FIRST HOMEWORK - CONVENER SOLUTIONS

MIKE ZABROCKI

Problem number 1 was to complete the online survey. People either did it or they didn't.

Problem number 2 was to explain why $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$. There were essentially 3 types of solutions that were complete.

- The first solution was by induction. We havn't covered this in class (we will) but some people knew that if you show $P(1)$ and $P(n) \Rightarrow P(n+1)$ for some statement $P(n)$ (where n here represents is an integer parameter) then $P(n)$ is true for all $n \geq 1$. That is they showed that $1^2 = 1 = \frac{1 \cdot 2 \cdot 3}{6}$ and if $1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}$, then
$$\begin{aligned} 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 &= (1^2 + 2^2 + \cdots + n^2) + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{(n+1)(n+2)(2n+3)}{6}. \end{aligned}$$
Therefore $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$ is true for all $n \geq 1$.
- There were also two really good solutions using collapsing sums. The first was to notice $(k+1)^3 - k^3 = 3k^2 + 3k + 1$. Therefore
$$\begin{aligned} 1^3 - 0^3 &= 3 \cdot 0^2 + 3 \cdot 0 + 1 \\ 2^3 - 1^3 &= 3 \cdot 1^2 + 3 \cdot 1 + 1 \\ 3^3 - 2^3 &= 3 \cdot 2^2 + 3 \cdot 2 + 1 \\ 4^3 - 3^3 &= 3 \cdot 3^2 + 3 \cdot 3 + 1 \\ &\vdots \\ (n+1)^3 - n^3 &= 3 \cdot n^2 + 3 \cdot n + 1 \end{aligned}$$
Now they added the numbers along the left up and noticed that everything canceled except $(n+1)^3 - 0^3$ and this was the LHS. on the right hand side they had $3(1^2 + 2^2 + 3^2 + \cdots + n^2) + 3(1 + 2 + 3 + \cdots + n) + (n+1)$. Therefore
$$(n+1)^3 = 3(1^2 + 2^2 + 3^2 + \cdots + n^2) + 3\frac{n(n+1)}{2} + (n+1)\ .$$
By rearranging terms then they could solve for $1^2 + 2^2 + 3^2 + \cdots + n^2$.
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{(n+1)^3 - 3\frac{n(n+1)}{2} - (n+1)}{3} = \frac{n(n+1)(2n+1)}{6}.$$

- Another solution using a collapsing sum was to use that $k(k+1)(k+2) - (k-1)k(k+1) = k(k+1)(k+2-(k-1)) = 3k(k+1)$. Then it is possible to add

$$1 \cdot 2 \cdot 3 - 0 \cdot 1 \cdot 2 = 3 \cdot (1 \cdot 2)$$

$$2 \cdot 3 \cdot 4 - 1 \cdot 2 \cdot 3 = 3 \cdot (2 \cdot 3)$$

$$3 \cdot 4 \cdot 5 - 2 \cdot 3 \cdot 4 = 3 \cdot (3 \cdot 4)$$

$$4 \cdot 5 \cdot 6 - 3 \cdot 4 \cdot 5 = 3 \cdot (4 \cdot 5)$$

$$\vdots$$

$$n(n+1)(n+2) - (n-1)n(n+1) = 3 \cdot (n(n+1))$$

If you add all of these equations on the LHS then you see that everything cancels except $n(n+1)(n+2) - 0 \cdot 1 \cdot 2$ and the right hand side sums to $3(1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1))$. Therefore

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

. But then

$$\begin{aligned}\frac{n(n+1)(n+2)}{3} &= 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) \\ &= (1^2+1) + (2^2+2) + (3^2+3) + \cdots (n^2+n) \\ &= (1^2+2^2+3^2+\cdots+n^2) + (1+2+3+\cdots+n) \\ &= (1^2+2^2+3^2+\cdots+n^2) + \frac{n(n+1)}{2}\end{aligned}$$

Again all they had to do was solve for $1^2+2^2+3^2+\cdots+n^2$.

$$1^2+2^2+3^2+\cdots+n^2 = \frac{n(n+1)(n+2)}{3} - \frac{n(n+1)}{2} = \frac{n(n+1)(2n+1)}{6} .$$

Problem number 3 had essentially the same three types of solutions: induction and two uses of collapsing sums that were used to solve problem 2. In this problem we were asked to find an equation for $1^3+2^3+3^3+\cdots+n^3$. Some people already knew the answer in class so we said that

$$1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$$

- Some people showed the first question by induction $1^3 = 1 = \frac{1^2 \cdot 2^2}{4}$ shows that the statement is true for $n=1$. If we assume that $1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$ is true for n then $1^3+2^3+3^3+\cdots+n^3+(n+1)^3 = \frac{n^2(n+1)^2}{4} + (n+1)^3 = \frac{(n+1)^2(n+2)^2}{4}$ hence $1^3+2^3+3^3+\cdots+n^3 = \frac{n^2(n+1)^2}{4}$ is true for all $n \geq 1$.

- One of the collapsing arguments used that $(k+1)^4 - k^4 = 4k^3 + 6k^2 + 4k + 1$ then $(n+1)^4 - 0^4 = \sum_{k=0}^{n}((k+1)^4 - k^4) = 4(1^3 + 2^3 + 3^3 + \cdots + n^3) + 6(1^2 + 2^2 + 3^2 + \cdots + n^2) + 4(1 + 2 + 3 + \cdots + n) + (n+1) = 4(1^3 + 2^3 + 3^3 + \cdots + n^3) + 6\frac{n(n+1)(2n+1)}{6} + 4\frac{n(n+1)}{2} + (n+1)$. If we rearrange terms then

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{(n+1)^4 - n(n+1)(2n+1) - 2n(n+1) - (n+1)}{4} = \frac{n^2(n+1)^2}{4}.$$

- The other collapsing argument noticed that $k(k+1)(k+2)(k+3) - (k-1)k(k+1)(k+2) = 4k(k+1)(k+2)$. Therefore $n(n+1)(n+2)(n+3) = \sum_{k=0}^{n}(k(k+1)(k+2)(k+3) - (k-1)k(k+1)(k+2)) = 4(1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \cdots + n(n+1)(n+2))$. Hence $1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \cdots + n(n+1)(n+2) = \frac{n(n+1)(n+2)(n+3)}{4}$. Now to find the sums of the cubes they noticed that

$$\begin{aligned}\frac{n(n+1)(n+2)(n+3)}{4} &= 1\cdot2\cdot3 + 2\cdot3\cdot4 + 3\cdot4\cdot5 + \cdots + n(n+1)(n+2)\\ &= (1^3 + 3\cdot1^2 + 2\cdot1) + (2^3 + 3\cdot2^2 + 2\cdot2) + (3^3 + 3\cdot3^2 + 2\cdot3) + \cdots + (n^3 + 3n^2 + 2n)\\ &= (1^3 + 2^3 + 3^3 + \cdots + n^3) + 3(1^2 + 2^2 + 3^2 + \cdots + n^2) + 2(1 + 2 + 3 + \cdots + n)\\ &= (1^3 + 2^3 + 3^3 + \cdots + n^3) + 3\frac{n(n+1)(2n+1)}{6} + 2\frac{n(n+1)}{2}\end{aligned}$$

Then by rearranging terms you can solve for $1^3 + 2^3 + 3^3 + \cdots + n^3$.

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n(n+1)(n+2)(n+3)}{4} - \frac{n(n+1)(2n+1)}{2} - n(n+1) = \frac{n^2(n+1)^2}{4}.$$