QUIZ 4 : MATH 1200 - PROBLEMS, CONJECTURES AND PROOFS

MARCH 30, 2009

You have 50 minutes to complete the following quiz. You may use any notes or books but please do not use any calculators, cell phones or internet devices.

(1) Let $S = \{1, 2, 3\}$, determine if the following statements are true or false. For each statement, justify in one sentence why your answer is correct.

(a) $\forall x \in S, x^2 - 6x + 11 = 6$ FALSE. $2^2 - 6 \cdot 2 + 11 = 3 \neq 6$ so the statement is not true for all values in S.

(b) $\forall x \in S, x^3 - 6x^2 + 11x = 6$ TRUE. $1^3 - 6 \cdot 1^2 + 11 \cdot 1 = 2^3 - 6 \cdot 2^2 + 11 \cdot 2 = 3^3 - 6 \cdot 3^2 + 11 \cdot 3 = 6$, so $x^3 - 6x^2 + 11x = 6$ is true for all values in S.

(c) $\exists x \in S, x^3 - 2x^2 + 4x = 3$ TRUE. $1^3 - 2 \cdot 1^2 + 4 \cdot 1 = 3$ so there is at least one value in S for which $x^3 - 2x^2 + 4x = 3$ is true.

(d) $\exists x \in S, x^2 + 2x - 4 = 3$ FALSE. $1^2 + 2 \cdot 1 - 4 = -1$, $2^2 + 2 \cdot 2 - 4 = 4$, $3^2 + 3 \cdot 2 - 4 = 11$. Since $x^2 + 2x - 4$ is never 3 for all values of x in S the statement is false.

(e) $\forall x \in S, \forall y \in S, x + y = 4$ FALSE. If $x = 1$ and $y = 1$, then $x + y \neq 4$ so $x + y = 4$ is not true for all values of x and y in S.

(f) $\forall x \in S, \exists y \in S, x + y = 4$ TRUE. Since $1 + 3 = 2 + 2 = 3 + 1 = 4$, then for each x value in S, there is a value y in S that makes $x + y = 4$ true.

(g) $\exists x \in S, \forall y \in S, x + y = 4$ FALSE. If $x = 1$ then $x + 1 = 2 \neq 4$. if $x = 2$, then $x + 1 = 3 \neq 4$. If $x = 3$, then $x + 2 = 5 \neq 4$. For each x in S, the statement $\forall y \in S, x + y = 4$ is false.

(h) $\exists x \in S, \exists y \in S, x + y = 4$ TRUE. If $x = 2$ and $y = 2$, then $x + y = 4$. So there is an $x \in S$ and a $y \in S$ such that $x + y = 4$.

(2) For each of the following questions, prove them or find a counterexample.

(a) Let $a, b \in \mathbb{Z}$. If a and b divide n then ab divides n.
The statement is false. If $a = 2$ and $b = 2$ and $n = 2$, then a and b divide n but $ab = 4$ which does not divide n.

(b) Let $a, b \in \mathbb{Z}$. If $a^2 = b^2$, then $a = b$ or $a = -b$.
This statement is true. If $a^2 = b^2$, then $0 = a^2 - b^2 = (a+b)(a-b)$. So either we have $a+b = 0$ or $a - b = 0$ and hence either $a = -b$ or $a = b$.

(c) Let $a, b \in \mathbb{Z}$. If $a < b$, then $a^2 < b^2$.
This statement is false. If $a = -1$ and $b = 0$, then $a < b$ but $a^2 = 1 > b^2 = 0$.

(3) Prove any *one* of the following statements. State clearly which statement you are showing. Use complete sentences:

(a) Let $n \in \mathbb{Z}$. Prove that if $2 - n^2 > 0$, then $n^3 - n = 0$.

(b) Let $x, y \in \mathbb{Z}$. Prove that if $xy + y$ is even, then either x and y are both even or both odd.

(c) Let $n \in \mathbb{Z}$. If 4 divides $n + 3$, then $\binom{n}{2}$ is even.

(a) If $2 - n^2 > 0$ and $n \in \mathbb{Z}$, then $-\sqrt{2} < n < \sqrt{2}$ and $n \in \mathbb{Z}$ and so $n = -1, 0, 1$. For each of these three values $n^3 - n = 0$.

(b) This statement is false in general. For example, if $x = 1$ and $y = 2$ then x and y are not both even or both odd, but $xy + y = 1 \cdot 2 + 2 = 4$ which is even.

(c) If 4 divides $n + 3$, then $n + 3 = 4k$ for some $k \in \mathbb{Z}$ and so $n = 4k - 3$. Since we know a formula for $\binom{n}{2}$ we have

$$\begin{aligned}
\binom{n}{2} &= \frac{n(n-1)}{2} \\
&= \frac{(4k-3)(4k-4)}{2} \\
&= (4k-3)(2k-2) \\
&= 2(4k-3)(k-1)
\end{aligned}$$

and this value is even.

(4) Say that you have an urn with n labeled balls numbered 1 through n. Let k be a natural number that is less than or equal to n.

(a) How many different ways are there to reach into the urn and pull out k of them and color them red? Explain your answer.

There are $\binom{n}{k}$ ways of choosing a subset of k things from a set of n distinguishable elements.

(b) How many ways are there are reaching into the urn and pulling out k of them and coloring them red such that the smallest one colored is 4? Explain your answer.

If the smallest ball that is colored is 4, then there are $k-1$ other elements which have labels between 5 and n. Hence there are $\binom{n-4}{k-1}$ ways of choosing a subset of k elements such the smallest is 4.

(c) How many ways are there are reaching into the urn and pulling out k of them and coloring them red such that the smallest label on a colored ball is i? Explain your answer.

If the smallest ball that is colored is i, then there are $k-1$ other elements which have labels between $i+1$ and n. Hence there are $\binom{n-i}{k-1}$ ways of choosing a subset of k elements such the smallest is i.

(d) Prove that for $n \geq k$,

$$\binom{n}{k} = \binom{n-1-k}{0} + \binom{n-k}{1} + \binom{n-k+1}{2} + \cdots + \binom{n-1}{k}$$

Let i be the smallest element which is not included in a given subset of balls from the urn. That is, the subset of balls which will be colored red is $\{1, 2, \ldots, i-1\}$ and then $k-i+1$ other balls which are greater than i. The number of such subsets is $\binom{n-i}{k-i+1}$ because there are this many ways of choosing the $k-i+1$ remaining balls from the set of balls labelled $\{i+1, i+2, \ldots, n\}$.

Now in any subset of the n balls in which k balls are colored red, the smallest one that is not colored is either 1, or 2, or 3 or any value up to $k+1$. Therefore by the addition principle we have the number of ways of coloring k balls red is:

$$\binom{n-1}{k} + \binom{n-2}{k-1} + \cdots + \binom{n-1-k}{0}.$$

By part (a) of this problem it is also $\binom{n}{k}$.