

MORE PROBLEMS

MARCH 27, 2018

- (1) Let \mathbb{N} represent the set of positive integers. Find all functions $f : \mathbb{N} \rightarrow \mathbb{N}$ such that

$$f(1) = 2, \text{ and } f(xy) = f(x)f(y) - f(x+y) + 1 \text{ for all } x, y \in \mathbb{N}.$$

Hint: Start by determining $f(2), f(3), f(4), \dots$. Make a conjecture and then use Mathematical Induction to prove your conjecture is correct.

- (2) Let S be a square region (in the plane) of side length 2 inches. Show that among any nine points in S there are three which are the vertices of a triangle of area $\leq \frac{1}{2}$ square inch.

- (3) This exercise provides a proof that $\sqrt{3} + \sqrt{2}$ is an irrational number. You may take as given (no proof required) that $\sqrt{3}$ and $\sqrt{2}$ are irrational numbers.

(a) Define what it means for a real number to be rational, and for a real number to be irrational.

(b) Prove that the sum and that the product of two rational numbers is rational.

(c) Verify that if $\sqrt{3} + \sqrt{2}$ is rational so is $\sqrt{3} - \sqrt{2}$.

Hint: What is their product?

(d) Verify that if $\sqrt{3} + \sqrt{2}$ is rational, so is $\sqrt{3}$.

(e) Given that $\sqrt{3}$ is not rational, what can you conclude about $\sqrt{3} + \sqrt{2}$? Explain your argument.

(f) Generalize. If x and y are irrational, what condition on $x^2 - y^2$ ensures that $x + y$ be irrational?

- (4) Consider sequences of 1s and 0s which we shall refer to as binary words. A word is called palindromic if it reads the same forwards as backwards. For example the word 0110110 is palindromic while the word 001110 is not. If u and v are words then uv is defined to be the concatenation of those words (put the two words next to each other). For example if $u = 0110110$ and $v = 001110$, then $uv = 0110110001110$.

(a) Show that if u and v are two palindromic words then it is not generally true that uv (the concatenation of the two words) is palindromic.

(b) Explain why if u and v are two palindromic words, then uvu is also palindromic.

(c) Is there a palindromic word with fifteen 0s and twenty three 1s? Explain.

- (5) Let $f(n) = \sum_{r=1}^n \frac{1}{4r^2-1}$. Compute $f(n)$ for $n = 1, 2, 3, 4$. Conjecture a formula and prove it by induction.