QUIZ 3 : MATH 1200- PROBLEMS, CONJECTURES AND PROOFS

FEBRUARY 18, 2009

You have 45 minutes to complete the following quiz. You may use any notes or books but please do not use any calculators.

(1) Show by induction that

$$1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1 = \binom{n+2}{3}.$$

Please include all necessary details and structure your induction proof clearly.
Hint: you may use $1 + 2 + 3 + \cdots + n = \binom{n+1}{2}$ as something that we have proven before.

Solution: To show this by induction we need a base case. If $n = 1$ then the left hand side of the equation is $1 \cdot 1 = 1$ and the equation is satisfied because $\binom{1+2}{3} = 1$.

Now assume that for some fixed n we have that $\sum_{i=1}^{n} i(n+1-i) = \binom{n+2}{3}$. Then

$$\begin{aligned}
\sum_{i=1}^{n+1} i(n+2-i) &= 1 \cdot (n+1) + 2 \cdot n + 3 \cdot (n-1) + \cdots + n \cdot 2 + (n+1) \cdot 1 \\
&= 1 \cdot (n+1) + 2 \cdot (n-1+1) + 3 \cdot (n-2+1) + \cdots + n \cdot (1+1) + (n+1) \cdot 1 \\
&= (1 \cdot n + 2 \cdot (n-1) + 3 \cdot (n-2) + \cdots + (n-1) \cdot 2 + n \cdot 1) + (1 + 2 + 3 + \cdots + n + (n+1)) \\
&= \binom{n+2}{3} + \binom{n+2}{2} \\
&= \binom{n+3}{3}
\end{aligned}$$

Therefore the equation holds also for $n+1$. We conclude by the principal of mathematical induction that $\sum_{i=1}^{n} i(n+1-i) = \binom{n+2}{3}$ is true for all $n \geq 1$.

(2) Consider the truth values of the following three statements.
- I love Kathy
- If I love Kathy, then I love Lucy
- the previous two statements are either both true or both false.

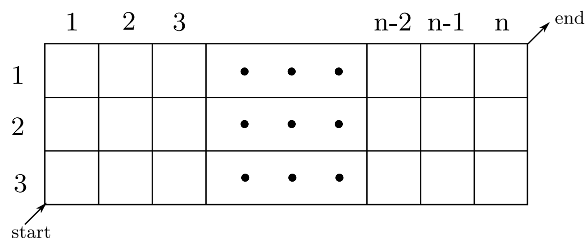
Let A represent "I love Kathy" and B represent "I love Lucy." Write each of the statements above as compound statements in terms of the truth values of these two simple statements. Make a truth table for the three statement in terms of the four possible ways of assigning truth values for A and B. If the third statement is true, can you determine the truth value of the other two statements?

Solution: The first statement is equivalent to the statement A, the second is equivalent to $A \Rightarrow B$ and the third is equivalent to $A \equiv (A \Rightarrow B)$ (other statements such as $(A \wedge (A \Rightarrow B)) \vee (\tilde{}A \wedge \tilde{}(A \Rightarrow B))$ are also equivalent, the simplest is $A \wedge B$, but I would not expect anyone write down this last one).

A	B	A	$A \Rightarrow B$	$A \equiv (A \Rightarrow B)$
T	T	T	T	T
T	F	T	F	F
F	T	F	T	F
F	F	F	T	F

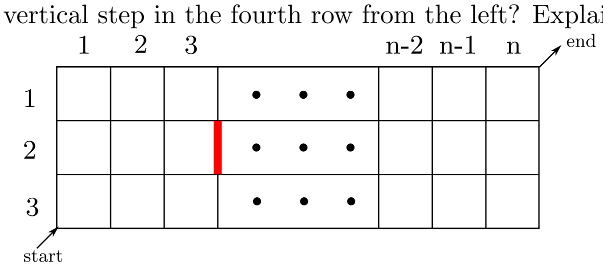
If the third statement is true then this happens only in the first line of the truth table and hence A is true and $A \Rightarrow B$ is also true.

(3) (a) How many paths are there in the following $3 \times n$ rectangle that start at the bottom left hand corner and end in the upper right using only single steps in the North and East direction? Explain your answer.



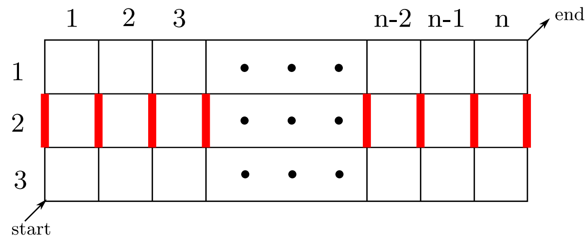
Solution: From the homework problem and the discussion in class we know that the number of paths in an $3 \times n$ grid using single north and east steps one at a time is equal to $\binom{3+n}{n} = \binom{n+3}{3}$.

(b) How many paths are there in the following $3 \times n$ rectangle that start at the bottom left hand corner and end in the upper right using only single steps in the North and East direction that pass through middle vertical step in the fourth row from the left? Explain your answer.



Solution: The number of paths from the corner labeled by 'start' to the bottom of the line marked in the image using single north and east steps is 4 because those paths pass within the bottommost 1×3 rectangle. The number of paths from the top of the segment marked in the image and the point labeled 'end' is $n - 2$ because it is equal to the number of paths in the topmost $1 \times (n - 3)$ rectangle. Since every path which passes through the marked segment consist of a path from start to the bottom of the segment, a north step, followed by a path to the upper right hand corner labeled by 'end,' by the multiplication principle the number of these paths is $4 \cdot (n - 2)$.

(c) How many paths are there in the following $3 \times n$ rectangle that start at the bottom left hand corner and pass through any of the center vertical lines in the diagram below? Add up each of the numbers of paths which pass through the highlighted vertical lines below. Explain your answer.



Solution: Every path which passes through the i^{th} segment from the left consists of a path in a $1 \times (i-1)$ rectangle (from the point labelled by 'start' to the bottom corner of the segment), a step up, followed by a path in a $1 \times (n-i+1)$ rectangle (from the top of the segment to the corner labeled by 'end'). The number of such paths is $i \cdot (n-i+2)$. Since there are $n+1$ segments and every path passes through exactly one of them, by the addition principle we see that the total number of paths from the point labeled 'start' to the point labelled 'end' using single north and east steps only is

$$1 \cdot (n+1) + 2 \cdot n + 3 \cdot (n-1) + \cdots + n \cdot 2 + (n+1) \cdot 1$$