the number of employees which are out sick in a given workday). It is precisely these types of situations that are the motivation for developing mathematical notation to make predictions about the future.

A *random variable* is a mathematical construction which represents a random experiment or process. Usually we represent random variables by capital letters $X, Y, Z, \ldots$. We will be thinking of random variables as roulette wheels which keep track of the possible outcomes and the frequency which those outcomes occur. Random variables have a range of possible values that they may take on (the numbers found on the edge of the wheel) and positive values representing the percentages that these values occur.

The *elementary outcomes* are the individual possible values that the random variable may take on. An *event* is a subset of elementary outcomes and the *field of events* is the set of all possible events for this random variable. The empty set and the set of all elementary outcomes are always included in the field of events. The field of events is assumed to be closed under intersection, union and complementation.

A *probability measure* is a function which associates to each event a number $P[A]$ which will be a value in the interval from 0 to 1. This number reflects the confidence that the event A will occur where a value of 0 represents no chance that the event will occur and 1 that the event happens every time the experiment is conducted. The roulette representation of a random variable records the In general, we must have that if A and B are mutually exclusive events (the intersection of A and B is empty), then

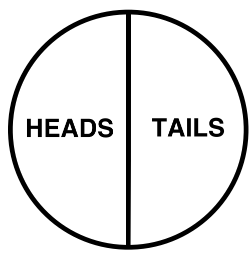
$$P[A \text{ or } \mathrm{B}] = \mathrm{P[A]} + \mathrm{P[B]}.$$

In the situations that we will encounter, we may generally calculate the probability of an event by

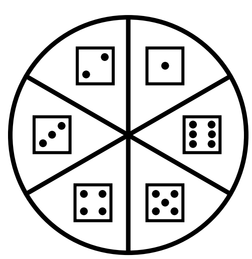
$$P[A] = \frac{\text{the number of outcomes considered as part of A}}{\text{the total number of elementary outcomes}}$$

A random variable for us will be represented as a wheel where the outside edge of this wheel is labelled with the elementary outcomes and the portion of the arc of the circle occupied by any given elementary outcome is equal to the probability that the outcome occurs. We imagine that this wheel spins with a pointer lying somewhere on the outer edge (which we will not draw) and that when the random experiment is conducted we imagine that it is equivalent to giving this wheel a spin a recording the outcome that is lying next to the pointer. Assuming that there are not too many outcomes and the probability that they occur are within a range of pen and paper, this roulette wheel is a visual presentation of the information contained in the random variable. If the number of outcomes of the random variable are too large to draw or their probabilities too small (e.g. a random variable representing the outcomes of a lottery where there are roughly 14 million possiblilities), then we do not actually draw the wheel and instead must imagine that small fractions of an enormously large wheel could be drawn.

Example: If our random variable represented the flipping of a coin then there are two possible elementary outcomes, { heads, tails } . There are four possible events which describes our field of events "heads," "tails," "either heads or tails," "neither heads nor tails." This random variable is visually represented by a roulette wheel with two equal sized regions. One half of the roulette wheel is labeled with 'heads' and the other half is labelled with 'tails.'



Example: If our random variable represented rolling a 6 sided die, the elementary outcomes would be the set $\{1, 2, 3, 4, 5, 6\}$. Examples of events might be "the roll was even" or "roll greater than 4" or more simply "a roll of 3." The roulette wheel representing a roll of a die has six regions. Each sixth of the roulette wheel is labeled with one of the elementary outcomes of the rolling of a die.



Odds are the not the opposite of evens

In gambling we also refer to the probability of an event through the odds. The *odds of an event* is the ratio of the probability of an event occurring to the event not occurring. Usually this is expressed in the form 'number to number' or 'number : number' (occassionally, and you need to be careful in this use, they are also expressed as 'number in number'). If the odds of an event are 'm to n' then if m is bigger than n then it will occur more than 50 percent of the time, if m is less than n then the event will occur less than 50 percent of the time. We may also talk about the odds against an event happening. If the odds for an event happening are 'm to n' then the odds against it happening are 'n to m.'

Example: Here is a mis-use of the vocabulary of odds: "Sir, the possibility of successfully navigating an asteroid field is approximately 3,720 to 1!" More precisely he should have said "the odds against successfully navigating an asteroid field are approximately 3,720 to 1" Never tell me the odds. The advantage of using this notation is that when we are speaking, it is generally understood if the event is likely or unlikely to occur, so mixing up the order of the numbers is generally not too confusing. However, it is important to be precise when describing the rules of a game and the odds of 5:4 or 4:5 are not far off.

Example: The OLG and most lottery boards state the odds of winning the lottery are '1 in number' and they really mean that the probability of winning is 1/number. When lottery boards say they are using the notation of odds, they usually aren't.

Example: The event of rolling a 3 on a die has probability 1/6. The odds that this event happens is '1 to 5' or '1 : 5' because the probability of the event happening is 1/6, the probability of the event not happening is 5/6, and so the odds are $\frac{1/6}{5/6} = 1/5$ which we then express as $1 : 5$ or 1 to 5.

Example: The odds of getting heads on the flip of a coin is $1 : 1$. In this case we say that "there are even odds" on the event.

How to go back and forth between odds and probability

If p is the probability of an event A, then the odds of A occurring are $p/(1-p)$ expressed in the form m to n where $p/(1-p) = m/n$

Example: The probability of rolling two 1's on a pair of dice is 1/36. The odds of this event occurring are $\frac{1/36}{1-1/36} = \frac{1/36}{35/36} = 1/35$. So we say that the odds of rolling snake eyes is 1 to 35.

Example: If we bet the next number on a roulette spin will be a fixed number that we choose from 1-36 or 0 (European roulette wheel), then the probability that our choice is right is 1/37. The odds that we win this bed is then $\frac{1/37}{36/37} = 1/36$ and so we say they are 1 to 36.

Example: The probability of dealing out two cards and both of them are Aces is $6/1326 = 1/221$. To find the odds we compute the fraction $\frac{1/221}{1-1/221} = \frac{1/221}{220/221} = 1/220$ and then we convert this to odds notation by $1 : 220$ or instead say 1 to 220.

If the odds of an event are m to n then the probability of the event occurring is $\frac{m}{m+n}$. This is because $p/(1-p) = m/n$ so $pn = m - mp$ and $pn + pm = m$ so $p = \frac{m}{m+n}$.

Example: If you look on betting websites they will often provide you with the odds of winning a given bet. For instance at the website

`http://en.wikipedia.org/wiki/Craps`

the odds of the Field bet are posted at $5 : 4$. Technically, what is being posted are the odds *against* winning the bet. The probability of winning the bet is $4/(4+5) = 4/9$.

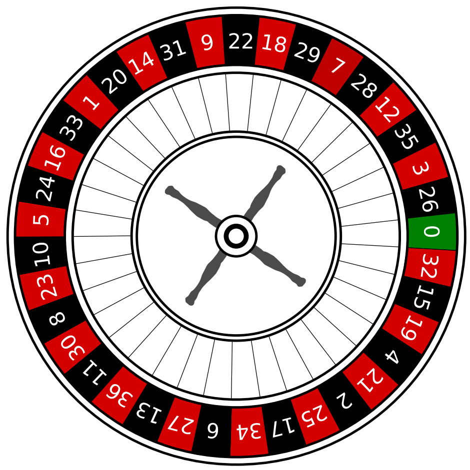
Odds on a payout

We also use the notation of odds to talk about payouts. When we say a bet pays 35 to 1 odds this means that the bet pays $35 for every $1 that was bet. When you bet that $1 and you win, you also get it back so you will actually be returned $36. If you lose that bet, then the $1 is gone.

At a gambling table the bets are often made by using chips. In those cases when the odds are listed where the second number is not 1 then it is most often the case that the bets must be made in multiples of that number. For example, on the Horn bet in craps the payout is $27 : 4$ on a 2 or 12. The bettor must place 4 chips on this bet and when the bet wins by getting the 2 or 12 he/she is returned 31 chips in total, 27 more than the 4 that were bet.

For casino bets the rules are set in the game so that the payout of a given bet favors the casino. If a bet is fair and the odds of winning the bet were $m : n$ and there is only one type of payout (sometimes there are different payouts for different outcomes and the situation is a lot more complicated), then the payout should be $n : m$. In a casino, since the bet is not fair, the payout will be somewhat less than $n : m$. How much less will indicate the size of the cut that the casino is taking. There is no set rule about how casinos determine what the payoff odds are. They set them so that the bets are close enough to a fair game that they will get people taking those bets, but if they can get away with giving a payout which is very unfavorable to a bettor they will set the odds as they like. On some level gamblers are still ‘customers’ in a casino and if another casino down the street is offering better odds then a casino will lose those customers.

For instance, in roulette we found that the odds of winning the bet on a single number is 1 : 36. The payout odds on this bet are 35 : 1. The house is taking a cut on this bet by making the payout odds 35 to 1 when the odds of winning the bet are 1 to 36. Similarly, the odds of winning the 'corner bet' in roulette (betting that one of 4 numbers will come up in the next roulette spin) is 1 : 8 but the payout of this bet is 7 : 1. What we call the 'house advantage' (see below) on these two bets is the same, but that is not easy to see just by looking at the odds alone.



Everyone expects to win, only the house does for sure

When the elementary outcomes of a random variable are all numbers then it makes sense to talk about the average value or the *expected value* of the random variable. For a random variable X with elementary outcomes $\{x_1, x_2, x_3, \ldots, x_n\}$, the expected value (denoted $E(X)$) is defined to be

$$E(X) = x_1 P(X = x_1) + x_2 P(X = x_2) + x_3 P(X = x_3) + \cdots + x_n P(X = x_n) \ .$$

This quantity has some intuitive meaning as sort of the average value that occurs if the experiment is repeated many times. Depending on what the random variable represents, this might not give us much intuition about the outcome of the experiment, but when the random variable represents a bet the expected value does give us some intuition about the bet.

When our random variable represents a gambling bet the expected value has a direct interpretation because in this case the elementary outcomes are the amount of money which is won or lost for each of the events in the bet. In this case, the expected value represents the amount of money that we should expect to win or lose on average per game that is played assuming that the bet is played many times. This does not mean that if we play the game many times that we should expect to win (the expected value) times (the number of bets), instead what it means is that the more times we play (the amount won or lost) divided by (the number of bets) should get closer and closer to the expected value. This will be made more precise when we study the weak law of large numbers.

If the expected value of a bet is positive, then the bet favors the player. If the expected value of a given bet is negative then the bet favors the entity the bet is made against (e.g. the house, the bookie, the opponent). If the expected value of the bet is 0 then we say it is a *fair bet*. When the expected value is negative, the *house advantage* is negative of the expected value expressed as a percentage of a $1 bet.

Example: The 'field bet' is made on a single roll of a pair of dice. On a $1 bet, if the dice add up to 2 or 12 the player wins $2. If the dice add up to 3, 4, 9, 10, 11 the player wins $1, otherwise the player loses the $1 bet. The probability that the roll will be 2 or 12 is 2/36. The probability that the roll is 3, 4, 9, 10, 11 is 14/36. The probability that the roll is 5,6,7 or 8 is 20/36. The expected value of this bet is

$$E(\text{field bet}) = 2 \cdot \frac{2}{36} + 1 \cdot \frac{14}{36} - 1 \cdot \frac{20}{36} = -\frac{2}{36} \approx -.0556$$

What this says is that the bet favors the house and the house advantage is approximately 5.6%.

Example: the expected value of a coin toss doesn't make sense because the elementary outcomes of a coin toss are 'heads' or 'tails.' Since these are not numerical values the expected value doesn't represent anything. However, if we put a bet on the outcome of a flip of a coin such as 'If the outcome is heads I owe you $100, and if the outcome is tails you owe me $100.' In this case the expected value of the bet is

$$E(\$100 \text{ on the flip of a coin}) = 100\frac{1}{2} - 100\frac{1}{2} = 0 \ .$$

This is what we would call a fair bet.

The expected value is an important single number that in a way summarizes something about the bet (or random variable) that is being considered, but a single number cannot tell us everything about a bet. Consider the following example:

Example: Say that you need to raise $1,000,000 by a deadline and choose to do it by betting. The positive outcomes of the following three bets will solve your problem by raising the $1,000,000. Some of the outcomes in the bets could cause you to be in debt for the rest of your life. Say that you are given the choice of the following:

- a bet where you are given a 1 in 11 chance of losing $10,000,000 but a 10 in 11 chance of winning the $1,000,000.

- a bet where you have a 1 in 2 chance of losing \$1,000,000 and a 1 in 2 chance of winning \$1,000,000.
- a 999,999 in 1,000,000 chance of losing \$1 and a 1 in a 1,000,000 chance of winning \$1,000,000.

In all three of these bets the expected value is 0, but the outcomes are very different. A lot of information about these bets is lost by considering only the expected values of the bets. In each case, there is some positive probability of achieving the \$1,000,000 outcome. In the first two bets you don't care what happens 'in the long run,' you probably would not be able to make that bet more than once.

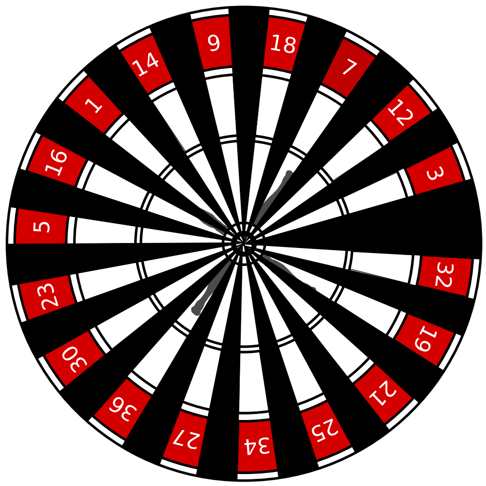
Probability when you know something

We know that the probability of any single number coming up on roulette is 1/37 (assuming a European roulette wheel) so the probability of seeing the number 14 (in particular) on the next roll is 1/37. Imagine a roulette wheel is rigged so that although we don't know the next spins value, we do know that the next spin is red. In this case, since 14 is red it is still possible that the result is 14, but the information that we are given has now changed the probability that 14 will be the next spin because it is a red value.

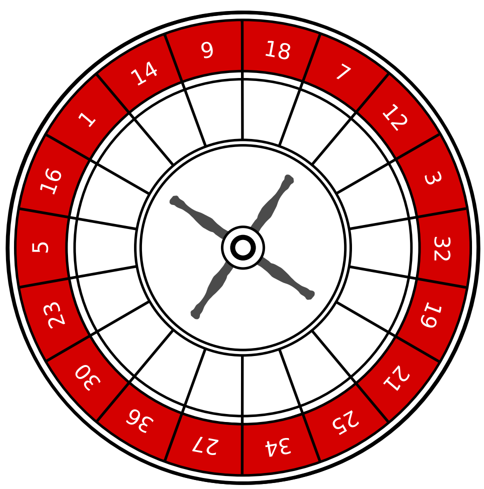
Since there are now 18 red values, one of them is to come up we can determine that the probability of getting a 14 *given* that the next spin is red is 1/18. This is called the *conditional probability* of the event that a 14 comes up given that the next value is red.

There are other situations where we are given some information about the outcome, for example what are the chances that it will rain in the next 24 hours given that it is cloudy now. Other times if we don't have the outcome we like, we repeat the experiment again until we have an outcome that falls into an event that we do like. For example, in the game of craps we might roll a pair of dice until their is either a 5 or a 7 showing, this would be a situation where we would ask the probability of an event given that the roll is either a 5 or a 7. Both of these are situations are examples we would like to determine the probability of an outcome given another outcome.

If we consider random variable as a wheel then the conditional probability can be represented by forbidding certain parts of the wheel of coming up or spinning the wheel again if the forbidden results happen to come up. The resulting wheel is then 'crippled' by blackening in or excluding certain results from arising. Consider the example mentioned above where we know that the next spin of the wheel is red. We would represent this on our roulette wheel by blackening in the possible outcomes that are forbidden as in the image below:



We imagine that if we spin this roulette wheel that we will respin if the outcome lies in one of the the blackened in areas. In this case we are left with 18 spots where the roulette wheel is allowed to stop and those 18 spots occupy still one 37^{th} of the total portion of the wheel. What we do to that wheel is take out the blackened in areas and stretch the remaining parts so that they take up a proportional amount of the resulting wheel.



The wheel above then represents the conditional random variable of the spin of the roulette wheel given the that the spin is red.