MATH 1200
Review

1. Prove that if $m, m+1, m+2$ are three consecutive integers, one of them is divisible by 3.

2. Let n be a positive integer. Prove that $n^3 - n$ is always divisible by 3 in each of these three different ways.

   - Factor $n^3 - n$ completely. The factors represent consecutive integers. Build your proof around this observation.
   - Consider the cases $n = 3k$, $n = 3k+1$, $n = 3k+2$ depending on the value of the remainder when n is divided by 3.
   - Use mathematical induction.

3. Let m and n be integers. Prove or disprove that if m and n are divisible by 3, then $m+n$ is divisible by 3.

4. Let m and n be integers. Prove or disprove that if $m+n$ is divisible by 3, then m and n are divisible by 3.

5. Prove that if n is any integer, the only common divisor of n and $2n+1$ is 1.

6. Prove that if n is an integer, either n^2 or $n^2 - 1$ is divisible by 4.

7. Let $a, b, c \in Z$. Prove that if $a \mid bc$ then $a \mid hcf(a,b) \cdot hcf(a,c)$.

8. Consider the following from **Kenneth H. Rosen**, *Elementary Number Theory and its Applications.*

   > Show that the congruence $x^2 \equiv 1 \pmod{2^k}$ has exactly four incongruent solutions, namely $x \equiv \pm 1$ or $\pm(1+2^{k-1}) \pmod{2^k}$, when $k > 2$. Show that when $k = 1$ there is one solution and when $k = 2$ there are two incongruent solutions.

   (a) Solve $x^2 \equiv 1 \pmod 2$ and $x^2 \equiv 1 \pmod 4$.

   (b) Verify that for $k > 2$, $x \equiv \pm 1$ and $\pm(1+2^{k-1}) \pmod{2^k}$ give four incongruent solutions to the congruence $x^2 \equiv 1 \pmod{2^k}$.

   (c) Prove that for $k > 2$, the congruence $x^2 \equiv 1 \pmod{2^k}$ has no other solutions than those given above in 2. .
   Hint: If $x^2 \equiv 1 \pmod{2^k}$, then $2^k \mid (x-1)(x+1)$. If $2^k \mid x-1$ or $2^k \mid x+1$ then $x \equiv \pm 1 \pmod{2^k}$. Otherwise, note that as x must be odd, you can write $x = 2y+1$ for some integer y. Consider the cases y odd, and y even, to obtain the remaining two solutions.

The following questions are taken from Liebeck, p. 115, #2, 6 and p. 144, # 4.

9. Let p be a prime number and k a positive integer.
   (a) Show that if x is an integer such that $x^2 \equiv x \bmod p$, then $x \equiv 0$ or $1 \bmod p$.
   (b) Show that if x is an integer such that $x^2 \equiv x \bmod p^k$, then $x \equiv 0$ or $1 \bmod p^k$.

10. Let p be a prime number, and let a be an integer that is not divisible by p. Prove that the congruence equation $ax \equiv 1 \bmod p$ has a solution $x \in \mathbb{Z}$.

11. Josephine lives in the lovely city of Blockville. Every day Josephine walks from her home to Blockville High School, which is located 10 blocks east and 14 blocks north from home. She always takes a shortest walk of 24 blocks.
    (a) How many different walks are possible?
    (b) 4 blocks east and 5 blocks north of Josephine's home lives Jemima, her best friend. How many different walks to school are possible for Josephine if she meets Jemima at Jemima's home on the way?
    (c) There is a park 3 blocks east and 6 blocks north of Jemima's home. How many walks to school are possible for Josephine if she meets Jemima at Jemima's home and they then stop in the park on the way?

12. Let $a, b, p, k \in \mathbb{Z}$.
    (a) Prove that if p is prime and $ab \equiv 0 \bmod p$, then either $a \equiv 0 \bmod p$ or $b \equiv 0 \bmod p$.
    (b) Prove that if p is prime such that $p > k \geq 1$, then $\binom{p}{k} \equiv 0 \bmod p$
    (c) Is it true that for any integer n, if $n > k \geq 1$, then $\binom{n}{k} \equiv 0 \bmod n$? Prove your claim.

13. Which of the following relations are equivalence relations on the set of real numbers $\mathbb{R}$? For the relations that are equivalence relations (if any), describe the equivalence classes
    (a) $x + y = 0$
    (b) $xy \geq 0$
    (c) $x = y$ or $x = -y$
    (d) $x = 2y$
    (e) $xy = 0$
    (f) $x = 1$
    (g) $x = 1$ or $y = 1$
    (h) $x^2 + x = y^2 + y$

14. If $p \geq q \geq 5$ and p and q are both primes, prove that $24 \mid p^2 - q^2$.

15. Consider the statement, the sum of any three consecutive positive integers is divisible by 3.

    (a) Sum $17, 18$, and 19 and verify that the resulting number is divisible by 3.

    (b) Prove the statement using Mathematical Induction.

    (c) Prove the statement by considering six cases, depending on the remainder when the smallest of the three numbers is divided by 3.

16. Use induction (mathematical or strong) to prove each of the following.

    (a) For every integer $n \geq 2$, $\binom{2}{2} + \binom{3}{2} + \ldots + \binom{n}{2} = \binom{n+1}{3}$

    (b) For all positive integers n, $2^{n-1} \geq n$.

    (c) Consider the sequence of numbers $a_0, a_1, a_2, a_3, \ldots$ given by $a_0 = 1$, $a_1 = 4$ and for any positive integer $k \geq 2$,

    $$a_k = 8a_{k-1} - 16a_{k-2}$$

    i. Evaluate a_2, a_3, a_4, a_5. Show your work.

    ii. Prove that for all integers $n \geq 0$, $a_n = 4^n$.

17. Compute the cube roots of $z = -1 - i$ and prove your solutions are correct.

18. Find the real and imaginary parts of $(a + ib)^3$ and use this to prove that if $\theta = \frac{2\pi}{3}$ then $3cos^2(\theta)sin(\theta) = sin^3(\theta)$ and $\sqrt{3}|cos(\theta)| = |sin(\theta)|$

19. Find the complex solutions of the following equations (if they exist). For those solutions that exist, prove the geometric representations.

    (a) $\bar{z} = i(z - 1)$

    (b) $z^2 \cdot \bar{z} = z$

    (c) $|z + 3i| = 3|z|$

20. Prove or disprove that, if $x + y$ is rational and x is irrational then y is irrational.

21. Prove or disprove that, if $x + y$ is irrational and x is irrational then y is rational.

22. Prove or disprove that, if xy is irrational and y is irrational then x is rational.

23. Prove or disprove, the sum of any four consecutive integers is even.

24. Prove that if n is an integer and $n^3 + 5$ is odd, then n is even.

25. If a, b, c are integers with $a^2 + b^2 = c^2$ prove that a, b, c are all even, or c is odd with one of a and b odd and the other even.