HOMEWORK #1 - MATH 4160

ASSIGNED: SEPT 13, 2012 DUE: SEPT 27, 2012

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers.

Stirling numbers and

(1) In class, we defined for $k > 0$, $(x)_k := x(x-1)(x-2)\cdots(x-k+1)$ (falling factorial). We may also define $(x)^{(k)} := x(x+1)(x+2)\cdots(x+k-1)$ (called a rising factorial). Both $(x)_k$ and $(x)^{(k)}$ are products of k terms. Prove the following by two means, by induction and by telescoping sums: For $k \geq 1$ and $n \geq 0$,

$$\sum_{i=1}^{n} (i)^{(k)} = (1)^{(k)} + (2)^{(k)} + \cdots (n)^{(k)} = \frac{(n)^{(k+1)}}{k+1} .$$

(2) For $n \geq 0$ and $1 \leq k \leq n$, let $s'(n,k) =$ the number of permutations of $[n]$ that have k disjoint cycles (these are called the signless Stirling numbers of the first kind). Explain why in 2-3 sentences that

$$s'(n,k) = (n-1)s'(n-1,k) + s'(n-1,k-1) .$$

Use this recursion to compute the values of $s'(n,k)$ for $1 \leq n \leq 5$ and $1 \leq k \leq n$. Show explicitly that $s'(4,2) = 11$ by listing the 11 permutations of $\{1,2,3,4\}$ into two cycles.

(3) Show that

$$(x)^{(n)} = \sum_{k=1}^{n} s'(n,k)x^k .$$

Enumeration problems. I am giving you the answer so the emphasis is on you coming up with a clear explanation of why the answer is true. I don't care what the answer is, I care why the answer is....

(1) How many 3 of a kind (no better poker pattern appears) hands are possible if 7 cards are dealt from a 52 card deck? A is high or low. Answer: 6,461,620

(2) Say that you have three urns containing balls. Urn A contains two red balls and three black balls. Urn B contains one red ball and four black balls. Urn C contains four red balls and one black ball. A ball is randomly chosen from each of the three urns. Find the probability that all three balls are the same color. Answer 0.16

(3) How many straight poker hands are there (not straight flush) from a deck of 51 cards with the queen of spades missing? A is high or low. Answer: 9,435

Combinatorial proofs: For the following problems give a combinatorial proof by describing a set that is counted by the left hand side of the equality and a set that is counted by the right hand side of the equality and explaining why these two sets are the same.

(1)
$$n^3 = (n)_1 + 3(n)_2 + (n)_3$$
(note the right hand side might be easier to explain as $(n)_1 + (n)_2 + (n)_2 + (n)_2 + (n)_3$).

(2)
$$\binom{2n}{2} = 2\binom{n}{2} + n^2$$

(3)
$$\binom{2n}{n}^2 = \sum_{k=0}^{n} \binom{2n}{k}\binom{2n-k}{k}\binom{2n-2k}{n-k}$$