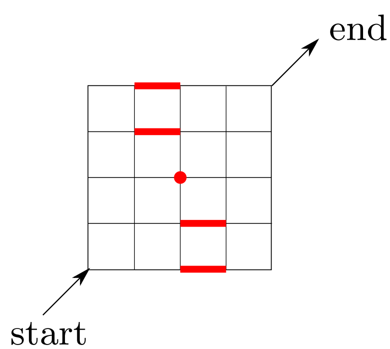
**Math 1200 section B - Problems, conjectures and proofs - An example path question**

Recall from in class and from homework 8 that we showed that the number of paths in an $n \times k$ rectangle from the bottom left to the upper right using only single steps in the north and east direction is equal to $\binom{n+k}{k}$.

One way of arriving at new binomial identities is to add up the total number of paths in two different ways.

Here is an exercise that I would like to use to illustrate this idea. Look at the image below and notice that the total number of paths from the point labeled *start* to the point labelled *end* can be calculated by the formula above to be $\binom{8}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{4 \cdot 3 \cdot 2 \cdot 1} = 70$. If you wish to draw all 70 of those paths to convince yourself that they exist, then take your time to order them carefully. You can also try to convince yourself that there are the same number of paths in that square as there are words in the letters $N$ and $E$ with 4 $N$s and 4 $E$s.




I also marked 4 horizontal steps and the center point of this grid to emphasize them. Notice that every path that passes from the *start* to *end* must pass uniquely through one of these 5 highlighted objects. I could have chosen other horizontal or vertical steps, but I just picked these because I wanted to come up with an interesting identity. As long as I choose a set of objects (segments or points) such that every path passes through exactly one of them then I know by the addition principle that the sum of the paths which pass through each of the objects is equal to the total number of paths.

To come up with a binomial identity we count the number of paths which pass through each of the horizontal steps and center point and then add them up.

The number of paths which pass through the topmost horizontal segment is $\binom{5}{1}\binom{2}{2} = 5$. This is because every path which passes through the topmost horizontal segment consists of a path in the $4 \times 1$ rectangle from start to the left hand corner of the horizontal segment and then continue onto the corner labeled *end*. Similarly the number of paths which pass through the horizontal segment in the second row from the top is $\binom{4}{1}\binom{3}{2} = 12$. The number of paths which pass through the center point on the grid is $\binom{4}{2}\binom{4}{2} = 36$. And the number of paths which pass through the horizontal segment in the fourth row from the top is $\binom{3}{2}\binom{4}{1} = 12$. The number of paths which pass through the horizontal segment in the bottom row is

$\binom{2}{2}\binom{5}{1} = 5$. We can conclude the concrete binomial identity

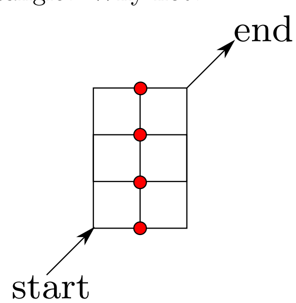
$$\binom{8}{4} = 2\binom{5}{1}\binom{2}{2} + 2\binom{4}{1}\binom{3}{2} + \binom{4}{2}^2 .$$

You should verify that both the left and the right hand side of these equations add up to 70.

**Exercises:**

(1) Draw a $2 \times 2$ square and highlight horizontal segments and the center point similar to what was done in the case of the $4 \times 4$ square in the example above. Write down a binomial identity that has the left hand side equal to $\binom{4}{2}$ based on this picture. Do something similar for a $6 \times 6$ square and arrive at a binomial identity for $\binom{12}{6}$. Generalize your binomial identity for $\binom{4n}{2n}$.

(2) In the picture below the center vertical line of the $3 \times 2$ rectangle has all of the vertices highlighted. How many paths pass through each of the highlighted vertices? Add those up. Do they add up to the number of paths in the $3 \times 2$ rectangle? Why not?




(3) Count the number of paths in the $6 \times 5$ grid below with only $N$ and $E$ steps by those that pass directly from *start* to *end* and in a second way by adding up the paths which pass through the highlighted vertices and segments.