AN INDUCTION PROOF

Prove the following identity by induction on r.

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

This problem seems to be harder than some of the others because I was doing it with the wrong identity. If you do induction on m or n it isn't as much as a problem because it can be done with the identity $\binom{n+1}{k} = \binom{n}{k} + \binom{n}{k-1}$ (which you should remember from the second homework assignment).

For this problem, the easiest ways seems to use the following intermediate result:

Lemma 1.

$$\sum_{a=0}^{n} \binom{a}{r} = \binom{n+1}{r+1} \quad .$$

Proof. If $n = 0$, then

$$\sum_{a=0}^{0} \binom{a}{r} = \binom{0}{r} = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{for } r > 0 \end{cases}$$

We also have that

$$\binom{0+1}{r+1} = \begin{cases} 1 & \text{if } r = 0 \\ 0 & \text{for } r > 0 \end{cases}$$

hence the left hand side and the right hand side of this identity are equal for the case of $n = 0$.

Assume that we have for a fixed n that

$$\sum_{a=0}^{n} \binom{a}{r} = \binom{n+1}{r+1} \quad .$$

Then

$$\sum_{a=0}^{n+1} \binom{a}{r} = \sum_{a=0}^{n} \binom{i}{r} + \binom{n+1}{r} = \binom{n+1}{r+1} + \binom{n+1}{r} = \binom{n+2}{r+1} = \binom{(n+1)+1}{r+1}$$

Hence the identity holds for $n+1$. Since it is true for a base case and if it is true for n then it holds for $n+1$ then by the principle of mathematical induction it is true for all n.

$\square$

Now to prove the identity I had to modify it slightly. See if you can spot the difference and when you read the proof, see if you can spot why the statement that I had before wasn't good enough.

Proposition 2. *For $m+n \geq 0$ and for $r \geq 0$,*

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

Proof. We will prove this by induction on r. First we try $r = 0$. On the left hand side of the equation we have

$$\sum_{k=0}^{0} \binom{m}{k} \binom{n}{0-k} = \binom{m}{0} \binom{n}{0} = 1$$

Similarly, on the right hand side of this identity we see

$$\binom{m+n}{0} = 1 \ .$$

Since these are both equal to 1 the identity is true for $r = 0$.

Now we will assume for a fixed r that the identity that we would like to show is true. That is assume that we know for any $m+n \geq 0$, that

$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}$$

or by replacing $x = m+n$,

$$\sum_{k=0}^{r} \binom{m}{k} \binom{x-m}{r-k} = \binom{x}{r}$$

Now we start with the right hand side of the identity we are showing with r replaced by $r+1$, then

$$\begin{aligned}
\binom{m+n}{r+1} &= \sum_{a=0}^{m+n-1} \binom{a}{r} \\
&= \sum_{a=0}^{m+n-1} \sum_{k=0}^{r} \binom{m}{k} \binom{a-m}{r-k} \\
&= \sum_{k=0}^{r} \binom{m}{k} \sum_{a=0}^{m+n-1} \binom{a-m}{r-k} \\
&= \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r+1-k}
\end{aligned}$$

Therefore the identity holds for r replaced by $r+1$. Since we have shown that since the identity is true for a fixed r, then it is true for r replaced by $r+1$ and we have shown it for the base case of $r = 0$, then by the principle of mathematical induction the identity holds for all $r \geq 0$. $\square$