A FEW WORDS ABOUT TELESCOPING SUMS

MIKE ZABROCKI

Say that you want to prove an identity of the form

$$a_1 + a_2 + a_3 + \cdots + a_n = b_n$$

where $a_1, a_2, a_3, \ldots$ and $b_0, b_1, b_2, \ldots$ are sequences of numbers and n is a non-negative integer with $b_0 = 0$. One way to go about this is to show that

$$b_r - b_{r-1} = a_r$$

for all $r \geq 1$, and then write down

$$\begin{aligned} b_1 - b_0 &= a_1 \\ b_2 - b_1 &= a_2 \\ b_3 - b_2 &= a_3 \\ &\vdots \\ b_{n-1} - b_{n-2} &= a_{n-1} \\ b_n - b_{n-1} &= a_n \end{aligned}$$

Now the sum of the expressions on the right hand side of this equation is

$$a_1 + a_2 + a_3 + \cdots + a_{n-1} + a_n$$

and the sum of the expressions on the left hand side of this equation is

$$(b_1 - b_0) + (b_2 - b_1) + (b_3 - b_2) + \cdots + (b_{n-1} - b_{n-2}) + (b_n - b_{n-1}) = b_n - b_0 = b_n \ .$$

We conclude therefore that

$$a_1 + a_2 + a_3 + \cdots + a_n = b_n \ .$$

To summarize what I have just expressed above, I will state it as the following theorem.

Theorem 1. *If* $a_1, a_2, a_3, \ldots$ *and* $b_0, b_1, b_2, b_3, \ldots$ *are two sequence of numbers satisfying*

$$b_r - b_{r-1} = a_r$$

for each $r \geq 1$ *and* $b_0 = 0$*, then*

$$a_1 + a_2 + a_3 + \cdots + a_n = b_n$$

for all $n \geq 1$*.*

Example There are lots of ways of proving the following identity.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \ .$$

Since $\frac{r(r+1)}{2} - \frac{(r-1)r}{2} = r$, we have

$$\begin{aligned}
\frac{1\cdot 2}{2} - \frac{0\cdot 1}{2} &= 1\\
\frac{2\cdot 3}{2} - \frac{1\cdot 2}{2} &= 2\\
\frac{3\cdot 4}{2} - \frac{2\cdot 3}{2} &= 3\\
&\vdots\\
\frac{(n-1)\cdot n}{2} - \frac{(n-2)\cdot(n-1)}{2} &= n-1\\
\frac{n\cdot(n+1)}{2} - \frac{(n-1)\cdot n}{2} &= n
\end{aligned}$$

The sum of the terms on the left hand side of these equations is $\frac{n(n+1)}{2}$ and the sum of the terms on the right hand side of these equation is $1 + 2 + 3 + \cdots + (n-1) + n$, therefore they are equal.

Example The formula for the sums of the squares of the first n integers also has a formula as a product.

$$1^2 + 2^3 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

The general formula for the sums of the k^{th} powers of the first n integers follows a pattern, but it is not easy to conjecture a general formula.

Calculate (using your best algebra skills ... I'm going to leave the calculation out of this example, but there is something to show here) that $\frac{r(r+1)(2r+1)}{6} - \frac{(r-1)(r-1+1)(2(r-1)+1)}{6} = r^2$ and then this shows that

$$\frac{1\cdot 2\cdot 3}{6} - \frac{0\cdot 1\cdot 1}{6} = 1^2$$
$$\frac{2\cdot 3\cdot 5}{6} - \frac{1\cdot 2\cdot 3}{6} = 2^2$$
$$\frac{3\cdot 4\cdot 7}{6} - \frac{2\cdot 3\cdot 5}{6} = 3^2$$
$$\vdots$$
$$\frac{n(n+1)(2n+1)}{6} - \frac{(n-1)n(2n-1)}{6} = n^2$$

The sum of the terms on the left hand side of these equations is $\frac{n(n+1)(2n+1)}{6}$ and the sum of the terms on the right hand side is $1^2 + 2^3 + 3^2 + \cdots + n^2$ and so they must be equal.

Example Define the Fibonacci sequence by $F_0 = 1$, $F_1 = 1$, and for $n \geq 0$, $F_{n+2} = F_{n+1} + F_n$. Say that we want to show that

$$F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1},$$

or in words "The sum of the first n Fibonacci numbers indexed by even n is the next Fibonacci number indexed by odd n." So we know that for $r \geq 1$, $F_{2r+1} - F_{2r-1} = F_{2r} + F_{2r-1} - F_{2r-1} = F_{2r}$. Therefore

$$\begin{aligned} F_3 - F_1 &= F_2 \\ F_5 - F_3 &= F_4 \\ F_7 - F_5 &= F_6 \\ &\vdots \\ F_{2n-1} - F_{2n-3} &= F_{2n-2} \\ F_{2n+1} - F_{2n-1} &= F_{2n} \end{aligned}$$

Since the sum of the left hand side of these equations is $F_{2n+1} - F_1 = F_{2n+1} - F_0$ and the sum of the right hand side of this equation is $F_2 + F_4 + F_6 + \cdots + F_{2n}$, we conclude that

$$F_0 + F_2 + F_4 + \cdots + F_{2n} = F_{2n+1} \ .$$

Example Here is a general identity that can be fairly useful:

$$1\cdot 2\cdots k + 2\cdot 3\cdots(k+1) \ + \ 3\cdot 4\cdots(k+2) \ + \ \cdots \ + \ n\cdot(n+1)\cdots(n+k-1)$$
$$= n\cdot(n+1)\cdots(n+k)/(k+1)$$

Observations: (1) if $k = 1$, then this identity reduces to $1+2+3+\cdots+n = n(n+1)/2$. (2) if $k = 2$, then this identity reduces to $1\cdot 2+2\cdot 3+3\cdot 4+n\cdot(n+1) = n(n+1)(n+2)/3$. (3) there is shorthand notation that makes this sum easier to work with. Let $(a)_k = a(a+1)(a+2)\cdots(a+k-1)$, then the identity becomes

$$(1)_k + (2)_k + (3)_k + \cdots + (n)_k = (n)_{k+1}/(k+1)$$

We note that

$$\begin{aligned} r{\cdot}(r+1)\cdots(r+k)/(k+1) \ &- \ (r-1)\cdot r\cdots(r+k-1)/(k+1) \\ &= r\cdot(r+1)\cdots(r+k-1)((r+k) \ - \ (r-1))/(k+1) \\ &= r\cdot(r+1)\cdots(r+k-1)\ . \end{aligned}$$

Therefore we have

$$1\cdot 2\cdots(k+1)/(k+1) - 0\cdot 1\cdots k/(k+1) = 1\cdot 2\cdots k$$
$$2\cdot 3\cdots(k+2)/(k+1) - 1\cdot 2\cdots(k+1)/(k+1) = 2\cdot 3\cdots(k+1)$$
$$3\cdot 4\cdots(k+3)/(k+1) - 2\cdot 3\cdots(k+2)/(k+1) = 3\cdot 4\cdots(k+2)$$
$$\vdots$$
$$(n-1)\cdot n\cdots(n+k-1)/(k+1) - (n-2)\cdot(n-1)\cdots(n+k-2)/(k+1) = (n-1)\cdot n\cdots(n+k-2)$$
$$n\cdot(n+1)\cdots(n+k)/(k+1) - (n-1)\cdot(n-2)\cdots(n+k-1)/(k+1) = n\cdot(n+1)\cdots(n+k-1)$$

The sum of the entries on the left hand side of these equalities is $n\cdot(n+1)\cdots(n+k)/(k+1)$ and the sum of the entries on the right hand side of these equalities is

$$1\cdot 2\cdots k \ + \ 2\cdot 3\cdots(k+1) \ + \ 3\cdot 4\cdots(k+2) \ + \ \cdots \ + \ n\cdot(n+1)\cdots(n+k-1),$$

therefore the two expressions are equal.

One final observation: It is always possible to express n^k as a sum in the notation $(n)_r$. $n^1 = (n)_1$, $n^2 = (n)_2 - (n)_1$, $n^3 = (n)_3 - 3(n)_2 + (n)_1$, $n^4 = (n)_4 - 6(n)_3 + 7(n)_2 - (n)_1$. This can be used to give a sum of $1^k + 2^k + 3^k + \cdots + n^k$. The coefficients in this expansion are known as the Stirling numbers of the second kind.

Prove the following identities using telescoping sums.

(1)
$$1 + 3 + 5 + \cdots + (2n-1) = n^2$$

(2)
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = 1 - \frac{1}{2^n}$$

(3)
$$x + x^2 + x^3 + \cdots + x^n = \frac{x^{n+1}-1}{x-1} - 1$$

(4)
$$1! \cdot 1 + 2! \cdot 2 + 3! \cdot 3 + \ldots + n! \cdot n = (n+1)! - 1$$

(5)
$$1^2 + 3^2 + 5^2 + \ldots + (2n-1)^2 = \frac{n(4n^2-1)}{3}$$

(6)
$$1^3 + 3^3 + 5^3 + \ldots + (2n-1)^3 = n^2(2n^2-1)$$

(7)
$$\frac{1}{1 \cdot 5} + \frac{1}{5 \cdot 9} + \frac{1}{9 \cdot 13} + \cdots + \frac{1}{(4n-3)(4n+1)} = \frac{n}{4n+1}$$

(8)
$$1 + 2 + 4 + \cdots + 2^n = 2^{n+1} - 1$$

If you have a problem with this last one, see the hint on the next page.

(9) Find a sequence of integers, $a_1, a_2, a_3, \ldots$, such that
$$a_1 + a_2 + a_3 + \cdots + a_n = n^3 \ .$$

(10) Find a sequence of integers, $a_1, a_2, a_3, \ldots$, such that
$$a_1 + a_2 + a_3 + \cdots + a_n = n(2n+1) \ .$$

(11) For each of the sums below, conjecture and prove a formula (ideally using telescoping sums, but you may have some ideas to prove the same thing a different way).
$$1 + 4 + 7 + \cdots + (3n-2) \ .$$
$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n-1)n}$$

Notice that some of the expressions that you are asked to prove don't quite fit the telescoping sums model because if you let the right hand side be b_n, then $b_0 \neq 0$. There are ways of fixing this. The first is to try to subtract something from both sides of the equation so that if you set b_n equal to the right hand side, then $b_0 = 0$. The second is to shift your indices on n so that n is replaced by $n - 1$ or $n + 1$.