

Math 2221 - Practice for Final - Zabrocki - April 3, 2008

- (1) Find the eigenvalues and the eigenvectors of the following matrix.

$$\begin{bmatrix} -16 & 9 & 45 \\ 0 & 2 & 0 \\ -6 & 3 & 17 \end{bmatrix}$$

Answer: The eigenvalues of the matrix are -1 and 2 . $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is a vector with eigenvalue

-1 . $\begin{bmatrix} 5x + y \\ 2y \\ 2x \end{bmatrix}$ are all the vectors with eigenvalue 2 .

- (2) What is the rank of the matrix?

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}$$

Find a basis for $Row(A)$, $Col(A)$ and $Null(A)$.

Solution: This matrix row reduces to

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can tell from the row reduced form of the matrix that

basis for $Row(A) = \{[1 \ 0 \ 3 \ 0 \ 0], [0 \ 1 \ -3 \ 0 \ -7], [0 \ 0 \ 0 \ 1 \ -2]\}$

$$\text{basis for } Col(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\text{basis for } Null(A) = \left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

- (3) Find a basis \mathcal{B} for $Null(A)$ where

$$A = \begin{bmatrix} 1 & -5 & 2 & 8 \\ 2 & -4 & 1 & 7 \\ -1 & -3 & 2 & 4 \\ 4 & -6 & 1 & 11 \end{bmatrix}$$

Show that the vector $v = \begin{bmatrix} -2 \\ 10 \\ 2 \\ 6 \end{bmatrix}$ is in $Null(A)$ and give $[v]_{\mathcal{B}}$.

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Answer: multiply A times v and you will see that the result is 0. This shows that v is in $Null(A)$. A row reduces to the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we determine that a basis of this space is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

The coordinates of v with respect to \mathcal{B} is $[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(4) Give the LU decomposition of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 6 \\ 1 & 3 & 4 & 7 & 12 \\ 1 & 4 & 6 & 12 & 24 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(5) What are the inverses of the following matrices? What are the determinants of the following matrices?

(a)

$$\begin{bmatrix} 3 & 2 \\ 1 & -7 \end{bmatrix}$$

Answer: determinant = -23,

$$A^{-1} = \begin{bmatrix} \frac{7}{23} & \frac{2}{23} \\ \frac{1}{23} & -\frac{3}{23} \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer: determinant = 1

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & 2 \\ 3 & 0 & -5 \end{bmatrix}$$

determinant = -1,

$$A^{-1} = \begin{bmatrix} -5 & 0 & 2 \\ -11 & -1 & 4 \\ -3 & 0 & 1 \end{bmatrix}$$

(6) Find a matrix P such that PAP^{-1} is diagonal where $A = \begin{bmatrix} 13 & 0 & -4 \\ -1 & 2 & 0 \\ 30 & 0 & -9 \end{bmatrix}$

Solution:

$$\begin{bmatrix} -5 & 0 & 2 \\ -11 & -1 & 4 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 0 & -4 \\ -1 & 2 & 0 \\ 30 & 0 & -9 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & 2 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Therefore $P = \begin{bmatrix} -5 & 0 & 2 \\ -11 & -1 & 4 \\ -3 & 0 & 1 \end{bmatrix}$

(7) Find a formula for A^k where $A = \begin{bmatrix} -11 & -30 \\ 6 & 16 \end{bmatrix}$

Solution:

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 5 - 4 \cdot 4^k & 10 - 10 \cdot 4^k \\ 2 \cdot 4^k - 2 & 5 \cdot 4^k - 4 \end{bmatrix}$$

(8) Give a basis for the following subspaces

(a)

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = 0 \text{ and } b + c = 0 \right\}$$

(subspace of 2×2 matrices).

Answer:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

(b)

$$\left\{ \begin{bmatrix} x + y + z \\ 3x - y - z \\ -2y + z \\ 2x + z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

(subspace of \mathbb{R}^4)

Answer:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

- (9) Which of the following two sets of vectors are bases for \mathbb{R}^3 ? Explain why they are or are not bases. If the set of vectors is a basis for \mathbb{R}^3 , then find the change of basis matrix from the standard basis matrix to this set of vectors.

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix} \right\}$

Answer: not a basis. These vectors are not linearly independent.

(b) $\left\{ \begin{bmatrix} 5 \\ 11 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}$

Answer: Is a basis the change of basis matrix from the standard basis to this is the inverse of the matrix

$$\begin{bmatrix} 5 & 0 & 2 \\ 11 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}$$

and this is

$$\begin{bmatrix} -1 & 0 & 2 \\ -1 & 1 & -2 \\ 3 & 0 & -5 \end{bmatrix}$$

- (10) Let $A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$ which we can show row reduces to the matrix

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & 7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Is the linear transformation T where $T(v) = Av$ 1-1? Is T an onto the subspace \mathbb{R}^4 ? Give a basis for the subspace $\{T(v) : v \in \mathbb{R}^5\}$ of \mathbb{R}^4 .

Answer: T is not 1-1. There are more columns than rows in the row reduced matrix so there are an infinite number of solutions to $Av = \mathbf{0}$. T is not onto the subspace \mathbb{R}^4 because the number of non-zero rows in the row reduced matrix is 3 and this is less than the number of elements in a basis for \mathbb{R}^4 . Note that $\{T(v) : v \in \mathbb{R}^5\}$ is the same as $Col(A)$ by definition, a basis is therefore given by

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}$$

- (11) If A is an invertible square matrix, what is $(A^T A^{-1})^T (((A^T A^{-1})^T A)^T A)^{-1}$ in terms of products of A , A^T , A^{-1} and $(A^T)^{-1}$?

Answer:

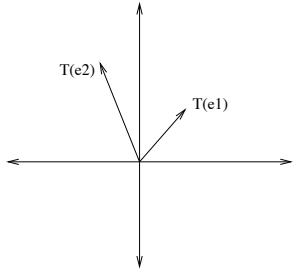
$$(A^T)^{-1} A (((A^T A^{-1})^T A)^T A)^{-1}$$

$$(A^T)^{-1} A (((A^T)^{-1} A A)^T A)^{-1}$$

$$(A^T)^{-1} A (A^T A^T A^{-1} A)^{-1}$$

$$(A^T)^{-1} A (A^T)^{-1} (A^T)^{-1}$$

- (12) Let T be a linear transformation on \mathbb{R}^2 such the image of the vectors $\mathbf{e}_1 = [1 \ 0]$ and $\mathbf{e}_2 = [0 \ 1]$ are shown in the picture. Draw approximately on the picture what the image of $[2 \ -1]$ under the action of T .



Answer:

