Consider the problem *Square Take-away* from Mason and followup questions from Tutorial 8.

> Take a rectangular piece of paper and remove from it the largest possible square. Repeat the process with the left-over rectangle. What different things can happen? Can you predict when they happen?

In particular,

1. What happens if the ratio of the two sides of the rectangle is a rational number?
2. What happens if the ratio of the two sides of the rectangle is $\sqrt{2}$? If the sides of the rectangle are of length a and b with $a > b$, then $\left(\frac{a}{b}\right)^2 = 2$. Interpret your answer to give a proof about the rationality of $\sqrt{2}$.

The answer to 1. is that the process terminates, i.e., eventually the rectangle one obtains is a square. If the sides are say m units and n units where m and n are positive integers then that square has side the greatest common divisor of m and n.

Here is a proof that the process terminates:

Consider a rectangle of size $s \times t$ units, with s, t positive integers, and $s \geq t$. If $s = t$ we are done. If $s < t$, after removing a square of dimension $s \times s$, one obtains a rectangle of size $(t - s) \times s$. What is important is not the actual values but rather that the new rectangle is smaller in one dimension than the original rectangle – it is smaller by at least one unit of length – and no larger in the other dimension. How many times can this process be iterated? It cannot go on forever as in that case (at least) one length would have to be reduced (by at least 1 unit) an infinite number of times. That cannot happen. As the process terminates, one eventually reaches a square. The smallest possible size for that square is 1×1 unit.

Why is the side of the square obtained the greatest common divisor of m and n?

If a rectangle has size $s \times t$ units, with s, t positive integers, and say, $s \geq t$, the rectangle obtained from it once a square is removed has size $s \times (t - s)$ units. Comparing lists of divisors gives $\gcd(s, t) = \gcd(s, t - s)$.

Now start with a rectangle of size $m \times n$. We know that at each stage the rectangles obtained have the property that the greatest common divisor of the lengths of their sides is unchanged, i.e., it remains $\gcd(m, n)$. The process terminates with a square of size $s \times s$. Comparing lists of divisors gives $\gcd(m, n) = \gcd(s, s) = s$.

What happens if one takes a rectangle where the ratio of its sides is not a rational number? A neat example is the case where the sides are in the golden ratio. At each stage the rectangle one is left with also has sides in the golden ratio. We never get squares so the process never terminates.

This suggests the possibility of using the outcome of the square take-away process as a means to prove that a particular number is irrational. That is what question 2. is about. If one starts with a rectangle whose sides are in ratio some real number ρ and proves that the process never terminates, then ρ must be irrational.

Start with a rectangle with sides a and b such that $\left(\dfrac{a}{b}\right)^2 = 2$. No assumption is made about a and b other than that they are positive real numbers. Apply the square take-away process to the rectangle. As (necessarily) $a > b$, the rectangle obtained has dimensions $b \times (a - b)$. As $a - b < b$ (this follows from $\left(\dfrac{a}{b}\right)^2 = 2$), at the next stage one obtains a rectangle of dimension $(a - b) \times (b - (a - b))$ or $(a - b) \times (2b - a)$. A simple calculation shows that if $\left(\dfrac{a}{b}\right)^2 = 2$, then $\left(\dfrac{2b - a}{a - b}\right)^2 = 2$, i.e., every other step we end up with a rectangle whose sides are in the same ratio as the original rectangle. Also in this case, as the square take-away process does not terminate. We interpret this as a proof of the irrationality of $\sqrt{2}$.

There is still one piece missing. We know if the sides of the original rectangle are commensurable, the square take-away process will terminate. If they are not, let's say that the sides have lengths a and b with $a < b$ and $\frac{a}{b}$ irrational. Applying the square take-away process leaves a rectangle with sides a and $b - a$. As $\frac{a}{b}$ is irrational, so is $\frac{b-a}{a}$. The new rectangle has incommensurable sides as well. Can one end up with a square? Since the sides of a square are commensurable the answer is no, as the take-away process always leaves us with rectangles having incommensurable sides.