MATH 1200: Review for Final

1. For the following relations on the complex numbers (A) show that they are an equivalence relation and (B) find a description of the equivalence classes.

   (a) $R_1 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : Re(z_1) = Re(z_2) \text{ and } Im(z_1) = Im(z_2)\}$

   (b) $R_2 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : Re(z_1) = Re(z_2)\}$

   (c) $R_3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : Im(z_1) = Im(z_2)\}$

   (d) $R_4 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : Re(z_1)Im(z_1) = Re(z_2)Im(z_2)\}$

   (e) $R_5 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1| = |z_2|\}$

   (f) $R_6 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |Re(z_1)| = |Re(z_2)| \text{ and } |Im(z_1)| = |Im(z_2)|\}$

   (g) $R_7 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : ax = y \text{ for some real number } a > 0\}$

   (h) $R_8 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : ax = y \text{ for some real number } a \neq 0\}$

2. Determine if the following relations on the real numbers are reflexive, symmetric and/or transitive and provide an explanation.

   (a) $R_1 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| = y\}$

   (b) $R_2 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : |x| \leq |y|\}$

   (c) $R_3 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\}$

   (d) $R_4 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y + 1 \text{ or } y = x + 1\}$

   (e) $R_5 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq 2y\}$

   (f) $R_6 = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \neq 2y\}$

3. Let n be a positive integer and a, b, c, d be integers. Provide a proof of the following statements.

   (a) $a \equiv a \ (mod\ n)$

   (b) if $a \equiv b \ (mod\ n)$, then $b \equiv a \ (mod\ n)$

   (c) if $a \equiv b \ (mod\ n)$ and $b \equiv c \ (mod\ n)$, then $a \equiv c \ (mod\ n)$

   (d) if $a \equiv b \ (mod\ n)$ then $a + c \equiv b + c \ (mod\ n)$

   (e) if $a \equiv b \ (mod\ n)$ then $a \cdot c \equiv b \cdot c \ (mod\ n)$

   (f) if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then $a + c \equiv b + d \ (mod\ n)$

   (g) if $a \equiv b \ (mod\ n)$ and $c \equiv d \ (mod\ n)$, then $a \cdot c \equiv b \cdot d \ (mod\ n)$

   (h) if $c \cdot a \equiv c \cdot b \ (mod\ n)$, then it is not necessarily the case that $a \equiv b \ (mod\ n)$

   (i) if $a \cdot b \equiv a \cdot c \ (mod\ n)$ and $gcd(a, n) = 1$, then $b \equiv c \ (mod\ n)$

4. Prove the following statements by induction.

   (a) Prove that the sum of the cubes of 12 consecutive positive integers is divisible by 36.

   (b) Show that if $a_n = 2a_{n-1} + (-1)^n$ and $a_0 = 2$ then show that $a_n = (5 \cdot 2^n + (-1)^n)/3$.

   (c) For all integer $n \geq 2$, $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right) \cdots \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$.

   (d) Prove that for all $n > 6$, $n^3 < 3^n < n!$.

   (e) Show that 8^n divides $(4n)!$ for all $n \geq 0$.

   (f) Let $a_1, a_2, \ldots, a_n$ be positive real numbers. Show that

$$\left(\sum_{i=1}^{n} a_i\right)\left(\sum_{i=1}^{n} \frac{1}{a_i}\right) \geq n^2$$

   (g) Show that for $n \geq 1$, $\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}^n = \begin{bmatrix} \frac{5^n+3}{4} & 3 \cdot \frac{5^n-1}{4} \\ \frac{5^n-1}{4} & \frac{3 \cdot 5^n+1}{4} \end{bmatrix}$.

   (h) Show that for $n > 0$, $1 + 2 + 4 + 5 + 7 + \cdots + (3n-1) + (3n+1) = 3n^2 + 3n + 1$.

5. (a) Find all values of $n > 0$ such that $(1-i)^n$ is imaginary.

   (b) Find all values of $n > 0$ such that $(1-i)^n$ is real.

   (c) Find all values of $z \in \mathbb{C}$ such that $z^2 = (1-i)$.

   (d) Find all values of $z \in \mathbb{C}$ such that $z^2 + 2z = -i$

   (e) Find all values of $z \in \mathbb{C}$ such that $\bar{z} = i(z-1)$

   (f) Find all values of $z \in \mathbb{C}$ such that $z^2\bar{z} = z$

   (g) Find all values of $z \in \mathbb{C}$ such that $|z + 3i| = 3|z|$

6. Let x, y be real numbers. For the following statements, either prove that they are true or provide a counterexample:

   (a) If $x + y$ is irrational, then at least one of x or y is irrational.

   (b) If $x + y$ is rational, then both x and y are rational.

   (c) If x is rational, then there exists a rational number y such that $x \cdot y = 1$.

   (d) Between any two rational numbers there is a rational number.

   (e) For all real numbers x, there is a y such that $x \cdot y$ is rational.

   (f) For all real numbers x, there is a y such that $x + y$ is an integer.

   (g) If x is a positive real number, then $x + 1/x \geq 2$.

   (h) For $n > 0$ and x and y are positive real numbers such that $xy > n^2$, then either $x > n$ or $y > n$