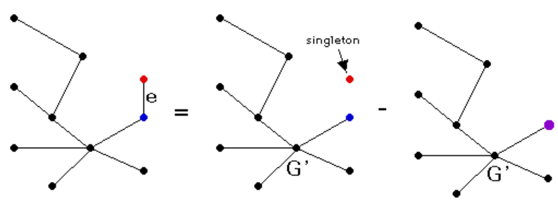
HOMEWORK #4 SOLUTIONS - MATH 3260

ASSIGNED: MARCH 19, 2003 DUE: APRIL 4, 2003 AT 2:30PM

(1) (a) Prove that the chromatic polynomial of any tree with s vertices is

$$k(k-1)^{s-1}$$

Solution: Pick any vertex of degree 1 of the tree and let e be the edge it is connected to. Let $G' = G/e$ and it will be a tree with $s-1$ vertices. $G-e$ is a tree with $s-1$ vertices and $s-2$ edges and and one disconnected vertex. $G-e$ is equal to G' union with a single disconnected vertex so it will have a chromatic polynomial $P_{G-e}(k) = kP_{G'}(k)$.



Therefore

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k) = kP_{G'}(k) - P_{G'}(k) = (k-1)P_{G'}(k)$$

Now since we can assume by induction that all trees with $s-1$ vertices have a chromatic polynomial $k(k-1)^{s-2}$ (with the base case of a single vertex has a chromatic polynomial of k) then

$$P_G(k) = (k-1)P_{G'}(k) = (k-1)k(k-1)^{s-2} = k(k-1)^{s-1}$$

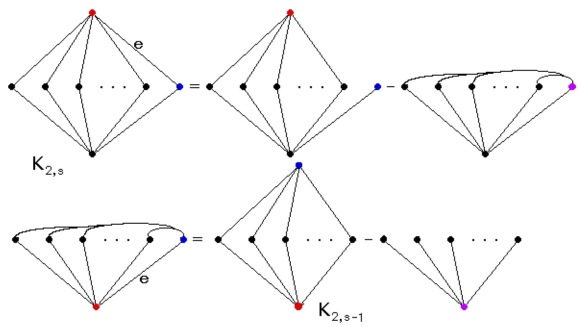
(b) Prove that the chromatic polynomial of $K_{2,s}$ is

$$k(k-1)^s + k(k-1)(k-2)^s$$

Solution: Observe from the diagram below that if we choose any edge from $K_{2,s}$ and apply our relation, then we find that

$$P_{K_{2,s}}(k) = (k-1)P_{K_{2,s-1}}(k) - (P_{K_{2,s-1}}(k) - P_{T_s}(k))$$

where T_s is a tree with s vertices in total. By the previous problem we know that T_s by the previous problem has chromatic polynomial equal to $k(k-1)^{s-1}$.



We will show the formula by induction on s. We know that $P_{K_{2,1}}(k) = k(k-1)^2 = k(k-1)+k(k-1)(k-2)$ satisfies the equation above. Assume that we know $P_{K_{2,s-1}}(k) = k(k-1)^{s-1} + k(k-1)(k-2)^{s-1}$. This means that

$$\begin{aligned} P_{K_{2,s}}(k) &= (k-2)P_{K_{2,s-1}}(k) + k(k-1)^{s-1} \\ &= (k-2)(k(k-1)^{s-1} + k(k-1)(k-2)^{s-1}) + k(k-1)^{s-1} \\ &= k(k-1)^{s-1}(k-2) + k(k-1)(k-2)^s + k(k-1)^{s-1} \\ &= k(k-1)^s + k(k-1)(k-2)^s. \end{aligned}$$

Therefore by induction we know the formula must hold for all $s \geq 1$.

(c) Prove that the chromatic polynomial of C_n is

$$(k-1)^n + (-1)^n(k-1)$$

Solution: From the diagram below we have the chromatic polynomial for C_n is the chromatic polynomial for P_n minus with the chromatic polynomial for C_{n-1}.

$$P_{C_n}(k) = P_{P_n}(k) - P_{C_{n-1}}(k).$$

We know that $P_{P_n}(k) = k(k-1)^n$.
We are going to show by inductioin on n that the chromatic polynomial is given by the equation above. For C_2, the chromatic polynomial is $k(k-1) = (k-1)^2 + (-1)^2(k-1)$. Assume that the chromatic polynomial for C_{n-1} is given by $(k-1)^{n-1} + (-1)^{n-1}(k-1)$. It folows that

$$\begin{aligned} P_{C_n}(k) &= P_{P_n}(k) - P_{C_{n-1}}(k) \\ &= k(k-1)^n - ((k-1)^{n-1} + (-1)^{n-1}(k-1)) \\ &= (k-1)^{n-1} + (-1)^n(k-1) \end{aligned}$$

Therefore by induction we know that the formula holds for all n.

(2) Let G be a simple graph with n vertices and m edges. Use induction on m, together with Theorem 21.1, to prove that
 (a) the coefficient of k^{n-1} is $-m$
 (b) the coefficients of $P_G(k)$ alternate in sign.

We know that $P_G(k)$ is a polynomial in k of degree equal to the number of vertices of G and the coefficient of k^n in $P_G(k)$ equals 1 (see p. 97).

(a) Solution: $G - e$ is a graph with n vertices and $m-1$ edges and G/e is a graph with $n-1$ vertices and $m-1$ edges. We know that $P_G(k) = P_{G-e}(k) - P_{G/e}(k)$. Assume by induction that any graph with $m-1$ edges has the property that the coefficient of k^{n-1} is equal to $m-1$. Therefore we have that the coefficient of k^{n-1} in $P_{G-e}(k)$ is $-(m-1)$ and the coefficient of k^{n-1} in $P_{G/e}(k)$ is equal to 1 because it is the leading term of the polynomial. Therefore the coefficient of k^{n-1} in $P_G(k)$ is $-(m-1) - 1 = -m$. Since we know that a graph with 0 edges and n vertices has chromatic polynomial equal to k^n (hence the coefficient of k^{n-1} is equal to 0) then by induction we know that it is true for all graphs that the coefficient of k^{n-1} will be negative the number of edges when n is the number of vertices.

(b) Solution: Assume that for every simple graph with fewer than n vertices and m edges has a chromatic polynomial with coefficients which alternates in sign. Therefore $P_{G-e}(k) = k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} + \cdots + (-1)^n a_0$ and $P_{G/e}(k) = k^{n-1} - b_{n-2}k^{n-2} + b_{n-3}k^{n-3} + \cdots + (-1)^{n-1} b_0$. Therefore

$$
\begin{aligned}
P_G(k) &= P_{G-e}(k) - P_{G/e}(k) \\
&= k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} + \cdots + (-1)^n a_0 \\
&\quad - \left(k^{n-1} - b_{n-2}k^{n-2} + b_{n-3}k^{n-3} + \cdots + (-1)^{n-1} b_0\right) \\
&= k^n - (a_{n-1}+1)k^{n-1} + (a_{n-2}+b_{n-2})k^{n-1} - (a_{n-3}+b_{n-3})k^{n-3} + \cdots + (-1)^n (a_0 + b_0)
\end{aligned}
$$

and so alternates in sign. Since a graph with n vertices and no edges is equal to k^n (and hence alternates in sign) then it follows by induction that the chromatic polynomial of every simple graph alternates in sign.

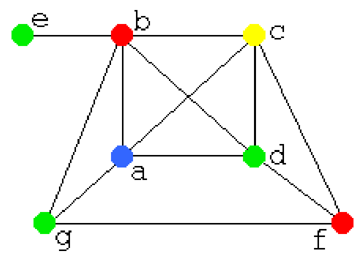
(3) Prove that, if $G = G(V_1, V_2)$ is a bipartite graph in which the degree of every vertex in V_1 is not less than the degree of each vertex in V_2 then G has a complete matching.

Solution: Let W be a subset of k vertices of V_1 and let U be the set of vertices of V_2 which are connected to W. Also set m equal to the maximum degree of a vertex in V_2 then every vertex of V_1 is at least of degree m. The number of edges adjacent to the vertices of W is at least km, since this is also equal to the number of edges adjacent to the vertices of U and there are at most $|U|m$ edges which are adjacent to the vertices of U (that is $km \le |U|m$) then $|U| \ge k$. Therefore by Hall's theorem we know that a complete matching exists.

(4) A schedule for finals is to be drawn up for a group of 7 classes, a through g. Two classes may not be scheduled at the same time if there exists a student in both classes. The table below shows the classes which may not be scheduled at the same time (marked with a •). What is the minimum number of time slots are needed to schedule all 7 classes?

	a	b	c	d	e	f	g
a		•	•	•			•
b	•		•	•	•		•
c	•	•		•		•	
d	•	•	•			•	
e		•					
f			•	•			•
g	•	•				•	

Solution: Draw the graph corresponding to this table. The number of time slots necessary is the chromatic number of this graph because if the graph can be colored with k colors then these classes can be scheduled in k time slots, one corresponding to each color. The graph listed in the table contains a subgraph K_4 consisting of the vertices $\{a, b, c, d\}$, therefore it requires at least 4 colors to color this graph. It can be done in exactly 4 colors since the graph can be colored as follows:



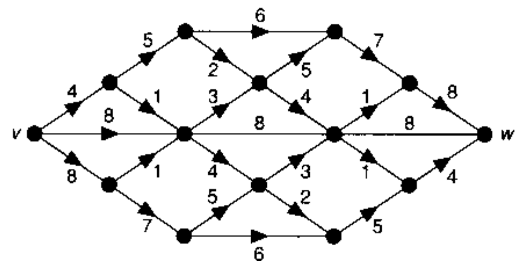
(5) Verify the statements of Corollary 26.2 when $E = \{a, b, c, d, e\}$ and $\mathcal{F} = (\ \{a, c, e\}, \{b, d\}, \{b, d\}, \{b, d\})$. This means that you must find a t such that a partial transversal of size t exists and verify for each $1 \leq k \leq 4$ that every subcollection of subsets contains at least $k + t - m$ elements.

Solution: For $k = 1, 2, 4$ each subcollection of k sets has at least k elements. The only time that the union of a sub-collection of k of these sets has less than k elements is when we take the last three sets. Their union only has 2 elements in it. This means that $3 + t - 4 = 2$, or that $t = 3$. Now a partial transversal of size 3 exists. Just take the first three sets ($\{a, c, e\}$, $\{b, d\}$, $\{b, d\}$) and the set $\{a, b, d\}$ is a partial transversal.

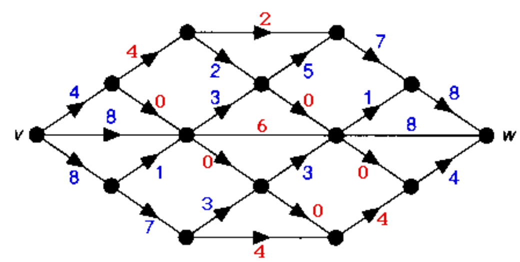
(6) Let E be the set $\{1, 2, \ldots, 6\}$. How many transversals do the following families have? Justify your answer.
  (a) $(\{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 5\}, \{4, 5, 6\}, \{5, 6, 1\})$
  (b) $(\{1\}, \{2, 3\}, \{1, 2\}, \{1, 3\}, \{1, 4, 6\})$
  (c) $(\{1, 2\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\})$
  (d) $(\{1, 3, 5\}, \{2, 4\})$
  (e) $(\{1, 3, 5\}, \{2, 3, 4\})$

Solution:
  (a) Each of the subsets of size 5 of the six element set $\{1, 2, 3, 4, 5, 6\}$ work as a transversal. Therefore there are 6 transversals in total.
  (b) There are no transversals because if you take the union of the first four subsets there are only 3 elements.
  (c) Each of the subsets of size 5 of the six element set $\{1, 2, 3, 4, 5, 6\}$ work as a transversal. Therefore there are 6 transversals in total.
  (d) Any element from $\{1, 3, 5\}$ and one element $\{2, 4\}$ is a transversal so there are 6 transversals in total.
  (e) If the transversal contains 3 then any of $\{1, 5, 2, 4\}$ can be the other element in the transversal. If the transversal does not contain 3, then there is one element from $\{1, 5\}$ and one element from $\{2, 4\}$ in the transversal and there are 4 transversal of this type. Therefore there are 8 transversal in total.

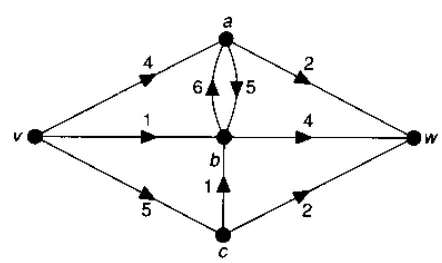
(7) Find a flow with value 20 in the network in the following figure. Is it a maximum flow?



Solution: The flow is shown below. The numbers in blue are the saturated edges, the numbers in red are the unsaturated edges. Other answers are possible. This is a maximum flow.

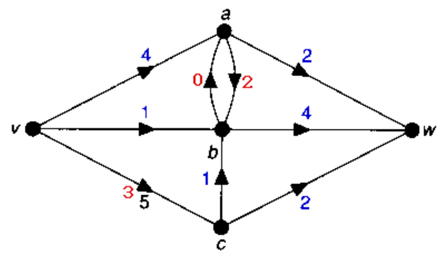


(8) List all of the cuts in the following network and find a minimum cut. Find a maximum flow and verify that this satisfies the max-flow min-cut theorem.



cut set	value
aw, bw, cw	8
va, vb, vc	10
va, vb, cb, cw	8
vc, bw, aw	9
aw, ab, vb, vc	13
there are others	

Below is a maximum flow with value 8 and this is equal to the value of all of the minimum cuts.


a
4
2
0
2
v
1
4
w
b
1
3
5
2
c