NOTES FROM THE FIRST TWO CLASSES

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main idea of this class

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$

to

$$1^r + 2^r + \cdots + n^r = ???$$

Just to show what we are up against:

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$
$$1^2 + 2^2 + 3^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$
$$1^3 + 2^3 + \cdots + n^3 = n^2(n+1)^2/4$$
$$1^4 + 2^4 + \cdots + n^4 = ???$$

but there is a sequence that continues:

$$1 + 2 + 3 + \cdots + n = n(n+1)/2$$

$$1 \cdot 0 + 2 \cdot 1 + 3 \cdot 2 + \cdots + n(n-1) = (n+1)n(n-1)/3$$

$$1 \cdot 0 \cdot (-1) + 2 \cdot 1 \cdot 0 + 3 \cdot 2 \cdot 1 + \cdots + n(n-1)(n-2) = (n+1)n(n-1)(n-2)/4$$

$$\vdots$$

$$1{\cdot}0{\cdot}(-1)\cdots(1-k)+2{\cdot}1{\cdot}0\cdots(2-k)+\cdots+n{\cdot}(n-1){\cdot}(n-2)\cdots(n-k) = (n+1)n(n-1)\cdots(n-k)/(k+2)$$

Proof either by (a) induction (b) telescoping sums

First class (a) the equality principle

If there is a bijection between a finite set A and a finite set B, then they have the same number of elements.

(b) the addition principle

say there are sets $A_1, A_2, \ldots, A_n$ with $|A_i| = a_i$ for $1 \leq i \leq n$ and all of the A_i are disjoint then the number of elements in $A_1 \cup A_2 \cup \cdots \cup A_n$ is

$$a_1 + a_2 + a_3 + \cdots + a_n$$

Example: Consider the set of words in 1 and 0 with three 1s and three 0s.

And paths in a 3×3

What about?

$$1^2 + 2^2 + \cdots + n^2 = n(n+1)(2n+1)/6$$

(c) multiplication principle

say there are sets $A_1, A_2, \ldots, A_n$ with $|A_i| = a_i$ for $1 \leq i \leq n$ and all of the A_i are disjoint then the number of elements in $A_1 \times A_2 \times \cdots \times A_n = \{(x_1, x_2, \ldots, x_n) \text{ where } x_i \in A_i\}$

is $a_1 a_2 \cdots a_n$

Example: lets say I was going to make a cereal with colored shape marshmellows
colors $= \{pink, yellow, orange, green, purple, red\}$
shapes $= \{hearts, moons, stars, clovers, horseshoes, balloons, pots\}$

I shouldn't have to list all possible marshmellows, $\{$*pink heart*, *pink moons*, *pink stars*, $\ldots$, *red pots*$\}$ instead it is much easier to say that there are 6 colors and 7 shapes so there are $6 \cdot 7 = 42$ marshmellows possible.

flavors $= \{chocolate, strawberry, peanutbutter\}$
eat it with $= \{fork, knife, spoon, chopsticks\}$
Then I could eat *chocolate purple ballons with a fork* (for example) but there should be $|colors| \cdot |shapes| \cdot |flavors| \cdot |\text{eat it with}| = 6 \cdot 7 \cdot 3 \cdot 4$ possibilities.

(d) division and subtraction - much harder, avoid doing it.

Application:
$S(n,k) =$ the number of set partitions of $\{1, 2, \ldots, n\}$ into k subsets
E.g.

$$\{123\}$$
$$\{12,3\}, \{13,2\}, \{1,23\}$$
$$\{1,2,3\}$$

$$\{1234\}$$
$$\{123,4\}, \{124,3\}, \{134,2\}, \{234,1\}, \{12,34\}, \{13,24\}, \{14,23\}$$
$$\{12,3,4\}, \{13,2,4\}, \{14,2,3\}, \{23,1,4\}, \{24,1,3\}, \{34,1,2\}$$
$$\{1,2,3,4\}$$

1			
1	1		
1	3	1	
1	7	6	1

but I can't do more of this table by hand because it there are too many set partitions of 5.

argue:

all set partitions of $\{1, 2, \ldots, n\}$ into k parts = the set partitions where n is by itself into $k-1$ other parts union the set partitions where n is with one of the other k parts of $\{1, 2, , n-1\}$ so

$$S(n, k) = S(n-1, k-1) + kS(n-1, k)$$

1					
1	1				
1	3	1			
1	7	6	1		
1	15	25	10	1	
1	31	90	65	15	1

$\cdots$

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first class: I covered

(1)

$$1{\cdot}0{\cdot}(-1)\cdots(1-k)+2{\cdot}1{\cdot}0\cdots(2-k)+\cdots+n{\cdot}(n-1){\cdot}(n-2)\cdots(n-k) = (n+1)n(n-1)\cdots(n-k)/(k+2)$$

(2)

addition and multiplication principle

(3)

definitions of $S(n, k)$ = the number of set partitions of $\{1, 2, \ldots, n\}$ into k parts. A set partition of $\{1, 2, \ldots, n\}$ is a division of $\{1, 2, \ldots, n\}$ into k nonempty and non-intersecting subsets

$$S(n, k) = S(n-1, k-1) + kS(n-1, k) \tag{1}$$

for $n > 1$ and $1 \leq k \leq n$ with the convention that $S(n-1, n) = 0$ and $S(n, 0) = 0$.

Proof. For shorthand, let $[n] := \{1, 2, ..., n\}$. The set partitions of $[n]$ into k parts can be divided into two sets, those that have n in a part by itself and those that have n in a part with other values from $[n-1]$. By the addition principle we have

$$S(n, k) = \text{ \# set partitions with } n \text{ in a set alone } + \text{ \# set partitions where } n \text{ is not alone}$$

The number of set partitions of $[n]$ into k parts with n in a part by itself is isomorphic to the set of set partitions of $[n-1]$ into $k-1$ parts by throwing away the set containing just n. This means that the number of set partitions of $[n]$ into k parts with n in a set all by itself is $S(n-1, k-1)$.

For a set partition P of $[n]$ with k parts and n is in a part with other elements, then let x be a value between 1 and k that indicates which of the k parts n is contained in and P' be the set partition of $[n-1]$ into k parts that is formed by removing n from P. Clearly if we know (x, P') then it is possible to recover P, and if we know P it is possible to recover both x and P'. Hence, there are the same number of these objects. Since there are k possible values of x and there are $S(n-1,k)$ possible set partitions P', then there are in total $kS(n-1,k)$ possible set partitions of $[n]$ into k parts where n is not in a part by itself.

Therefore (1) holds true. □

This recursion allows us to compute more of the table than before.

1					
1	1				
1	3	1			
1	7	6	1		
1	15	25	10	1	
1	31	90	65	15	1

⋯

There is an application for set partitions in terms of algebra.

Define for k and integer with $k > 0$, set:

$$(x)_k = x(x-1)(x-2)\cdots(x-k+1)$$

such that there are k terms in the product.

Examples: $(x)_1 = x$, $(x)_2 = x(x-1)$, $(x)_3 = x(x-1)(x-2), \ldots$

This is new notation that makes some of our formulas simpler.

Example: Remember the identity that we

$$1\cdot 0\cdot(-1)\cdots(1-k)+2\cdot 1\cdot 0\cdots(2-k)+\cdots+n\cdot(n-1)\cdot(n-2)\cdots(n-k) = (n+1)n(n-1)\cdots(n-k)/(k+2)$$

which is kind of horrible notation is equivalent to

$$(1)_{k+1} + (2)_{k+1} + \cdots + (n)_{k+1} = (n+1)_{k+2}/(k+2)$$

Now it arises that the table of numbers $S(n,k)$ appear in the expansion of x^n in terms of $(x)_k$. In particular we have

$$x^n = \sum_{k=1}^{n} S(n,k)(x)_k \tag{2}$$

Example:

$$(x)_1 = x^1$$

$$(x)_1 + (x)_2 = x + x(x-1) = x + x^2 - x = x^2$$

$$(x)_1 + 3(x)_2 + (x)_3 = x(x-1)(x-2) + 3x(x-1) + x = x^3$$

$$\begin{aligned}(x)_1 + 7(x)_2 + 6(x)_3 + (x)_4 &= x + 7x(x-1) + 6x(x-1)(x-2) + x(x-1)(x-2)(x-3) \\ &= x + 7(x^2 - x) + 6(x^3 - 3x^2 + 2x) + x^4 - 6x^3 + 11x^2 - 6x \\ &= x^4\end{aligned}$$

So it should seem surprising that it is even possible to give a formula for x^n in terms of $(x)_k$, and hopefully it is even more surprising that these coefficients are counted by combinatorial objects called set partitions.

Here is the quick proof that this formula holds:

Proof. We will prove this by induction on n. We have already shown the base case for $n = 1, 2, 3, 4$ above.

Assume that (2) holds for some fixed n. Then we have

$$x^{n+1} = x^n \cdot x = \sum_{k=1}^{n} S(n,k)(x)_k \cdot x \tag{3}$$

$$= \sum_{k=1}^{n} S(n,k)(x)_k(x-k+k) \tag{4}$$

$$= \sum_{k=1}^{n} S(n,k)(x)_k(x-k) + \sum_{k=1}^{n} kS(n,k)(x)_k \tag{5}$$

$$= \sum_{k=1}^{n} S(n,k)(x)_{k+1} + \sum_{k=1}^{n} kS(n,k)(x)_k \tag{6}$$

$$= \sum_{k=2}^{n+1} S(n,k-1)(x)_k + \sum_{k=1}^{n} kS(n,k)(x)_k \tag{7}$$

$$= S(n,n)(x)_{n+1} + \sum_{k=2}^{n} S(n,k-1)(x)_k + \sum_{k=2}^{n} kS(n,k)(x)_k + S(n,1)(x)_1 \tag{8}$$

$$= S(n,n)(x)_{n+1} + \sum_{k=2}^{n} (S(n,k-1) + kS(n,k))(x)_k + S(n,1)(x)_1 \tag{9}$$

$$= S(n,n)(x)_{n+1} + \sum_{k=2}^{n} S(n+1,k)(x)_k + S(n,1)(x)_1\ . \tag{10}$$

Some comments about this calculation: from step (6) to step (7) we did a shift of indices $k \to k-1$ (but they are the same sum). From step (7) to (8) we broke off the $k = n+1$ term of the first sum and the $k = 1$ term of the second sum. From step (9) to (10) we applied (1) with $n \to n+1$. Now recall that $S(n,n) = S(n+1,n+1) = 1$ and $S(n,1) = S(n+1,1) = 1$, hence we can rewrite the first and last term so that they are consistent with the other terms in this sum and hence we have shown

$$x^{n+1} = \sum_{k=1}^{n+1} S(n+1,k)(x)_k$$

which is equation (2) with $n \to n+1$.

Therefore by the principle of mathematical induction, (2) is true for all $n \geq 1$. $\square$

Because of the equation

$$\sum_{i=1}^{n} (i)_{k+1} = (1)_{k+1} + (2)_{k+1} + \cdots + (n)_{k+1} = (n+1)_{k+2}/(k+2)$$

that we wrote down above, this allows us to sum powers of i^r.

$$\sum_{i=1}^{n} i^1 = \sum_{i=1}^{n} (i)_1 = (n+1)_2/2 = (n+1)n/2$$

$$\sum_{i=1}^{n} i^2 = \sum_{i=1}^{n} ((i)_1 + (i)_2) = \sum_{i=1}^{n} (i)_1 + \sum_{i=1}^{n} (i)_2 = (n+1)_2/2 + (n+1)_3/3$$

With a little algebra we can show:

$$(n+1)_2/2+(n+1)_3/3 = (n+1)n/2+(n+1)n(n-1)/3 = (n+1)n(1/2+(n-1)/3) = n(n+1)(2n+1)/6$$

$$\sum_{i=1}^{n} i^3 = \sum_{i=1}^{n} ((i)_1 + 3(i)_2 + (i)_3) = (n+1)_2/2 + 3(n+1)_3/3 + (n+1)_4/4$$

The right hand side is a polynomial in n of degree 4 and we can calculate directly that,

$$(n+1)_2/2+3(n+1)_3/3+(n+1)_4/4 = (n+1)n/2+(n+1)n(n-1)+(n+1)n(n-1)(n-2)/4 = n^2(n+1)^2/4$$

And the formula for the sum of the 4^{th} powers of i is

$$\sum_{i=1}^{n} i^4 = \sum_{i=1}^{n} ((i)_1 + 7(i)_2 + 6(i)_3 + (i)_4) = (n+1)_2/2 + 7(n+1)_3/3 + 6(n+1)_4/4 + (n+1)_5/5$$

and the right hand side in the form it is in is cleaner than calculating the polynomial:

$$(n+1)_2/2 + 7(n+1)_3/3 + 6(n+1)_4/4 + (n+1)_5/5 = n(n+1)(2n+1)(1-3n+3n^2)/30\ .$$

What is great about what we have done is here is that it is difficult to conjecture the right hand side of this sum or for higher powers (so that one might prove it by some other means). Instead here we have proven an explicit formula which works for all powers of r, that is:

$$\sum_{i=1}^{n} i^r = \sum_{k=1}^{r} S(r,k)(n+1)_{k+1}/(k+1)\ .$$