SECOND PODCAST - MODULAR ARITHMETIC AND CARD SHUFFLING

MIKE ZABROCKI

Hi, My name is Mike Zabrocki and welcome to my second podcast for Math 2590.

Tonight I am going to tell you what perfect shufing a deck of cards has to do with mathematics and modular arithmetic.

This podcast is a bit more technical than the last and to follow it properly it is recommended that you have: a deck of cards and a pen and paper.

Last week I tried this experimentally and I found that with a deck of 4 cards and I shufe the deck 4 times, then I get back to the original order.

I also tried with a deck of 6 cards and I shufe the deck 3 times, then I get back to the original order.

If I start with a deck of 8 cards and shufe it perfectly 6 times then I get back to the original order.

4, 3, 6. Already these seem to be puzzling numbers. What sort of pattern can this be?

I asked you to compute as an exercise the number of perfect shufes needed to bring a deck with 10, 12 and 14 cards back to original order.

If you did it correctly then you should have found that with 10 cards it took 10 shufes to get back to the original order, with 12 cards it took 12 shufes to get back to the original order, and with 14 cards it took 4 shufes to get back to the original order.

Now here is a puzzle: 4, 3, 6, 10, 12, 4, what kind of pattern is that?

We will have to work a long time to do this with a full deck of 52 cards.

Care to gure out how many perfect shufes get us back to the original order?

Try to do a perfect shufe with a full deck of 52 cards.

Go ahead. Take a deck of cards and perfect shufe it once.

Don't worry, I'll wait while you try to do that. (insert long pause here)

Are you done yet? No? OK, I'll give you a little more time. (insert long pause here)

OK, I'm tired of waiting. I hope you are done.

Alright, Now that you shufed the deck once, I need you to keep doing those perfect shufes.

Make sure that you don't make a mistake. Keep doing it until you see the cards back in their original order.

How long do you think is going to take? Should I go get a coffee while you are working?

The truth is, we don't have to do those perfect shufes with a full deck. I claim we can calculate this.

Think about what happens when we shufe a deck of cards.

To picture it better, lets number the positions of the cards 1 through 52.

For the rst half of the deck, the card in position 1 goes to position 2, the card in position 2 goes into position 4, the card in position 3 goes into position 6, and so on.

Card 26 is the last card in the rst half of the deck will go into position 52.

For these cards the position is always multiplied by 2.

What about the second half of the deck? The card that was position 27 goes into position 1, the card that was in position 28 goes into position 3, the card that was in position 29 goes into position 5, and so on.

The last card that was in position 52 now goes into position 51.

It is only slightly harder to gure out an equation, but it is not hard to verify that for the cards in the second half of the deck we can calculate the new position of the card by mulitiply by 2 and subtract 53.

Check to see if what I am saying makes sense. Card 27 will go into position 1, and 27 times 2 is 54 and 54 minus 53 is 1.

Lets try that again. Card 28 will go into position 3. We also have that 28 times 2 is 56 and 56 minus 53 is 3.

The next card is in position 29 and after shufing it goes to position 5.

Then also check that 29 times 2 is 58 and 58 minus 53 is also 5.

Here is where we can use a bit of mathematics to solve this problem.

The formula says that the card in position x goes to 2 times x if it is in the rst half of the deck and it goes to 2 times x minus 53 if it is in the second half of the deck.

These two equations can be captured by a single equation if we have the concept of modular arithmetic.

Recall that in our lesson last week when we talked about modular arithmetic, if we work "(mod m)" then we are working with the numbers 1 through m and if we go beyond m, m+1 is the equivalent to 1, m+2 is equivalent to 2, m+3 is equivalent to 3 and so on.

Both of my equations, 2 times x and 2 times x minus 53 are the same formula if we work "(mod 53)" so what we do is we give one equation for the position of a shufed card "(mod 53)."

What I am saying is that the card that starts in position x, after a single shufe will be in position 2 times x (mod 53).

If a single shufe of the deck takes the card at position x into position 2x (mod 53), then where does the card at position x go after 2 shufes?

Our discussion has lead us to say that a shufe of the deck has the effect of multiply the position of the card by 2 (mod 53).

Therefore 2 shufes of the deck will be multiply be 2 one more time (mod 53) and the card in position x will be sent to position 4x (mod 53).

If I shufe the deck 3 times, this will multiply the position of the card yet again by 2 so the card at position x will be sent to position 8x (mod 53).

If we shufe the deck a bunch of times then for each time we shufe the deck, we multiply the position of the card by 2 (mod 53).

We have now an equation that describes what happens to the cards after a bunch of shufes, this is described as multiply by a power of 2 (mod 53).

So now we do a calculation in order to gure our how many shufes it takes to get back to the original order.

This takes experimenting with modular arithmetic a bit to gure out the answer, but this is best done by making a table of the powers of 2 mod 53.

Rather than ask you to compute a table of the powers of 2 mod 53, I'll just tell you that it takes 51 shufes to get back to the original order.

::::Actually the last part of the script is obscured by a "that number is out of service."

Reference: George Andrews, Number Theory, Dover (1994).