MATH 1200: Review for Midterm

1. Consider the following statement for integers x and y.

   If $5x + 7y$ is even, then both x and y are even or both x and y are odd.

   (a) Prove the statement using a proof by contradiction.

   (b) Prove the statement by writing $5x+7y = 2z$ for some integer z and manipulating this expression to represent $x - y$ as a multiple of 2.

2. Prove or disprove, the sum of any four consecutive integers is even.

3. Prove or disprove, the sum of any six consecutive integers is odd.

4. Let a and b be integers. Prove or disprove that, if ab is even and a is odd, then b must be even.

5. Let a, b, c, d be integers. Prove or disprove that, if $a - b$ is even and $c - d$ is even then $ac - bd$ is even.

6. Let a, b, c, d be integers. Prove or disprove that, if $a + b$ is even and $c + d$ is even then $ac + bd$ is even.

7. For each of the following statements, either provide a proof of the statement or an example showing the statement is false:

   (a) If a and b are both rational and $b \neq 0$ then a/b is rational.

   (b) If a and b are both irrational and $b \neq 0$ then a/b is rational.

   (c) If a and b are both rational then ab is rational.

   (d) If a and b are both irrational then ab is rational.

   (e) If a and b are both irrational then ab is irrational.

   (f) If a is irrational and b is rational such that $b \neq 0$ then ab is irrational.

8. Prove that there is no complex number z, such that $|z| - z = i$.

9. Find $z \in \mathbb{C}$ such that

   (a) $z = i(z - 1)$

   (b) $z^2 \cdot \bar{z} = z$

10. Find $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$

11. Compute the complex 5th roots of $z = -1 - i$ and express your solutions in both normal and polar form.

12. Find the complex solutions of the following equations and prove the geometric representation of these solutions

    (a) $|z - i| = 2$

    (b) $(z + 6)^3 = i$

13. Prove by induction that for every integer $n \geq 1$,

$$1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n}$$

14. Prove by induction that if q is rational and $n \in \mathbb{Z}^+$ then $(\frac{q-1}{q^2+1})^n$ is rational.

15. Prove that for every positive integer n,

$$1 \cdot 2 + 2 \cdot 3 + \cdots + n(n+1) = n(n+1)(n+2)/3 \ .$$

16. Use Mathematical Induction to prove that if n people stand in a line, n a positive integer at least 2, and the first person in the line is a woman and the last person in line is a man, somewhere in the line there is a woman directly in front of a man.

17. Prove by induction that $(1 + a)^n \geq 1 + na$ for $n \geq 1$, where $a > -1$ is a fixed real number.

18. The triangle inequality says that for any two complex numbers x and y, $|x + y| \leq |x| + |y|$. Show that for any n complex numbers $x_1, x_2, \ldots, x_n$, with $n \geq 2$ any integer,

$$|x_1 + x_2 + \cdots + x_n| \leq |x_1| + |x_2| + \cdots + |x_n|$$