(1) (a) Prove that if an integer n has the form $6q + 5$ for some $q \in \mathbb{Z}$, then n also is of the form $3k + 2$ for some $k \in \mathbb{Z}$.
(b) Prove or disprove the converse of the statement in part (a).

(2) Let x be a real number. Prove or disprove, $x = 2$ if and only if $x^3 - x^2 - x = 2$.

(3)

$$1 \quad 2 \quad 3 \quad \cdots \quad n \quad (n+1) \quad \cdots \quad (2n-2) \quad (2n-1) \quad (2n)$$

Consider the natural numbers from 1 to $2n$. Pair off these numbers as above, 1 and $(2n)$, 2 and $(2n-1)$, 3 and $(2n-2)$, $\ldots$, n and $(n+1)$, and evaluate the products of the pairs, $1 \times (2n)$, $2 \times (2n-1)$, $3 \times (2n-2)$, $\ldots$, $n \times (n+1)$. Prove that for no value of n are two of these n products equal.

(4) Consider the function $f : \mathbb{R} \to \mathbb{R}$, where $f(x) = x^3 - 2x^2 - 3x$.
(a) Prove or disprove that f is $1-1$.
(b) Prove or disprove f is onto.

(5) Consider the statement, the sum of any three consecutive positive perfect cubes is divisible by 9.
(a) Sum the cubes of 4, 5 and 6 and verify that the resulting number is divisible by 9.
(b) Prove the statement using mathematical induction.
(c) Prove the statement by considering three cases, depending on the remainder when the smallest number cubed is divided by 3.

(6) Read the following excerpt from Stillwell, *Mathematics and its History*, and use it as required to answer the following questions. Note that p and q represent positive integers.

> As early as 2000 B.C., the Babylonians could solve pairs of simultaneous equations of the form
>
> $$\begin{aligned} x + y &= p \\ xy &= q \end{aligned}$$
>
> which are equivalent to the quadratic equations
>
> $$x^2 + q = px$$
>
> The original pair was solved by a method that gave the two roots of the quadratic:
>
> $$x, y = \frac{p}{2} \pm \sqrt{\left(\frac{p}{2}\right)^2 - q}$$
>
> when both [of these roots] were positive (the Babylonians did not admit negative numbers). The steps in the method were:
>
> (i) Form $\dfrac{x+y}{2}$
> (ii) Form $\left(\dfrac{x+y}{2}\right)^2$
> (iii) Form $\left(\dfrac{x+y}{2}\right)^2 - xy$
> (iv) Form $\sqrt{\left(\dfrac{x+y}{2}\right)^2 - xy} = \dfrac{x-y}{2}$
> (v) Find x, y by inspection of the values in (i), (iv)
>
> Of course these steps were not expressed in symbols but only applied to specific numbers. Nevertheless, a general method is implicit in the many specific cases solved.

(a) Why is solving the system

$$\begin{aligned} x + y &= p \\ xy &= q \end{aligned}$$

equivalent to solving the quadratic equation

$$x^2 + q = px\,?$$

Hint: Compare the solution of the system with the two roots of the quadratic.

(b) Solve $x^2 + 3 = 4x$ using the Babylonian method. Clearly indicate each step of the procedure.

(7) Consider the following statement.

> Let p and q be integers. If both $x^2 + px - q = 0$ and $x^2 + px + q = 0$ have integer solutions then there exist integers a and b such that $a^2 + b^2 = p^2$.

The following leads to a proof.

(a) State the quadratic formula and use the quadratic formula to verify that both $x^2 + 5x - 6 = 0$ and $x^2 + 5x + 6 = 0$ have integer solutions.

(b) Recall that two integers have the *same parity* if both are even or both are odd. Let $m^2 = p^2 + 4q$ and $k^2 = p^2 - 4q$ with p, q, m, k integers. Prove that m and k have the same parity.

(c) Verify the identity,

$$\left(\frac{m+k}{2}\right)^2 + \left(\frac{m-k}{2}\right)^2 = \frac{m^2 + k^2}{2}.$$

(d) Prove the statement. Give formulas for the values of a and b.

(e) Verify that in the case $p = 13$, $q = 30$, the formula you obtain in (d) gives the values 12 and 5 for a and b.

(8) (a) Prove that the sum of the squares of four consecutive even integers is divisible by 8.

(b) Is the sum of the squares of four consecutive odd integers ever divisible by 8? Justify your answer.

(9) Let

$$f(n) = \frac{1}{1 \cdot 3} + \frac{1}{3 \cdot 5} + \frac{1}{5 \cdot 7} + \cdots + \frac{1}{(2n-1)(2n+1)}.$$

(a) Evaluate $f(n)$ for $n = 1, 2, 3, 4$. Write your answer as an ordinary fraction.

(b) Write down a formula for $f(n)$ and use mathematical induction to prove that it is correct.

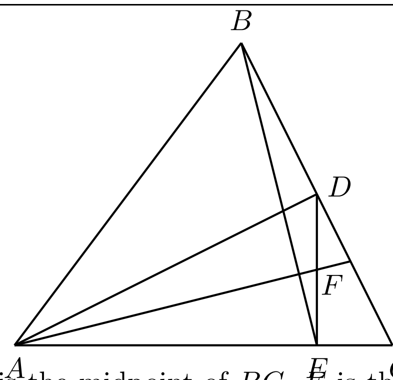
(10) Recall the following.

Let (x_1, y_1) and (x_2, y_2) be two points in the Cartesian plane.

- The distance between them is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$.
- The midpoint of the segment between them is $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$.
- The slope of the line through them is $\frac{y_2 - y_1}{x_2 - x_1}$.

Two lines are perpendicular when the product of their slopes is -1.

The foot of the perpendicular from the point (x_1, y_1) to the horizontal axis has coordinates $(x_1, 0)$.



In triangle ABC, $AB = AC$, D is the midpoint of BC, E is the foot of the perpendicular drawn from D to AC, and F is the midpoint of DE. The goal is to prove that AF is perpendicular to BE.

Let $A = (0,0)$, $B = (4a, 4b)$, $C = (4c, 0)$.
(a) Explain why $a^2 + b^2 = c^2$.
(b) Find the slopes of the line $\overleftrightarrow{AF}$ and of the line $\overleftrightarrow{BE}$.
(c) Explain why $\overleftrightarrow{AF} \perp \overleftrightarrow{BE}$.

(11) Define numbers $A_{n,k}$ by $A_{n,0} = A_{n,n} = n+2$ and if $1 \le k \le n$, by $A_{n,k} = A_{n-1,k-1} + A_{n-1,k}$.
(a) Evaluate these numbers for $n = 1, 2, 3, 4, 5$ (or more) and arrange them in a triangle.
(b) Compare your triangle in (a) to Pascal's triangle. Use ordinary language to make the comparison.
(c) Is $A_{n,k} = C_{n+2,k+1}$? Justify your answer.

(12) Consider the following question from Larson, *Problem-Solving Through Problems*.

> Which is larger $\sqrt[3]{60}$ or $2 + \sqrt[3]{7}$? (Cubing each number leads to complications that are not easily resolved. Consider instead the more general problem: Which is larger, $\sqrt[3]{4(x+y)}$ or $\sqrt[3]{x} + \sqrt[3]{y}$, where $x, y \ge 0$? Take $x = a^3, y = b^3$.)

(a) In general, for $x, y \ge 0$, $\sqrt[3]{4(x+y)} \ge \sqrt[3]{x} + \sqrt[3]{y}$. Which values of x and y can be used to establish which of $\sqrt[3]{60}$ and $2 + \sqrt[3]{7}$ is bigger? Which is bigger? Explain why.
(b) Following Larson's suggestion, let $x = a^3$ and $y = b^3$. Show that for $a, b \ge 0$, $4(a^3 + b^3) \ge (a+b)^3$.
Hint: First verify that $4(a^3 + b^3) - (a+b)^3 = 3(a+b)(a-b)^2$.
(c) Explain how using (b) you can conclude that for $x, y \ge 0$, $\sqrt[3]{4(x+y)} \ge \sqrt[3]{x} + \sqrt[3]{y}$.

(13) Prove that $n^3 - n$ is divisible by 3 for all integers n.
Hint: Consider three cases, $n = 3k$ for some integer k, $n = 3k+1$ for some integer k, $n = 3k+2$ for some integer k.

(14) If a, b, c are positive real numbers, and $a < b + c$, carefully show that

$$\frac{a}{1+a} < \frac{b}{1+b} + \frac{c}{1+c} .$$

(15) Use Mathematical Induction to prove that

$$\frac{5^n}{n!} < 625 = 5^4$$

for all positive integers n.
Note: You may have to verify this directly for values of n in addition to $n = 1$.

(16) Let $\mathbb{N}$ represent the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(1) = 2, \text{ and } f(xy) = f(x)f(y) - f(x+y) + 1 \text{ for all } x, y \in \mathbb{N}.$$

Hint: Start by determining $f(2), f(3), f(4), \ldots$. Make a conjecture and then use Mathematical Induction to prove your conjecture is correct.

(17) Recall that a function $f : A \to B$ is one-to-one provided whenever $f(a) = f(a')$ then $a = a'$. Let $\Phi(x) : \mathbb{R} \to \mathbb{R}$ be defined by

$$\Phi(x) = \begin{cases} \sqrt{x} + 1 & \text{if } x \geq 0 \\ \dfrac{1}{x^2+1} & \text{otherwise} \end{cases} .$$

Is Φ a one-to-one function? Justify your answer.

(18) This exercise provides a proof that $\sqrt{3} + \sqrt{2}$ is an irrational number. You may take as given (no proof required) that $\sqrt{3}$ and $\sqrt{2}$ are irrational numbers.
  (a) Define what it means for a real number to be rational, and for a real number to be irrational.
  (b) Prove that the sum and that the product of two rational numbers is rational.
  (c) Verify that if $\sqrt{3} + \sqrt{2}$ is rational so is $\sqrt{3} - \sqrt{2}$.
  Hint: What is their product?
  (d) Verify that if $\sqrt{3} + \sqrt{2}$ is rational, so is $\sqrt{3}$.
  (e) Given that $\sqrt{3}$ is not rational, what can you conclude about $\sqrt{3}+\sqrt{2}$? Explain your argument.
  (f) Generalize. If x and y are irrational, what condition on $x^2 - y^2$ ensures that $x+y$ be irrational?

(19) (a) If nine objects are distributed among four boxes, prove that one box has three (or more) objects in it.
  Hint: It is perhaps easier to explain why the contrapositive of this statement is true.
  (b) Let S be a square region (in the plane) of side length 2 inches. Show that among any nine points in S there are three which are the vertices of a triangle of area $\leq \frac{1}{2}$ square inch.

(20) Let $D(n, k)$ be the number of subsets of size k of the set $\{1, 2, 3, \ldots, n\}$.
  (a) Find the 10 subsets of size 3 of the set $\{1, 2, 3, 4, 5\}$.
  (b) Find the 6 subsets of size 2 of the set $\{1, 2, 3, 4\}$.
  (c) Find the 4 subsets of size 3 of the set $\{1, 2, 3, 4\}$
  (d) Explain why in general if $n \geq 2$ and $1 \leq k \leq n-1$, that $D(n, k) = D(n-1, k-1) + D(n-1, k)$.
  Note: Use the definition of $D(n, k)$ as a count. You will not receive credit for a proof based on a known formula for $D(n, k)$.
  (e) Explain why for $n \geq 0$, $D(n, 0) = 1$ and $D(n, n) = 1$.

(21) Consider sequences of 1s and 0s which we shall refer to as binary words. A word is called palindromic if it reads the same forwards as backwards. For example the word 0110110 is palindromic while the word 001110 is not. If u and v are words then uv is defined to be the concatenation of those words (put the two words next two each other). For example if $u = 0110110$ and $v = 001110$, then $uv = 0110110001110$.
  (a) Show that if u and v are two palindromic words then it is not generally true that uv (the concatenation of the two words) is palindromic.
  (b) Explain why if u and v are two palindromic words, then uvu is also palindromic.
  (c) Is there a palindromic word with fifteen 0s and twenty three 1s? Explain.

(22) (a) If x=1, find a value of $\epsilon > 0$ such that $|x| > \epsilon$.
  (b) If x= .02, find a value of $\epsilon > 0$ such that $|x| > \epsilon$.
  (c) Show that if x is a real number $x \neq 0$, then there is an ϵ greater than 0 such that $|x| > \epsilon$.
  (d) Use the result in (c) to prove that if $|x| \leq \epsilon$ for all real $\epsilon > 0$, then $x = 0$.