Homework Assignment no. 3

Date: Due December 3, 2012

Your assignment should include complete sentences and explanations and not just a few equations, tables or numbers. A solution will not receive full credit unless you explain what your answer represents and where it came from. You may discuss the homework with other students in the class, but please write your own solutions.

(1) The statements below are all true, provide a proof of why they are true.
  (a) Assume a, b, c are all integers, if ab, ac, bc are all odd, then a, b, c are odd.
  (b) For integers a and b, $a - b$ is odd if and only if $a + b$ is odd.
  (c) If y is an integer, then $y^3/3 + y^2 - 10y/3 + 2$ is an integer.
  (d) The sum of the squares of two odd integers cannot be a perfect square.
  (e) If x and y are positive real numbers, then $\sqrt{x+y} \neq \sqrt{x} + \sqrt{y}$.
  (f) The product of a rational number and an integer is a rational number.

(2) The following statements are either true or false. For each statement that is true give an explanation why. For each statement that is false give a counterexample.
  (a) For integers $m > 0$ and $a, b, c, d \in \mathbb{Z}$, if $m|(a - b)$ and $m|(c - d)$, then $m|(ac - bd)$.
  (b) For $m > 0$ and $a, b \in \mathbb{Z}$, if $m|ab$, and m does not divide a, then $m|b$.
  (c) For positive real numbers x, y, $\sqrt{xy} \leq \frac{x+y}{2}$
  (d) For every positive integer n, $n^3 - 10n$ is divisible by 3.

(3) Let X be the set of subsets of $\{1, 2, 3, \ldots, 10\}$ with less than 5 elements (e.g. $\{2, 3, 4, 5\}$ has less than 5 elements so it is a set in X while $\{2, 4, 5, 6, 8, 10\}$ is not a set in X since it has 6 numbers in it).

Let Y be the set of non-empty subsets S of $\{1, 2, 3, \ldots, 10\}$ where the largest number in S minus the smallest number in the set S is greater than 0 and less than 5 (e.g. the set $\{2, 3, 4, 5\}$ is a set in Y, but $\{3, 8, 9\}$ is not a set in Y because the smallest is 3 and the largest is 9).

One of the following two statements is true and the other is false. Prove the one that is true and give a counterexample to the one that is false.
  (a) For every set S in X, there exists a set T of Y such that S and T have exactly two elements in common.
  (b) For every set S in Y, there exists a set T of X such that S and T have exactly two elements in common.