ASSIGNMENT 6

DUE: APRIL 17

Questions:

(1) Did you complete the online course evaluation for math 1200? If not, go to `http://courseevaluations.yorku.ca` now before you do anything else. You have from March 15 to April 5 to submit your evaluation.

(2) Prove or disprove each of the following statements
  (a) If $a - b$ is even and $c - d$ is even, then $ac - bd$ is even.
  (b) If a^2 is divisible by n, and b^2 is divisible by n, then ab is divisible by n.
  (c) Let p be a prime, a^2 is divisible by p if and only if a is divisible by p
  (d) If x and y are in $\mathbb{Q}$, and $x < y$, then there is a z in $\mathbb{Q}$ such that $x < z < y$.
  (e) For every positive integer n, $n^2 - n + 17$ is prime.

(3) Let n be an integer. Justify the following statements.
  (a) The last digit of n is even if and only if n is divisible by 2.
  (b) The last two digits of n are divisible by 4 if and only if n is divisible by 4.
  (c) The last three digits of n are divisible by 8 if and only if n is divisible by 8.
  (d) The last k digits of n are divisible by 2^k if and only if n is divisible by 2^k.

(4) Recall that we call a function $f : A \to B$ 'injective' or '1-1' if for all $x, y \in A$, if $f(x) = f(y)$, then $x = y$. And we call a function 'surjective' or 'onto' if for every $y \in B$, there is an $x \in A$ such that $f(x) = y$. Consider the function from $f : \mathbb{Z} \to \mathbb{Z}$ where $f(x) = x^3 - 4x$ and $g : \mathbb{R} \to \mathbb{R}$ where $g(x) = x^3 - 1$.
  (a) Is f injective? Why or why not? If it is demonstrate or explain why. If not, give an example of where it fails to be injective.
  (b) Is f surjective? Why or why not? If it is demonstrate or explain why. If not, give an example of where it fails to be surjective.
  (c) Is g injective? Why or why not? If it is demonstrate or explain why. If not, give an example of where it fails to be injective.
  (d) Is g surjective? Why or why not? If it is demonstrate or explain why. If not, give an example of where it fails to be surjective.

(5) Write the *negation* and *contrapositive* (if possible) of each of the following statements:
  (a) For every $q \in \mathbb{Z}$, there exists some $p \in \mathbb{Q}$ such that $q^3 \geq p$.
  (b) If $x < \log y$ then for all $z > 0$, $zx < \log(y^z)$.
  (c) If n is odd then there is some integer k such that $n = 2k + 1$.
  (d) If $(x, y) \in A \times B$ or $(x, y) \in A \times C$ then $(x, y) \in A \times (B \cup C)$.

(6) Show that for all $n, r \geq 0$,

$$\binom{n}{0} + \binom{n+1}{1} + \binom{n+2}{2} + \cdots + \binom{n+r}{r} = \binom{n+r+1}{r}.$$

Find a set of lattice paths which are counted by $\binom{n+r+1}{r}$, find a subset of those paths that count $\binom{n}{0}$, $\binom{n+1}{1}$ and more generally $\binom{n+k}{k}$. Use this to show the identity.

(7) Prove by induction that
  (a) By induction on r, the identity in the previous problem.
  (b) By induction on n, the identity in the previous problem.
  (c) For all $n \geq 0$ that $5^{2n} - 1$ is divisible by 24.

(d) For every $n \geq 0$, $3^n > 2n$.

(8) Consider the following definition of a certain type of integer:

A positive integer n is called *ips* if there exists a positive integer k such that $n = k^2$ and it is called not *ips* otherwise.

(a) Which of the following integers are or are not ips: -9, 1, 10, 49. Explain your answer by stating why each number does or does not satisfy the definition.
(b) For each positive integer n, let D_n be the complete set of pairs (a, b) where a and b are positive integers such that $n = ab$. What are D_9 and D_{12}?
(c) Explain why if (a, b) is an element of D_n, then (b, a) is in D_n.
(d) Explain why if n is not ips and (a, b) is in D_n, then $a \neq b$.
(e) Explain why if n is an ips, then there is an integer a such that (a, a) is in D_n.