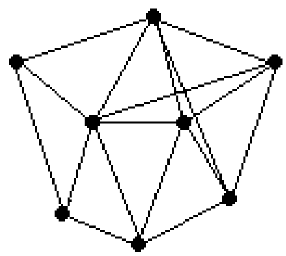
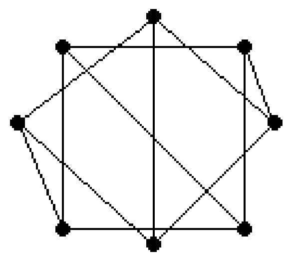
HOMEWORK #3 SOLUTIONS - MATH 3260

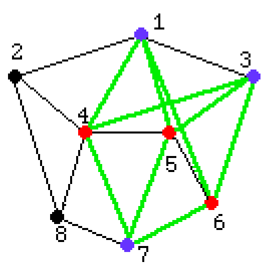
ASSIGNED: FEBRUARY 26, 2003 DUE: MARCH 12, 2003 AT 2:30PM

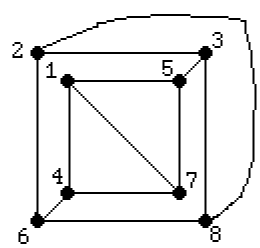
(1) Show either that each of the following graphs are planar by drawing them in a way that the vertices do not cross or give a subgraph which is homeomorphic to $K_{3,3}$ or K_5.

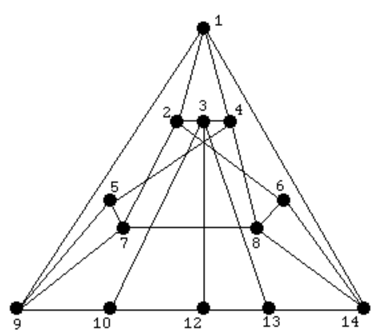


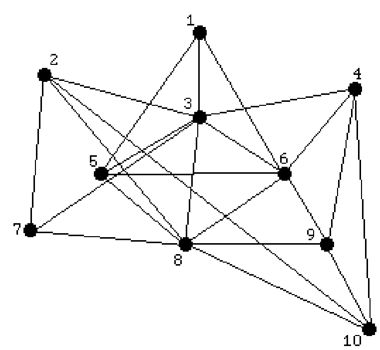


For the first graph we see below on the left a subgraph isomorphic to $K_{3,3}$. For the second graph it is planar and we draw the isomorphic graph in the plane below.

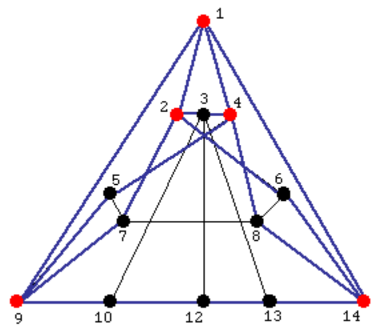


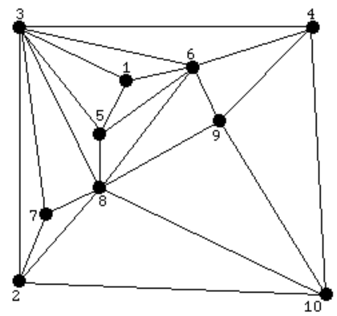




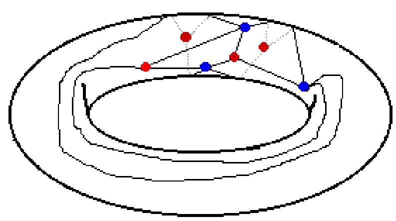


The graph on the left is not planar and we can show it by isolating the subgraph on the left. All of the black vertices are of degree 2 in the subgraph and so can be ignored when we observe that the blue edges and red vertices form a graph homeomorphic to K_5. The graph on the right is planar and we show it by giving a graph which is isomorphic and has no crossings in the plane.





(2) (a) Show that $K_{3,4}$ can be drawn in a torus without crossings by drawing this image. It will take some thought to determine how to arrange the vertices in a nice way.



(b) Corollary 14.3 on page 71 in the text gives a bound on the surface that a graph G can be embedded in. What does this say about the genus of the graphs $K_{r,s}$ for $r + s \geq 4$? What does this say in particular about $K_{3,3}$ and $K_{3,4}$?

Solution: The number of edges in $K_{r,s}$ is rs and the number of vertices is $r+s$, therefore this corollary says that

$$g(K_{r,s}) \geq \lceil (rs - 3r - 3s)/6 + 1 \rceil$$

and in particular for $K_{3,3}$ and $K_{3,4}$

$$g(K_{3,3}) \geq \lceil (9 - 9 - 9)/6 + 1 \rceil = 0$$

and

$$g(K_{3,4}) \geq \lceil (12 - 9 - 12)/6 + 1 \rceil = 0.$$

This doesn't tell us much since we already know that $g(K_{3,3}) = g(K_{3,4}) = 1$.

(c) Modify the proof of Corollary 14.3 to obtain a better lower bound for g if the graph does not contain any triangles. Show that this implies

$$g(K_{r,s}) \geq \left\lceil \frac{(r-2)(s-2)}{4} \right\rceil$$

In particular, explain in words what this inequality tells you about the graphs $K_{3,3}$ and $K_{3,4}$.

Solution: If the graph does not contain any triangles, then every face has at least 4 edges and since every edge adjoins exactly two faces we have that $4f \leq 2m$ where f is the number of faces and m is the number of edges. Euler's theorem says that

$$2 - 2g = n - m + f \leq n - m + \frac{m}{2} = n - \frac{m}{2}$$

This implies that

$$2 - n + \frac{m}{2} \leq 2g$$

or more precisely,

$$g \geq 1 - \frac{n}{2} + \frac{m}{4}.$$

For $K_{r,s}$, $n = r + s$ and $m = rs$, therefore we have that

$$g(K_{r,s}) \geq 1 - \frac{r+s}{2} + \frac{rs}{4} = \frac{1}{4}\left(4 - 2(r+s) + rs\right) = \frac{(2-r)(2-s)}{4}.$$

Since the genus is an integer we can say more precisely that

$$g(K_{r,s}) \geq \left\lceil \frac{(2-r)(2-s)}{4} \right\rceil$$

In particular this says that

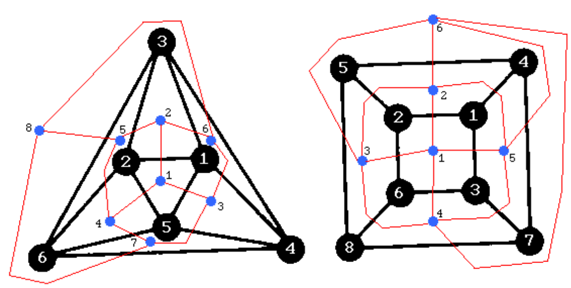
$$g(K_{3,3}) \geq \left\lceil \frac{(2-3)(2-3)}{4} \right\rceil = 1$$

and

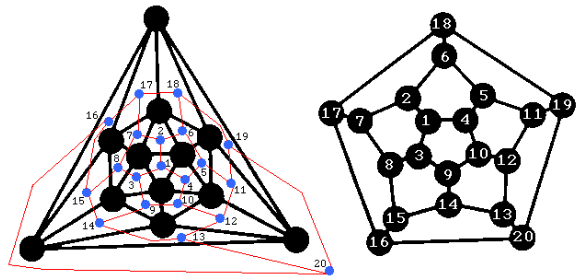
$$g(K_{3,4}) \geq \left\lceil \frac{(2-3)(2-4)}{4} \right\rceil = 1.$$

This says that $K_{3,3}$ and $K_{3,4}$ are not planar. It does not say that they can be embedded into a torus, it says that they MIGHT be able to be embedded into a torus (but we already showed that they can in part (a)).

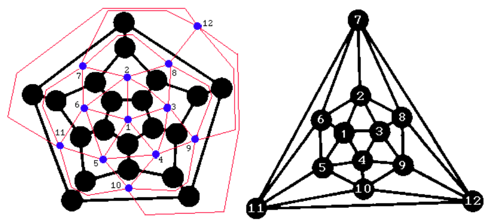
(3) Show that the dual of the cube graph is the octahedron graph and that the dual of the dodecahedron graph is the icosahedron graph. Give the number of vertices, edges and faces for each of these four graphs.



The cube has 8 vertices, 12 edges and 6 faces. The octahedron has 6 vertices, 12 edges and 8 faces.

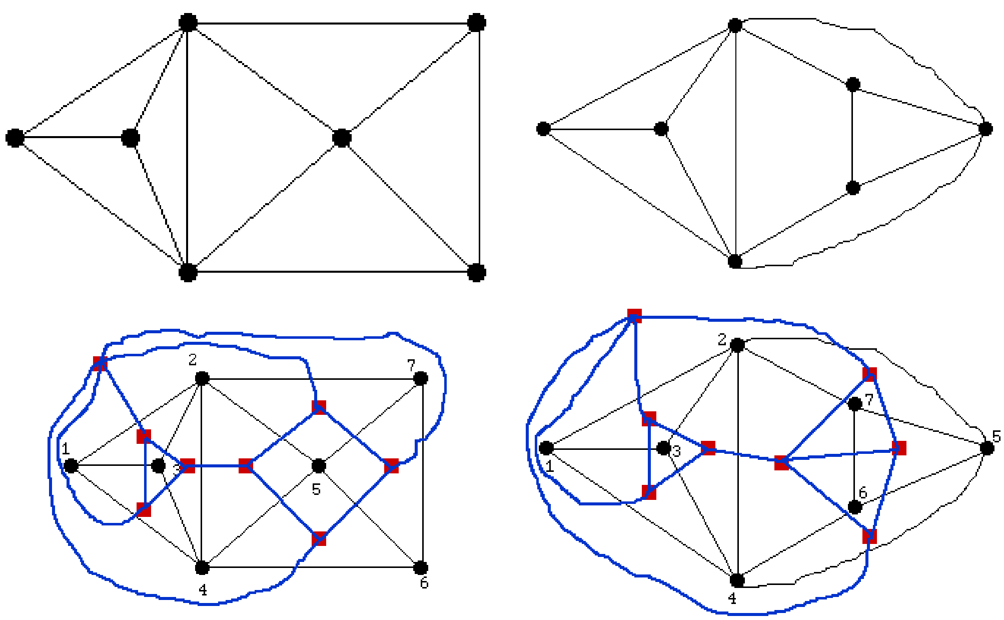


The icosahedron has 12 vertices, 30 edges, and 20 faces.



The dodecahedron has 20 vertices, 30 edges, and 12 faces.

(4) Show that the following two graphs are isomorphic but that their geometric duals are not isomorphic.



Note the reason that the duals of these graphs cannot be isomorphic is that the one on the right has two vertices of degree 4 while the one on the left has all vertices of degree 3 except for 1 which is of degree 5.

(5) Let G be a simple planar graph. Demonstrate what conditions on G are necessary to guarantee that the dual of G has the following properties.

(a) G^* has no edges to and from the same vertex (no loops).

Answer: G should have no edges which for which border only one face.

(b) G^{**} is isomorphic to G.

Solution: G is connected (see theorem 15.2).

(c) G^* is bipartite.

Answer: G is Eulerian.

(d) G^* is Eulerian.

Answer: G is bipartite and connected. The solution to the last two problems is in the book it is problem number 15.9.

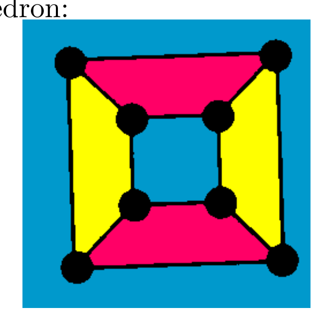
Hint: These last two follow from Theorem 5.1 and Corollary 6.3 in the book.

(6) Find the chromatic number of

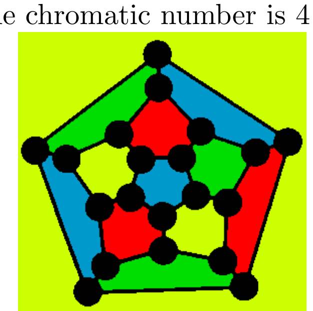
(a) each of the Platonic graphs.

General remark: Recall that a bipartite graph has the property that every cycle even length and a graph is two colorable if and only if the graph is bipartite.

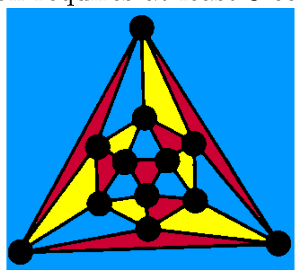
Solution: All of the platonic solids are planar so the chromatic number is less than or equal to 4 for each of them. The tetrahedron has chromatic number 4 since it is isomorphic to K_4. The octahedron is not bipartite so the chromatic number is > 2 and it is equal to 3 since the faces of the cube may be colored as in the diagram below and the cube is dual to the octahedron:



The cube is bipartite and hence has chromatic number 2 (see justification that the k-cube is bipartite below). The icosahedron is 4-colorable but not 3-colorable so the chromatic number is 4. This is easier to see by looking at the face coloring of the dodecahedron (since the dodecahedron is dual to the icosahedron, the face coloring of the dodecahedron is equivalent to the vertex coloring of the icosahedron). The center face of the dodecahedron is one color and the surrounding faces cannot be colored in less than 3 colors, therefore the chromatic number is 4 for the icosahedron.



The chromatic number of the dodecahedron is 3. Observe in the image below that we can face color the icosahedron with 3 colors, we can vertex color the dodecahedron with 3 colors. The dodecahedron requires at least 3 colors since it is not bipartite.



In summary, the *tetrahedron* has chromatic number 4, *cube* has chromatic number 2, *octahedron* has chromatic number 3, *icosahedron* has chromatic number 4, *dodecahedron* has chromatic number 3.

(b) the complete graph K_n

Solution: The chromatic number is n. The complete graph must be colored with n different colors since every vertex is adjacent to every other vertex.

(c) the complete bipartite graph $K_{r,s}$, $r, s \geq 1$.

Solution: The chromatic number is 2. A bipartite graph is always 2 colorable, since the set of r vertices can be colored black while the set of s vertices can be colored white and none of the black vertices are adjacent and none of the white vertices are adjacent. It requires at least two colors because all of the vertices in one of the partition sets are adjacent to all of the others.

(d) the complete tripartite graph $K_{r,s,t}$, $r, s, t \geq 1$.

Solution: The chromatic number is 3. A complete tripartite graph requires at least three colors since this graph consists of a bunch of triangles with each vertex of the triangle in one of the three different sets. It can be done with exactly 3 since the vertices in the r-vertex set can be one color, in the s-vertex set a second color and the t-vertex set a third color.

(e) the wheel graph W_n.

Solution: The chromatic number is 3 if n is odd and 4 if n is even. Center will be one color. The outside of the wheel is a cycle of length $n - 1$ which can be colored with 2 colors if n is odd and it will take 3 colors if n is even (none of these colors can be the same as the center vertex).

(f) the k-cube Q_k.

Solution: The chromatic number is 2 since Q_k is bipartite. This is not difficult to see since the vertices are labeled with sequences of k 0s and 1s. One set will be the sequences with an even number of 1s and the other set will be the sequences with an odd number of 1s. There will never be an edge between two sequences with an even number of 1s (or an odd number of 1s) because edges (u, v) in the graph Q_k exist only if the number of 1 in u is one more or one less than in v.