Math 2221 - Practice for Final - Zabrocki - April 3, 2008

(1) Find the eigenvalues and the eigenvectors of the following matrix.

$$\begin{bmatrix} -16 & 9 & 45 \\ 0 & 2 & 0 \\ -6 & 3 & 17 \end{bmatrix}$$

Answer: The eigenvalues of the matrix are -1 and 2. $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ is a vector with eigenvalue -1. $\begin{bmatrix} 5x+y \\ 2y \\ 2x \end{bmatrix}$ are all the vectors with eigenvalue 2.

(2) What is the rank of the matrix?

$$A = \begin{bmatrix} 1 & -2 & 9 & 5 & 4 \\ 1 & -1 & 6 & 5 & -3 \\ -2 & 0 & -6 & 1 & -2 \\ 4 & 1 & 9 & 1 & -9 \end{bmatrix}$$

Find a basis for $Row(A), Col(A)$ and $Null(A)$.
Solution: This matrix row reduces to

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 0 \\ 0 & 1 & -3 & 0 & -7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We can tell from the row reduced form of the matrix that

$$\text{basis for } Row(A) = \left\{ \begin{bmatrix} 1 & 0 & 3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & -3 & 0 & -7 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 1 & -2 \end{bmatrix} \right\}$$

$$\text{basis for } Col(A) = \left\{ \begin{bmatrix} 1 \\ 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ 5 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$\text{basis for } Null(A) = \left\{ \begin{bmatrix} -3 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 7 \\ 0 \\ 2 \\ 1 \end{bmatrix} \right\}$$

(3) Find a basis $\mathcal{B}$ for $Nul(A)$ where

$$A = \begin{bmatrix} 1 & -5 & 2 & 8 \\ 2 & -4 & 1 & 7 \\ -1 & -3 & 2 & 4 \\ 4 & -6 & 1 & 11 \end{bmatrix}$$

Show that the vector $v = \begin{bmatrix} -2 \\ 10 \\ 2 \\ 6 \end{bmatrix}$ is in $Null(A)$ and give $[v]_{\mathcal{B}}$.

Answer: multiply A times v and you will see that the result is 0. This shows that v is in $Null(A)$. A row reduces to the matrix

$$\begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 2 & -1 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

From this we determine that a basis of this space is

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

The coordinates of v with respect to $\mathcal{B}$ is $[v]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

(4) Give the LU decomposition of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 6 \\ 1 & 3 & 4 & 7 & 12 \\ 1 & 4 & 6 & 12 & 24 \end{bmatrix}$$

Answer:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 0 \\ 1 & 3 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 & 3 \\ 0 & 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(5) What are the inverses of the following matricies? What are the determinants of the following matrices?

(a)

$$\begin{bmatrix} 3 & 2 \\ 1 & -7 \end{bmatrix}$$

Answer: determinant = -23,

$$A^{-1} = \begin{bmatrix} \frac{7}{23} & \frac{2}{23} \\ \frac{1}{23} & -\frac{3}{23} \end{bmatrix}$$

(b)

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Answer: determinant = 1

$$A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

(c)

$$\begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & 2 \\ 3 & 0 & -5 \end{bmatrix}$$

determinant = -1,

$$A^{-1} = \begin{bmatrix} -5 & 0 & 2 \\ -11 & -1 & 4 \\ -3 & 0 & 1 \end{bmatrix}$$

(6) Find a matrix P such that PAP^{-1} is diagonal where $A = \begin{bmatrix} 13 & 0 & -4 \\ -1 & 2 & 0 \\ 30 & 0 & -9 \end{bmatrix}$

Solution:

$$\begin{bmatrix} -5 & 0 & 2 \\ -11 & -1 & 4 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 13 & 0 & -4 \\ -1 & 2 & 0 \\ 30 & 0 & -9 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 1 & -1 & 2 \\ 3 & 0 & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Therefore $P = \begin{bmatrix} -5 & 0 & 2 \\ -11 & -1 & 4 \\ -3 & 0 & 1 \end{bmatrix}$

(7) Find a formula for A^k where $A = \begin{bmatrix} -11 & -30 \\ 6 & 16 \end{bmatrix}$

Solution:

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$$

$$P^{-1} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$$

$$A^k = \begin{bmatrix} 5 - 4 \cdot 4^k & 10 - 10 \cdot 4^k \\ 2 \cdot 4^k - 2 & 5 \cdot 4^k - 4 \end{bmatrix}$$

(8) Give a basis for the following subspaces

(a)

$$\left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a + d = 0 \text{ and } b + c = 0 \right\}$$

(subspace of 2×2 matrices).

Answer:

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}$$

(b)

$$\left\{ \begin{bmatrix} x + y + z \\ 3x - y - z \\ -2y + z \\ 2x + z \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$$

(subspace of $\mathbb{R}^4$)

Answer:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(9) Which of the following two sets of vectors are bases for $\mathbb{R}^3$? Explain why they are or are not bases. If the set of vectors is a basis for $\mathbb{R}^3$, then find the change of basis matrix from the standard basis matrix to this set of vectors.

(a) $\left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 8 \end{bmatrix} \right\}$

Answer: not a basis. These vectors are not linearly independent.

(b) $\left\{ \begin{bmatrix} 5 \\ 11 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} \right\}$

Answer: Is a basis the change of basis matrix from the standard basis to this is the inverse of the matrix

$$\begin{bmatrix} 5 & 0 & 2 \\ 11 & 1 & 4 \\ 3 & 0 & 1 \end{bmatrix}$$

and this is

$$\begin{bmatrix} -1 & 0 & 2 \\ -1 & 1 & -2 \\ 3 & 0 & -5 \end{bmatrix}$$

(10) Let $A = \begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 4 & 7 & -4 & -3 & 9 \\ 6 & 9 & -5 & 2 & 4 \\ 0 & -9 & 6 & 5 & -6 \end{bmatrix}$ which we can show row reduces to the matrix

$$\begin{bmatrix} 2 & 5 & -3 & -4 & 8 \\ 0 & -3 & 2 & 5 & 7 \\ 0 & 0 & 0 & 4 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Is the linear transformation T where $T(v) = Av$ $1-1$? Is T an onto the subspace $\mathbb{R}^4$? Give a basis for the subspace $\{T(v) : v \in \mathbb{R}^5\}$ of $\mathbb{R}^4$.

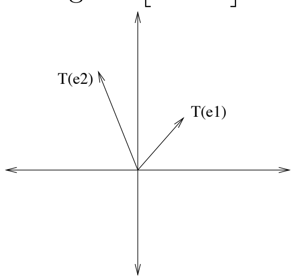
Answer: T is not $1-1$. There are more columns than rows in the row reduced matrix so there are an infinite number of solutions to $Av = \mathbf{0}$. T is not onto the subspace $\mathbb{R}^4$ because the number of non-zero rows in the row reduced matrix is 3 and this is less than the number of elements in a basis for $\mathbb{R}^4$. Note that $\{T(v) : v \in \mathbb{R}^5\}$ is the same as $Col(A)$ by definition, a basis is therefore given by

$$\left\{ \begin{bmatrix} 2 \\ 4 \\ 6 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 7 \\ 9 \\ -9 \end{bmatrix}, \begin{bmatrix} -4 \\ -3 \\ 2 \\ 5 \end{bmatrix} \right\}$$

(11) If A is an invertible square matrix, what is $(A^TA^{-1})^T(((A^TA^{-1})^TA)^TA)^{-1}$ in terms of products of A, A^T, A^{-1} and $(A^T)^{-1}$?
Answer:

$$(A^T)^{-1}A(((A^TA^{-1})^TA)^TA)^{-1}$$
$$(A^T)^{-1}A(((A^T)^{-1}AA)^TA)^{-1}$$
$$(A^T)^{-1}A(A^TA^TA^{-1}A)^{-1}$$
$$(A^T)^{-1}A(A^T)^{-1}(A^T)^{-1}$$

(12) Let T be a linear transformation on $\mathbb{R}^2$ such the image of the vectors $\mathbf{e}_1 = \begin{bmatrix}1 & 0\end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix}0 & 1\end{bmatrix}$ are shown in the picture. Draw approximately on the picture what the image of $\begin{bmatrix}2 & -1\end{bmatrix}$ under the action of T.



Answer: