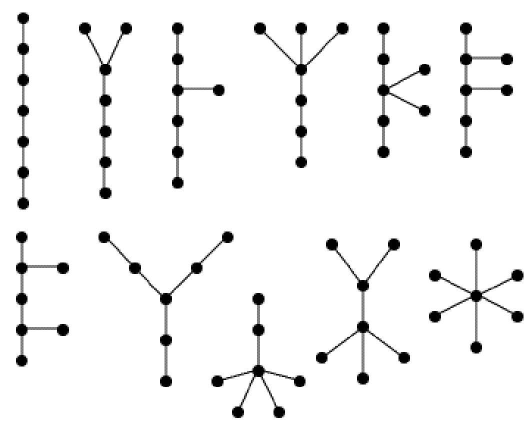
HOMEWORK #4 - MATH 4160

ASSIGNED: WEDNESDAY NOVEMBER 14, 2012   DUE: NOVEMBER 29, 2012

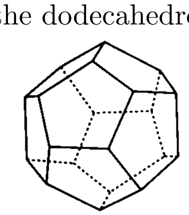
Write your homework solutions neatly and clearly. Provide full explanations and justify all of your answers.

Pólya enumeration problems.

(1) For each of the following trees (they are called trees because there are no loops and only branches), how many different ways are there of coloring the vertices with k colors so that two colorings of the graphs are the same if there is a mapping (not rigid) of vertices to vertices and edges to edges such that the colorings are equal?



(2) The dodecahedron has 12 pentagonal faces, 30 edges and 20 vertices. Explain clearly how you know that the of the motions of the dodecahedron form a group of order 60.



(a) How many motions of the dodecahedron leave precisely two vertices fixed (no edges or faces)? Why?
(b) How many motions of the dodecahedron leave precisely two edges fixed (no vertices or faces)? Why?
(c) How many motions of the dodecahedron leave precisely two faces fixed (no vertices or edges)? Why?

(d) How do you know that there are no other motions of the group except for the identity and those motions mentioned in part (a),(b),(c) ?

(3) If the motions of the dodecahedron are represented as a permutation of the faces, pick one permutation which fixes exactly two vertices and give the permutation represented in cyclic notation. Pick one permutation which fixes exactly two edges and give it in cyclic notation. Find one permutation which fixes exactly two faces and and give it in cyclic notation.
(4) How many ways are there of coloring the faces of the dodecahedron with black and white?
(5) How many ways are there of coloring the faces of the dodecahedron with black and white using four black faces and eight white faces?

Generating functions

(1) Recall that we have shown that $B(x) = e^{e^x-1}$ is the exponential generating function for the number of set partitions. Also recall that the Stirling numbers of the second kind are defined by $S(0,0) = 1$, $S(n,0) = S(0,n) = 0$ for $n \geq 1$ and $S(n,k) = S(n-1,k-1) + kS(n-1,k)$. Show that
$$e^{u(e^x-1)} = 1 + \sum_{n\geq 1}\sum_{k=1}^{n} S(n,k)u^k \frac{x^n}{n!} \ .$$
Note that one way of doing this (and there might be many ways) would be to let $B(x,u) = \sum_{n\geq 0}\sum_{k=1}^{n} S(n,k)u^k \frac{x^n}{n!}$ and show that $\frac{\partial}{\partial x}B(x,u) = ue^x B(x,u)$ and then show that the coefficient of x^0 is $1 = S(0,0)$ (similar to the way that we derived the formula for the exponential generating function of the Bell numbers). However, there are other ways of arriving at this result.
(2) Use this last result to show that
$$S(n,k) = \frac{1}{k!}\sum_{r=0}^{k}(-1)^{k-r}\binom{k}{r} r^n \ .$$
Using this formula calculate $S(4,k)$ for $k \in \{1,2,3,4\}$.
(3) Recall that the (unsigned) Stirling numbers of the first kind were satisfy the recursive formula $s'(n,0) = 0$ and $s(0,n) = 0$ for $n > 0$, $s'(0,0) = 1$ and for $n \geq 1$ and $1 \leq k \leq n$,
$$s'(n,k) = (n-1)s'(n-1,k) + s'(n-1,k-1).$$
Use this recursion to show that
$$e^{-ulog(1-x)} = 1 + \sum_{n\geq 1}\sum_{k=1}^{n} s'(n,k)u^k \frac{x^n}{n!} \ .$$