HOMEWORK #2 - MATH 4160

ASSIGNED: SEPT 27, 2012 DUE: OCT 16, 2012

Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers.

(1) Give a proof of the following generating function formulas.

(a) Let F_n be the Fibonacci sequence $F_0 = 1$, $F_1 = 1$ and $F_{n+1} = F_n + F_{n-1}$ for $n \geq 1$ (that we will do in class and has a generating function of $\sum_{n\geq 0} F_n x^n = \frac{1}{1-x-x^2}$), find (and prove) a formula for

$$\sum_{n\geq 0} F_{2n+1} x^n \ .$$

(b) If $L_0 = 1$, $L_1 = 3$, $L_{n+1} = L_n + L_{n-1}$ for $n \geq 1$ (these are called the Lucas numbers), show that

$$\sum_{n\geq 0} L_n x^n = \frac{1+2x}{1-x-x^2} \ .$$

(2) Find a formula for the generating function

$$(1)_k + (2)_k x + (3)_k x^2 + (4)_k x^3 + (5)_k x^4 + \cdots = \sum_{n\geq 0} (n+1)_k x^n$$

(3) Use the last result and the formula $n^r = \sum_{k=1}^{r} S(r,k)(n)_k$ to find a formula for the generating function

$$\sum_{n\geq 0} (n+1)^r x^n$$

Use your formula to verify in particular that

$$\sum_{n\geq 0} (n+1) x^n = \frac{1}{(1-x)^2}, \qquad \sum_{n\geq 0} (n+1)^2 x^n = \frac{1+x}{(1-x)^3}, \qquad \sum_{n\geq 0} (n+1)^3 x^n = \frac{1+4x+x^2}{(1-x)^4},$$

Combinatorial problems and generating functions.

(1) Find the generating function for the ways of distributing n loonies to 5 people satisfying the following conditions:

(a) the first two people have at most 8 loonies each

(b) the first two people have at most 8 loonies together

(c) the first two people have together at most 8 loonies and an even number

(d) the first two people have together either at most 8 loonies or an even number

Check your answer on the computer to verify your answer that if there are 75 loonies to distribute to 5 people then for condition (a) there are 190,566 ways; for condition

(b) there are 114,060 ways; for condition (c) there are 62,900 ways; for condition (d) there are 802,781 ways.

(2) How many 9 digit numbers are there whose sum of digits is equal to 24? Answer: 5,949,615

Combinatorial identities:

(1) Give a formula for the generating function $P(x) = \sum_{n\geq 0} \binom{2n}{2} x^n$, $Q(x) = \sum_{n\geq 0} \binom{n}{2} x^n$ and $R(x) = \sum_{n\geq 0} n^2 x^n$. Show that $P(x) = 2Q(x) + R(x)$ and this implies that the coefficient of x^n in the LHS is equal to the coefficient of x^n in the right hand side and hence
$$\binom{2n}{2} = 2\binom{n}{2} + n^2 \ .$$

(2) Let $B_k(x) = \sum_{n\geq 0} \binom{n+k-1}{k-1} x^n$. Show that $B_k(x)B_\ell(x) = B_{k+\ell}(x)$. Take the coefficient of x^n in both sides of this equation and explain what identity this shows about binomial coefficients.